The use of BMAD in simulating transverse and longitudinal dynamics in RHIC

H. Lovelace III

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Collider Accelerator Department

Brookhaven National Laboratory

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Henry Lovelace III

Collider-Accelerator Department
Brookhaven National Laboratory
Upton, N.Y. 11973

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Henry Lovelace III
C-AD
Brookhaven National Laboratory

Abstract

In accelerator physics, models of a given machine are used to predict the behaviors of the beam, magnets, and radiofrequency cavities. The use of the computational model has become wide spread to ease the development period of the accelerator lattice. There are various programs that are used to create lattices and run simulations of both transverse and longitudinal beam dynamics. The programs include Methodical Accelerator Design(MAD) MAD8, MADX, Zgoubi, Polymorphic Tracking Code (PTC), and many others. In this discussion the BMAD (Baby Methodical Accelerator Design) is presented as an additional tool in creating and simulating accelerator lattices for the study of beam dynamics in the Relativistic Heavy Ion Collider (RHIC).

Introduction

A sequence of elements and there attributes that a particle traverses is known as a lattice. In closed geometry lattices, such as storage rings and synchrotrons, the FODO (focusing, drift, defocusing, drift) lattice system is commonly used. Variations may occur in the lattice, however, the derivation of the closed orbit and other transverse beam dynamics still hold true. The local reference orbit describes the coordinate system that is used for BMAD particle tracking.
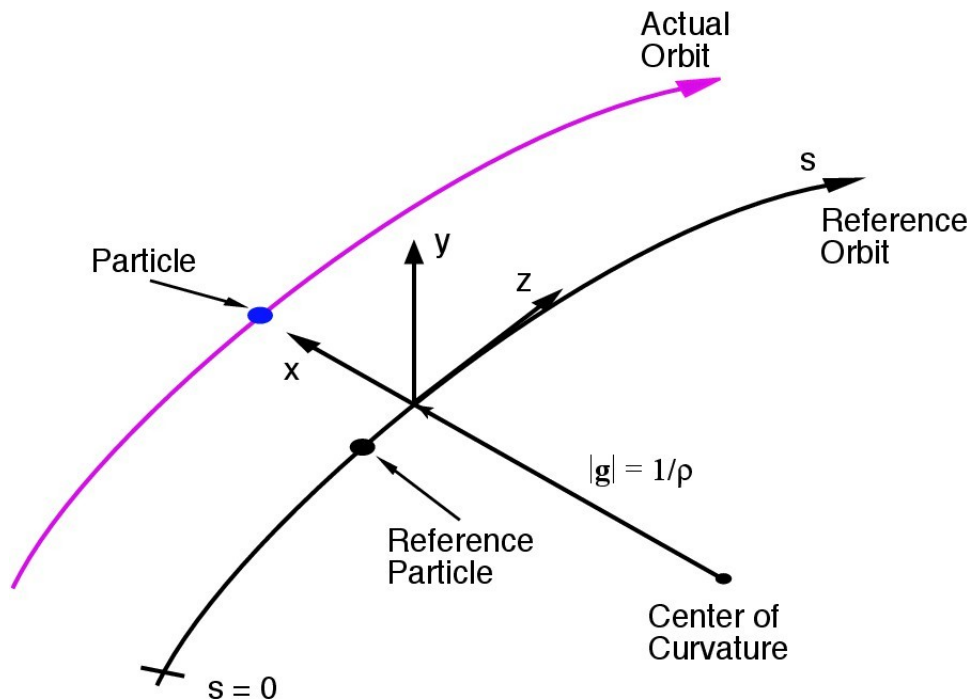
Hill's equation, which was first used to describe the three body problem of orbits of the Moon around the Earth, are now used to describe the motion of a particle in an accelerator. Hill's equation is a second order ordinary differential equation,

$$x'' + K_x(s)x = 0$$

where

$$x' \equiv dx/ds$$

$$K_x(s) \equiv B'/(B\rho) + \rho^{-2}$$

$$B' \equiv \partial B_y / \partial x$$

where B is the magnetic field strength, and ρ the radius of the circular lattice. The solution of Hill's equation in phase amplitude format is

$$x(s) = A\sqrt{\beta(s)}\cos(\varphi(s) + \delta)$$

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos(\varphi(s) - \varphi_0)$$

where ε is the emittance, β is the amplitude of the oscillation, φ is the phase and φ_0 phase offset. Taking the derivative of the solution we have,

$$x'(s) = \frac{-A}{\sqrt{\beta(s)}}[\alpha(s)\cos(\varphi(s) + \delta) + \sin(\varphi(s) + \delta)]$$

$$\alpha(s) \equiv -\beta'(s)/2$$

Rewriting Hill's equation in terms of A we have the Courant-Snyder invariant,

$$\varepsilon = A^2 = \gamma(s)x^2 + 2\alpha(s)xx' + \beta x'^2$$

we have α, β, and γ which, including ε, are the ellipse parameters. Note that BMAD does not use Hill's equation for tracking. BMAD uses the six dimensional phase space coordinates (x, p_x, y, p_y, z, p_z) with transfer maps. The PTC /FPP engine within BMAD allows the calculation of both Taylor maps (Truncated Power Series Algebra) and Lie algebra operations. FPP is a package that is used to produce and analize the Taylor approximation of the map. BMAD has the ability to track particles using symplectic and non-symplectic maps. An example of how to calculate the Courant-Snyder invariant from the transfer map follows.

Let *M* be a the one turn map of somewhere in the lattice.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad ad - bc = 1$$

There is a symplectic matrix *S* that satisfies this condition,

$$M = SRS^{-1}$$

$$R = \begin{bmatrix} \cos(v) & \sin(v) \\ -\sin(v) & \cos(v) \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} S_{22} & -S_{12} \\ S_{21} & -S_{11} \end{bmatrix}$$

M can now be written as

$$M = \cos(\mu) I + \sin(\mu) \begin{bmatrix} -S_{11}S_{21} - S_{12}S_{22} & S_{11}^2 + S_{12}^2 \\ -S_{21}^2 - S_{22}^2 & S_{11}S_{21} + S_{12}S_{22} \end{bmatrix}$$

$$M = \begin{bmatrix} \cos(\mu) + \alpha sin(\mu) & \beta \sin(\mu) \\ -\gamma sin(\mu) & \cos(\mu) - \alpha sin(\mu) \end{bmatrix}$$

This is the de Moivre formula for the one degree of freedom sympletic model. The angle of the rotation is known as the fractional tune when measured in revolutions.

$$det\ M = \gamma\beta - \alpha^2 = 1$$

The Courant-Snyder invariant can now be written as

$$\varepsilon = \left(S_{21}^2 + S_{22}^2 \right) x^2 + 2 \left(-S_{11}S_{21} - S_{12}S_{22} \right) xp + \left(S_{11}^2 + S_{22}^2 \right) p^2$$

The operating point of a circular accelerator is defined as

$$v = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

where s is the longitudinal coordinate position in the beam pipe. The chromaticity of the beam is defned as

$$\xi = \frac{\Delta v}{\Delta p / p}$$

where p is the longitudinal momentum of the particle. The revolution frequency of the particle is defined as the

$$f_{rev} = \frac{1}{T_o} = \frac{c\beta}{C}$$

where T_0 is the period of one revolution, C is the circumfernce of the accelerator, c is the speed of light, and β is the ratio of particle velocity and speed of light . The RF-frequency is determined by

$$f_{rf} = h f_{rev}$$

where h is the harmonic number. From the dipole bending, dipersion occurs which causes horizontal orbit deviations. Dispersion can be defined as

$$\Delta x = D(x,s)(\Delta p / p_0)$$

where x is the orbit position and D is the dispersion function. In BMAD, the dispersion calculation is taken a step further with the use of the define dispersion equations

$$\eta_x(s) = \frac{dx}{dp_z} \quad \eta_y(s) = \frac{dy}{dp_z} \quad \eta_z(s) = \frac{dz}{dp_z}$$

$$\eta'_x(s) \equiv \frac{d\eta_x}{ds} = \frac{dx'}{dp_z} \quad \eta'_y(s) \equiv \frac{d\eta_y}{ds} = \frac{dy'}{dp_z}$$

where η and η' describe the dispersion and dispersion derivative.

The energy of the particle beam is defined by the product of the species mass in GeV and the Lorentz factor. The momentum of the beam can be calculated using the relation,

$$p = \frac{E_{tot} * \sqrt{\left(1.0 - \left(mc^2 / E_{tot}\right)^2\right)}}{c}$$

where p is momentum, m is the mass of the particle, E_{tot} is the total beam energy, and c is the speed of light. It is useful for heavy ions to describe both the energy and momentum per nucleon. In a bunch, the particles differ in momenta. The closed orbit differs due to the difference in momenta and can be expressed by a factor,

$$\alpha_p = \frac{\frac{\Delta C}{C}}{\frac{\Delta p}{p_0}} \quad \alpha_p = \frac{1}{C} \int_s \left\langle \frac{D}{\rho} \right\rangle ds \approx \frac{\sum_{(all\,dipole)} D\theta_i}{C}$$

where the D is the dipersion, ρ is the bending radius, and θ_i is the bending angle. Both the bending radius and angle are attributes of the dipole. Transition gamma can then be defined as,

$$\gamma_t \equiv \frac{1}{\sqrt{\alpha_p}}$$

and the amount of slippage is,

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \quad .$$

A subroutine has been written to calculate this momentum compaction in a closed geometry lattice.

```
!----------------------------------------------------------------------
!+
! subroutine alpha_pcalc(lat, alpha_p)
!
! Routine to calculate the momentum compaction factor for a given lattice. Geometry
!        does not need to be closed. Radiation_integrals and twiss parameters
!        must be calculated before this subroutine is called. This is an
!        approximation. !
! Input:
!   lat     -- lat_struct!
! Output:
!   alpha_p -- real(rp)!-
subroutine alpha_pcalc(lat, alpha_p)

type (lat_struct), target :: lat
type (ele_pointer_struct), allocatable :: eles(:)
type (branch_struct), pointer :: branch
integer :: ix_branch, n_ele_track, n_loc1, j, n, ix
integer, allocatable :: ix_dipole(:)
real(rp) dispersion
real(rp), intent(out):: alpha_p
logical err_flag

 if (lat%param%particle == electron$ .or. lat%param%particle == positron$) then
  alpha_p = mode%synch_int(1) / lat%param%total_length
 else
 branch => lat%branch(ix_branch)
 n_ele_track = branch%n_ele_track
  call lat_ele_locator ("sbend::*", lat, eles, n_loc1, err_flag)
  allocate (ix_dipole(count(branch%ele%key == sbend$)))
  j = 0
  do n = 1, branch%n_ele_max
   if (branch%ele(n)%key /= sbend$) cycle
   j = j + 1
   ix_dipole(j) = n
  enddo
  do n = 1, size(ix_dipole)
   ix = ix_dipole(n)
   dispersion = (branch%ele(ix)%value(angle$) * branch%ele(ix)%a%eta) + dispersion
  enddo
  alpha_p = dispersion / lat%param%total_length
 endif
end subroutine alpha_pcalc
!----------------------------------------------------------------------
```

Closed orbit calulations are done using absolute time tracking. This means that the particle on the first turn enters into the RF cavity with a phase equal to zero, on subsequent turns the phase will increase.

BMAD also has the ability to calculate dynamic aperture of a given lattice. The dynamic aperture is the maximum phase-space amplitude within which particles do not get lost as a consequence of single particle dynamics effect[1]. For proton rings, the concern is only the at injection due to the bunch adiabatically damping during acceleration. For simulations, the number of turns is 2000.

BMAD Comparison

The lattices that were used for the simulations were parsed from lattices that were created in the MADX format using the Universal Accelerator Parser(UAP). The UAP has the ability to translate between Accelerator MarkUp Language (AML), BMAD, MAD8, and Extended Standard Input Format (XSIF). Radiofrequency cavities were added to the lattice to give a more realistic approach to the simulations. The species of the particle used in the two lattices that were parsed are proton and gold stripped of all electrons. The two lattices are injection with gamma equal to 25.379, and at store with gamma equal to 271.635 of the blue ring only for protons in Run 17. For gold, injection gamma is 10.520 and store gamma 107.3961. The particle mass is taken to be 0.938 GeV for protons and for gold, 183.43 GeV. The lattice length is 3833.845 m. The number of magnet elements are exactly the same as the MADX lattice with the same s-coordinates.

The single particle and first turn tracking were implemented for both lattices. In using the parser, the strengths of all the quadrupole and dipole magnets within the MADX code are kept the same with the exception of the sextupole magnet strengths which were changed to provide for the lattice a chromaticity that is approximately 2 at both injection and store. The calculated natural chromaticity of the blue proton ring is -80.581 horizontal, and -15.953 vertical at injection and at store -70.697 and -63.413.For gold, the natural chromaticity is -84.563 horizontal, and -8.273 vertical at injection and at store -101.662 horizontal and -80.435 vertical. The natural chromaticity was found by setting all of the sextupole magnet strengths to zero.
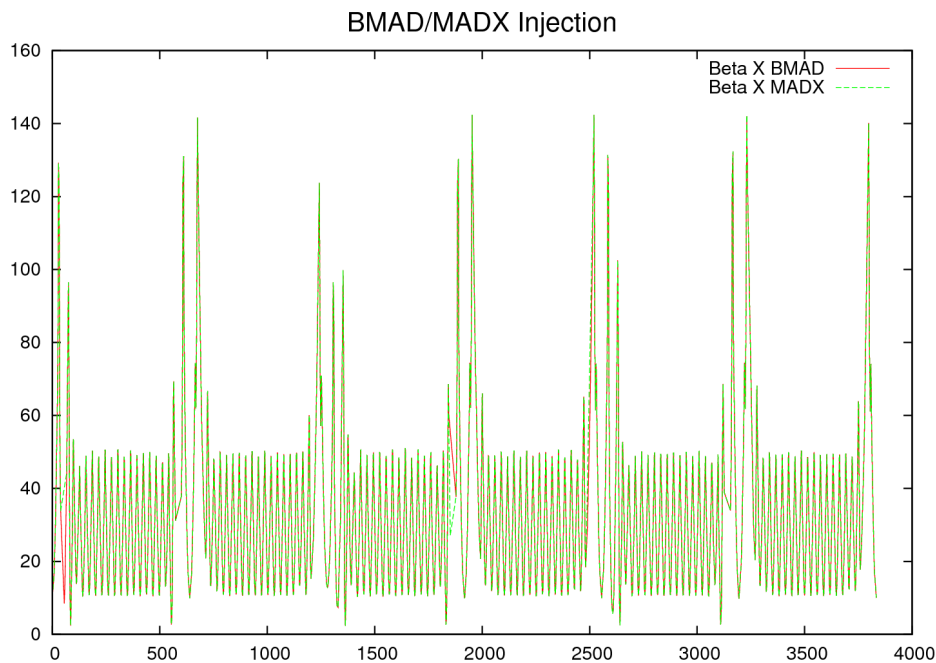

The betatron tunes at injection are 28.697, horizontal and 29.687, vertical. The amplitude of the beta function is taken at each element.

BMAD/MADX store
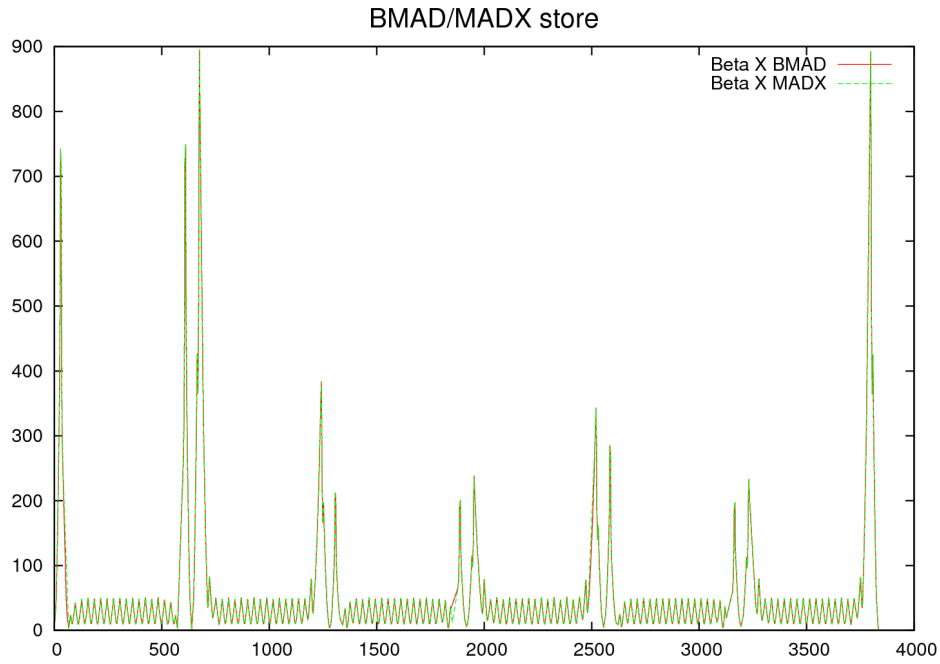
The tunes calculated at store are 28.689, horizontal and 29.684, vertical. The value for the tunes in both planes and for both lattices are in close agreement with the lattices that were used for parsing without retuning. With both tune calculations, at injection and at store, the error is less than 1%. The percent error in the measurement of the horizontal beta amplitudes at the two IR 6 o'clock and 8 o'clock at store energy are both less than 1%, $1.62 * 10^{-3}$ and $2.36 * 10^{-3}$, respectively.

Revolution Frequency during Energy Ramp
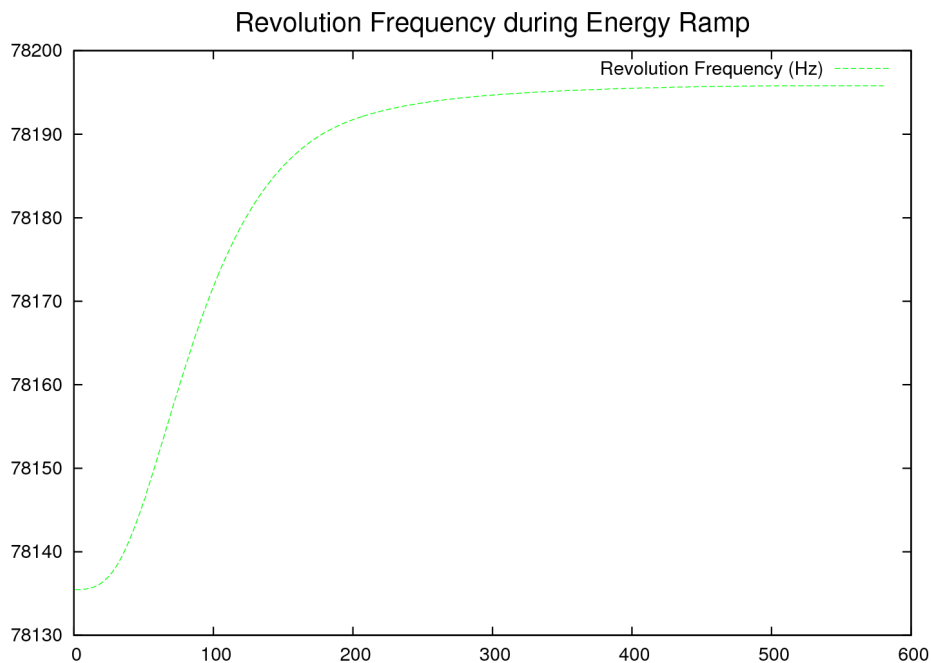
The BMAD calculated revolution frequency at injection is 78135.559 Hz and at store 78195.757 Hz. The actual injection and store revolution frequency measurements are 78135.474 Hz and 78195.756 Hz. The reason for the discrepencies between revolution frequency at injection is that the circumference is lengthened due to the helical dipoles for proton by 4.16 mm at injection.

Orbit/Dispersion at Injection
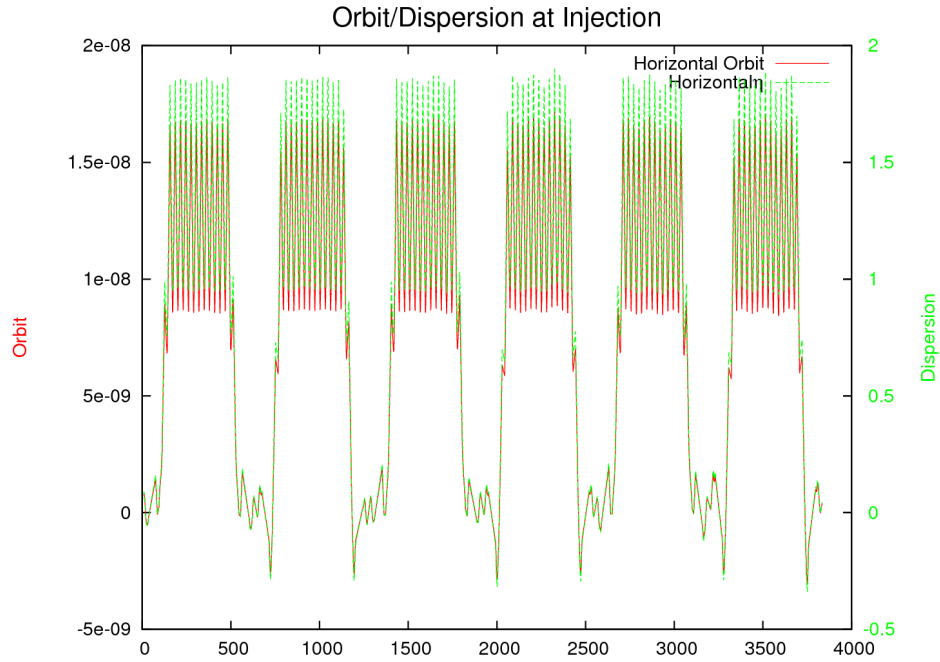
The lattices were improved with the inclusion of two RF gaps at injection and three at storage energy. These gaps represent the storage, acceleration, and Landau cavities used in ramping process. The three frequencies used in this process are 9.376 MHz and 28.129 MHz at injection, and 197.053 MHz is used for the storage gap. The corresponding harmonics are 120, 360, and 2520, respectively.

Orbit/Dispersion at Store
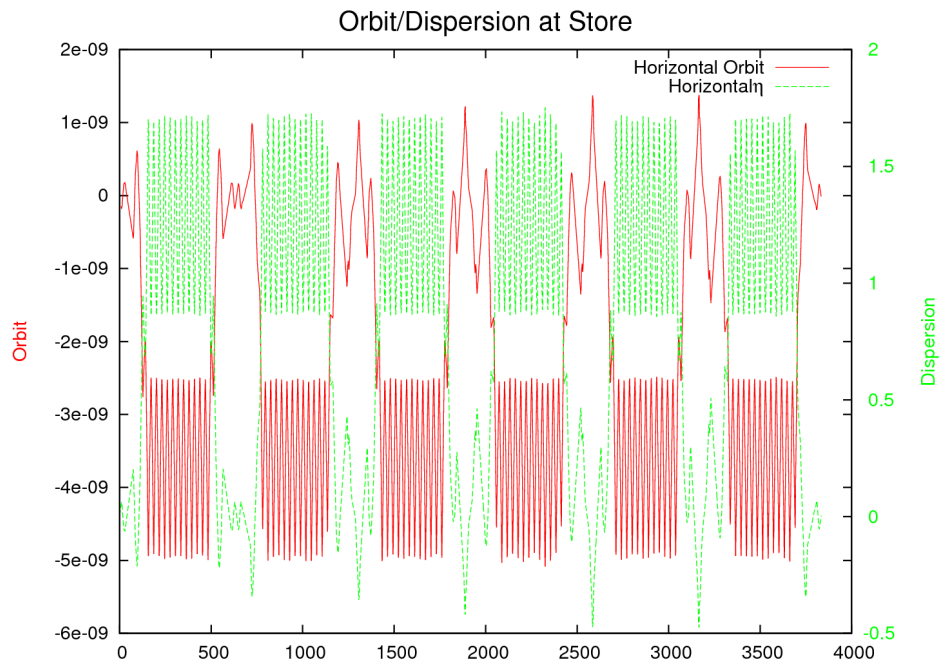
The effect of the RF on the close orbit is that in regions of zero dispersion the orbit excursion is at a minimum. The negative orbit excursion at store energy is due to negative momentum deviation. With increasing the energy at store or decreasing dipole magnet strength the orbit excursion can be made positive.

The phase space can also be studied using BMAD code. Multi-turn tracking calulation of the 6 dimensional phase space can be made where an initial vector can be given and a number of turns specified. Phase ellipse focusing and defocusing can be demonstrated. In the first plot, a single particle is plotted for 1000 turns through the focusing quadrupole BO7_QF4, for the second plot the particle is plotted trhough the defocusing quadrupole BO10_QD1. Both plots are at injection energy.
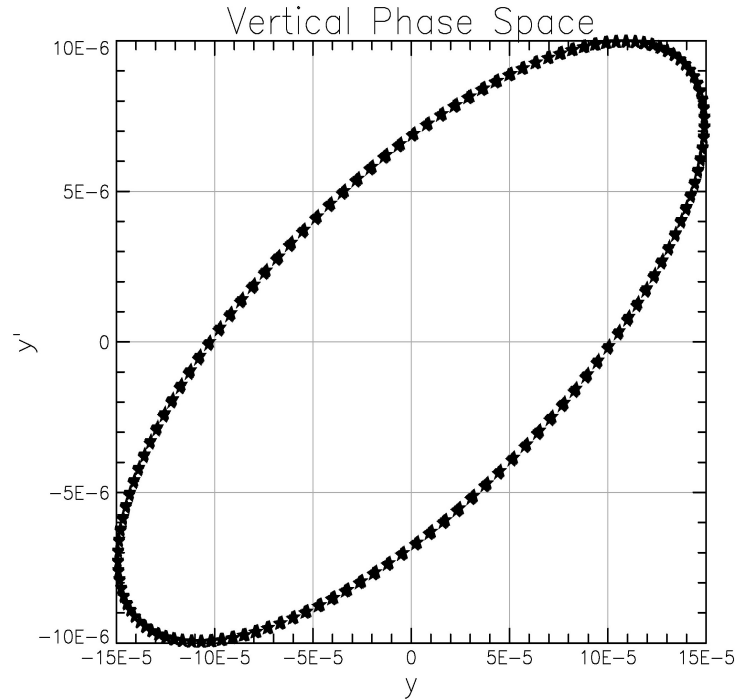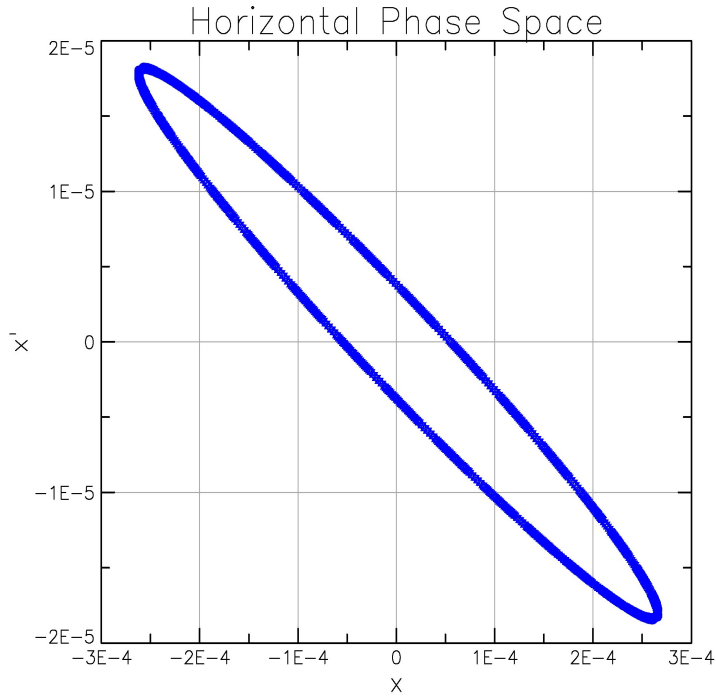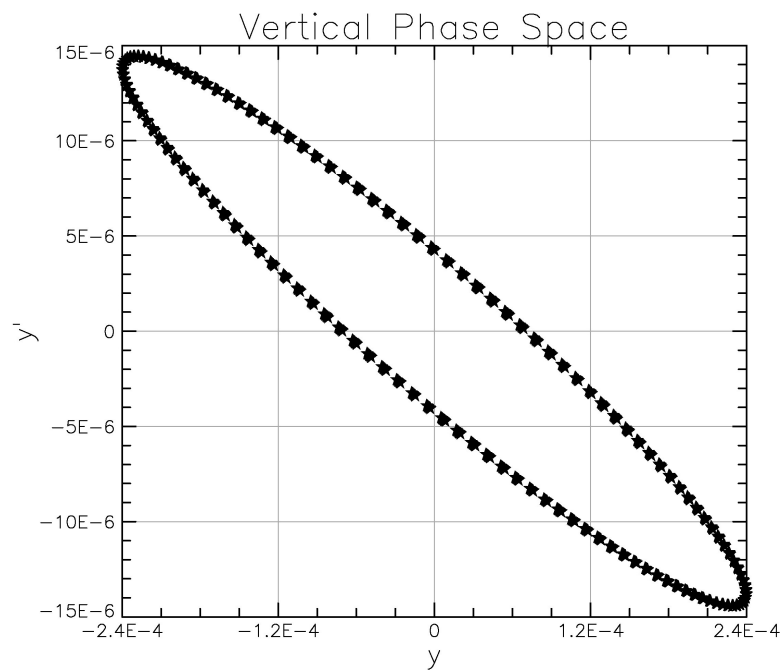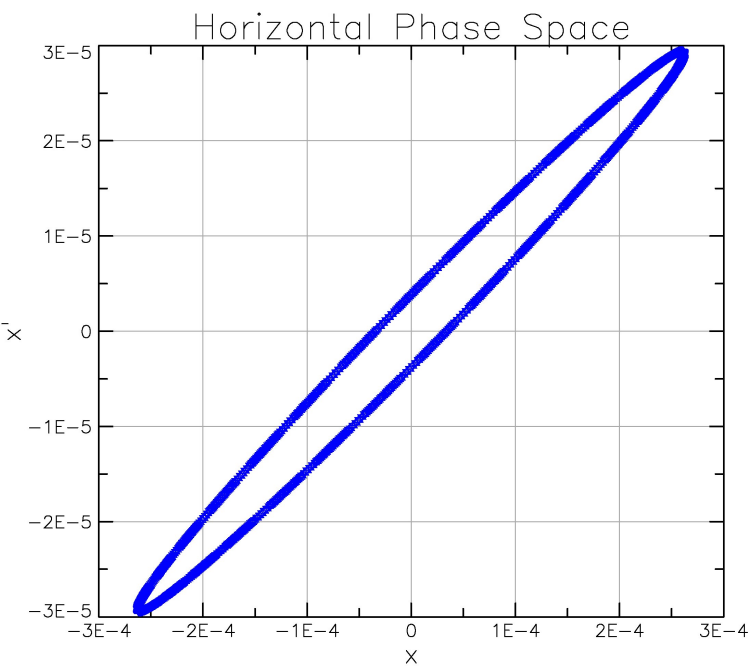
For the gold lattice, the tunes at injection are 28.2441 horizontally and 29.2249 vertically are different from the MADX lattice tunes 28.2990 horizontally and 29.1634 vertically. This discrepency is due to diffences in the quadrupole strengths. With adjustments to the quadrupole strength the tunes can be made to match.

BMAD/MADX at Injection Gold

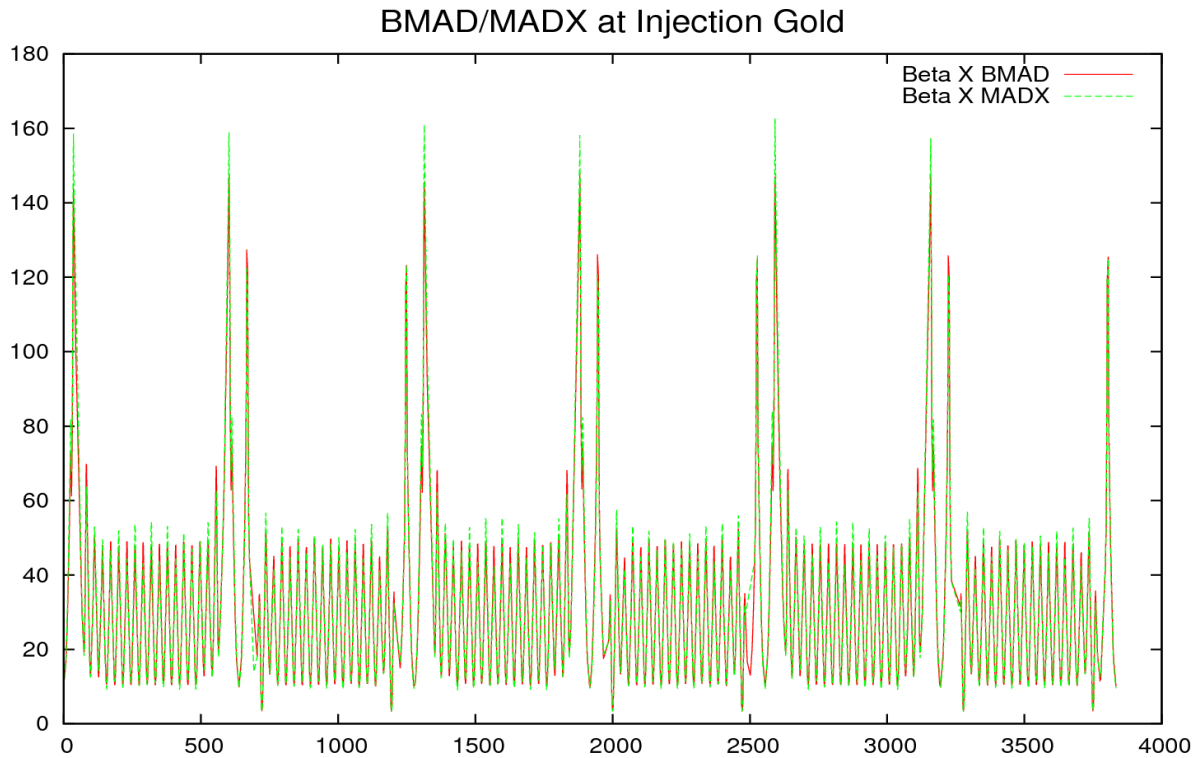At store the tunes are 28.2420 horizontally and 29.21889 vertically. The MADX tunes are 28.2340 horizontally and 29.2280 vertically. There is a -3.419% error between the fraction part of the horizontal tune and a 3.995% error in fraction part of the vertical. As shown in both plots the beta amplitude calculated for gold at both injection and store closely match the results generated from MADX. The percent error in the measurement of the horizontal beta amplitudes at the two IR 6 o'clock and 8 o'clock at store energy are both less than 2%, 1.777% and 1.632%, respectively. The increase in percent error, in comparison to the protons, is due to the tune difference. The transition gamma for the RHIC design lattice is 22.89, for the calculated transition is 22.911 at injection energy which has an error less than a .1%.

BMAD/MADX at store Gold
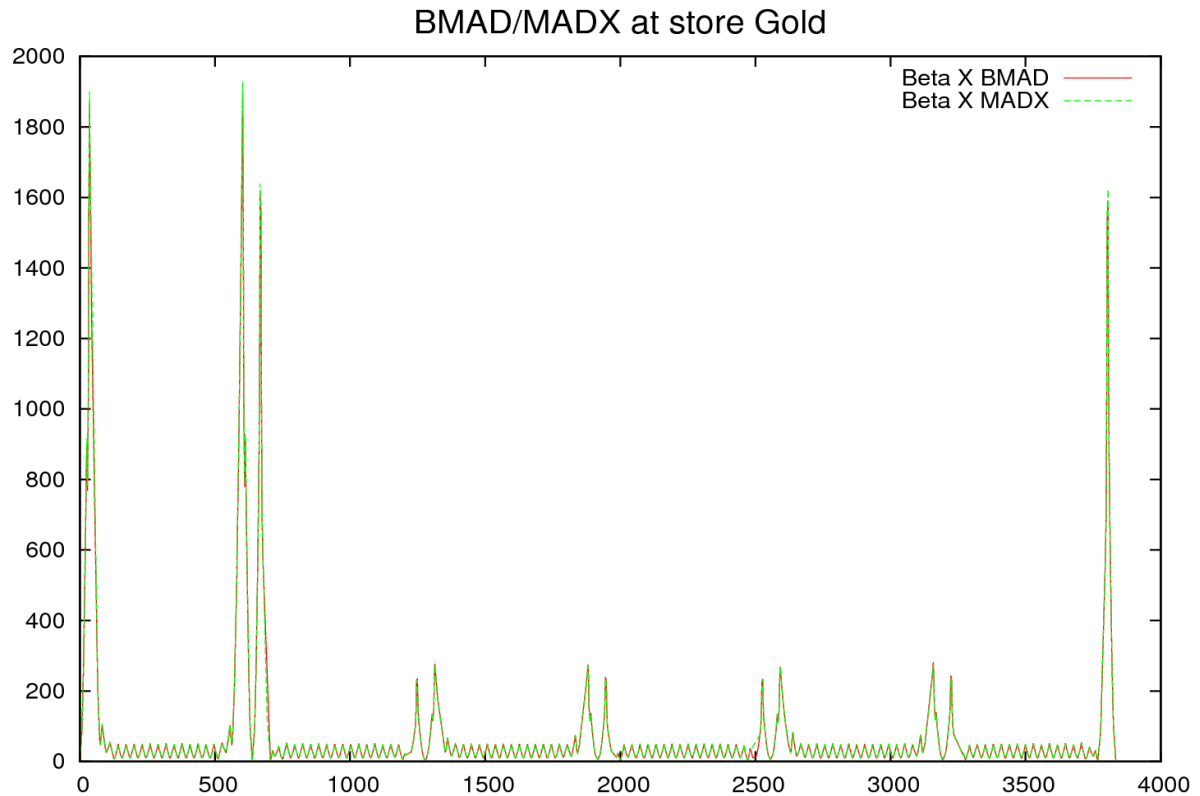
The BMAD calculated revolution frequency at injection is 77842.234 Hz and at store 78192.897 Hz. The actual injection and store revolution frequency measurements are 77842.232 Hz and 78192.897 Hz.
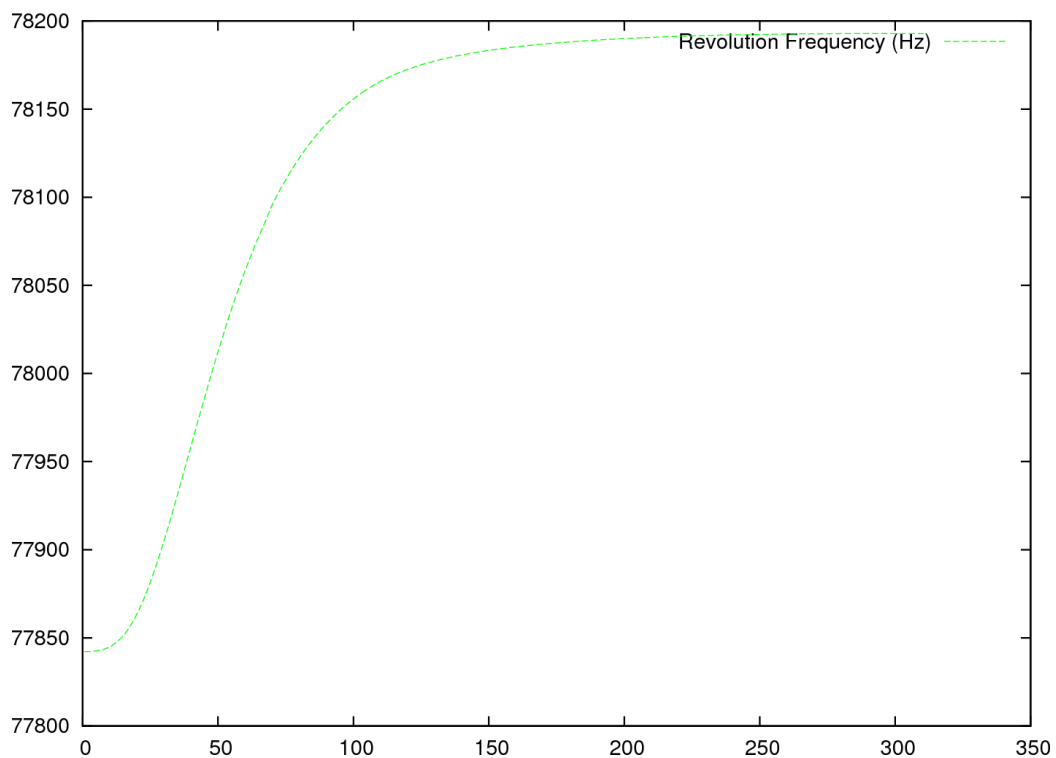
The gold lattices were improved with the inclusion of one RF gaps at injection and two at storage energy. These gaps represent the storage and acceleration cavities used in ramping process. The two frequencies used in this process are 28.129 MHz at injection, and 197.053 MHz at store. The

corresponding harmonics are 360, and 2520, respectively.
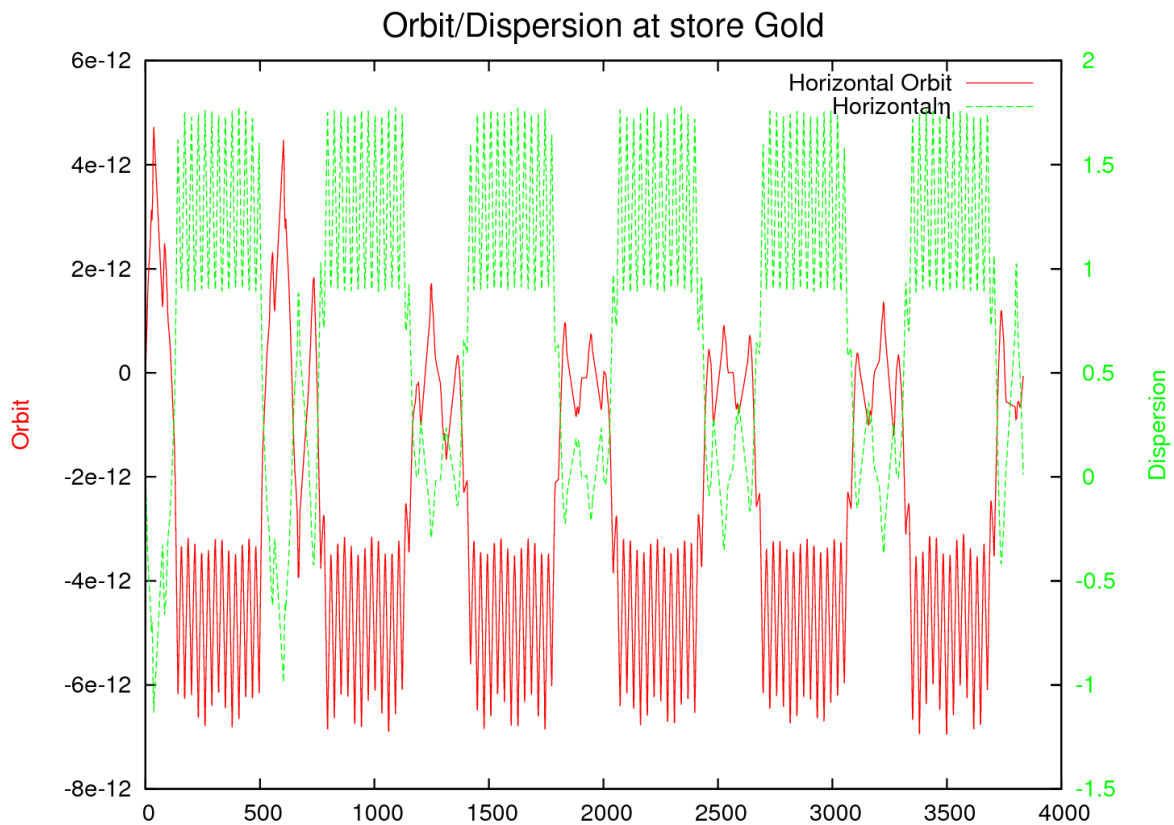
Orbit/Dispersion at Injection Gold

Orbit/Dispersion at store Gold

The phase space can also be studied using BMAD code. Multi-turn tracking calulation of the 6 dimensional phase space can be made where an initial vector can be given and a number of turns

specified. Phase ellipse focusing and defocusing can be demonstrated. In the first plot, a single particle is plotted for 1000 turns through the focusing quadrupole YO1_QF20, for the second plot the particle is plotted through the defocusing quadrupole YO1_QD11. Both plots are at injection energy.

phase_space.dat
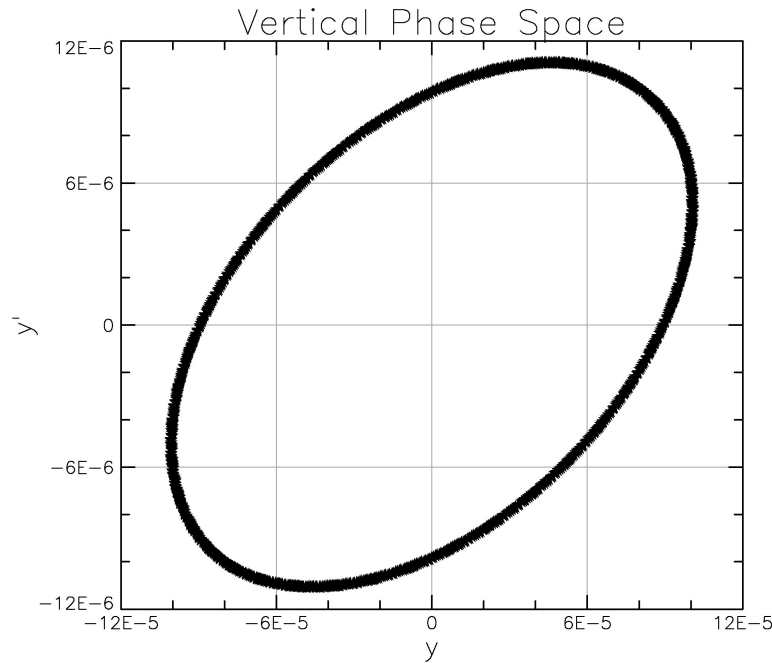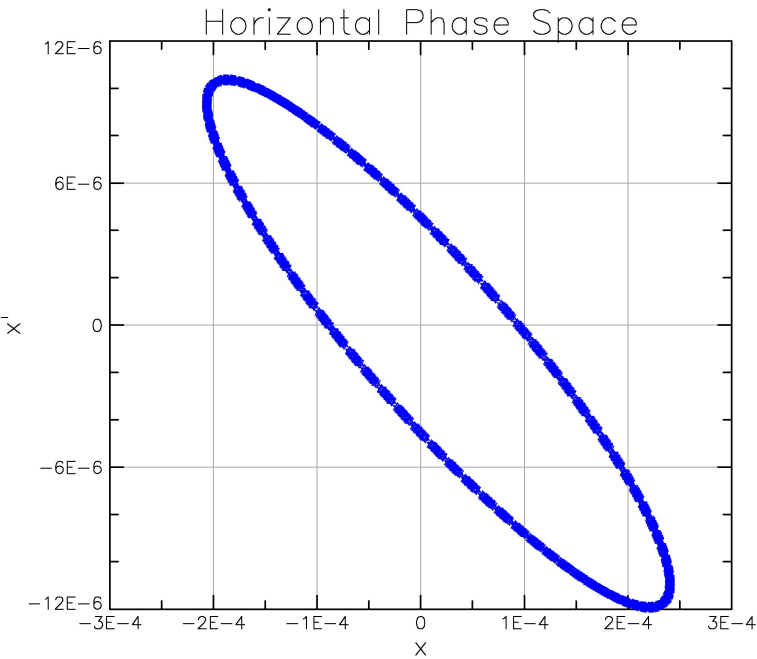
phase_space.dat
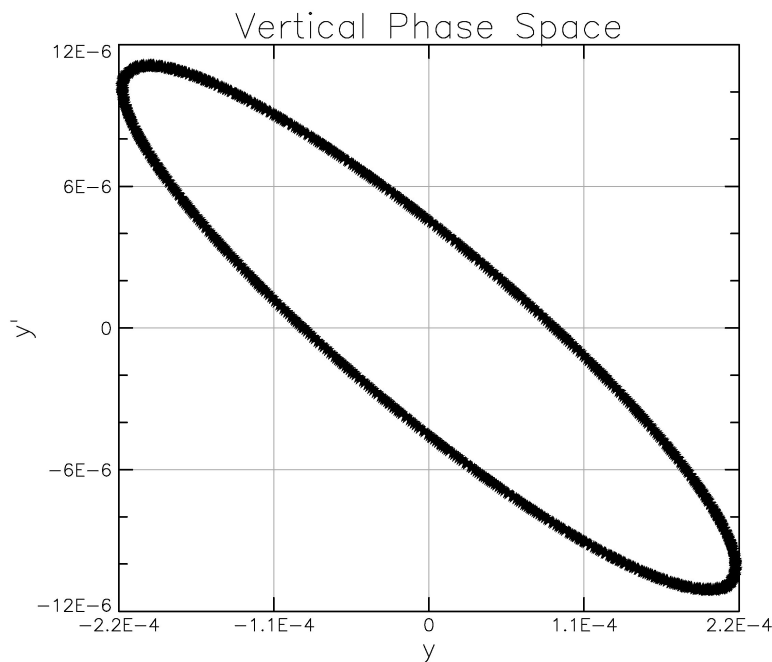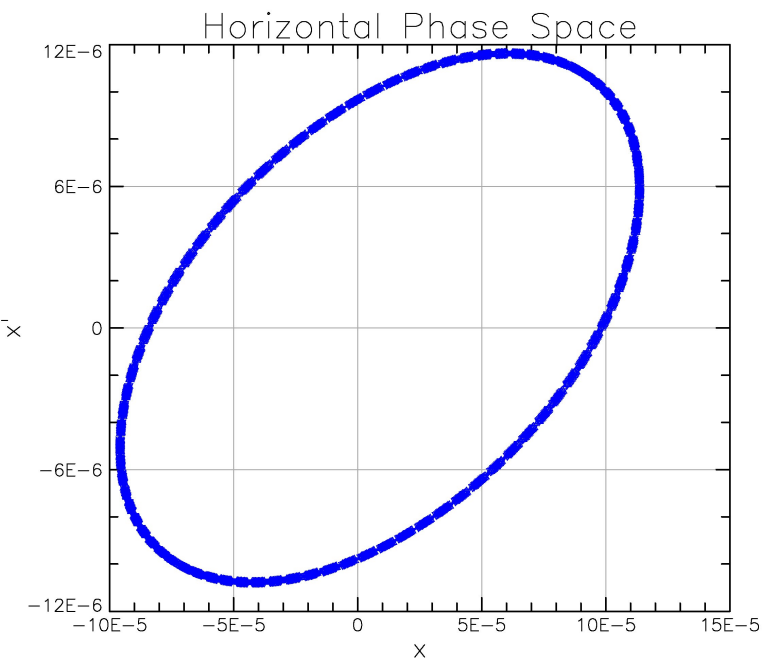
In RHIC, the first 1024 turns are tracked to give an approximation of the number of amplitude of the oscillations taken in a turn by turn method. A maximum amplitude plot can be created with a given initial amplitude. An energy difference is given for each plot. The initial amplitude given to the proton is .5 mm in both planes. The points are generated when the particle becomes lost within the 2000 turns that are tracked with the RF off.

Dynamic Aperture Blue Proton Injection
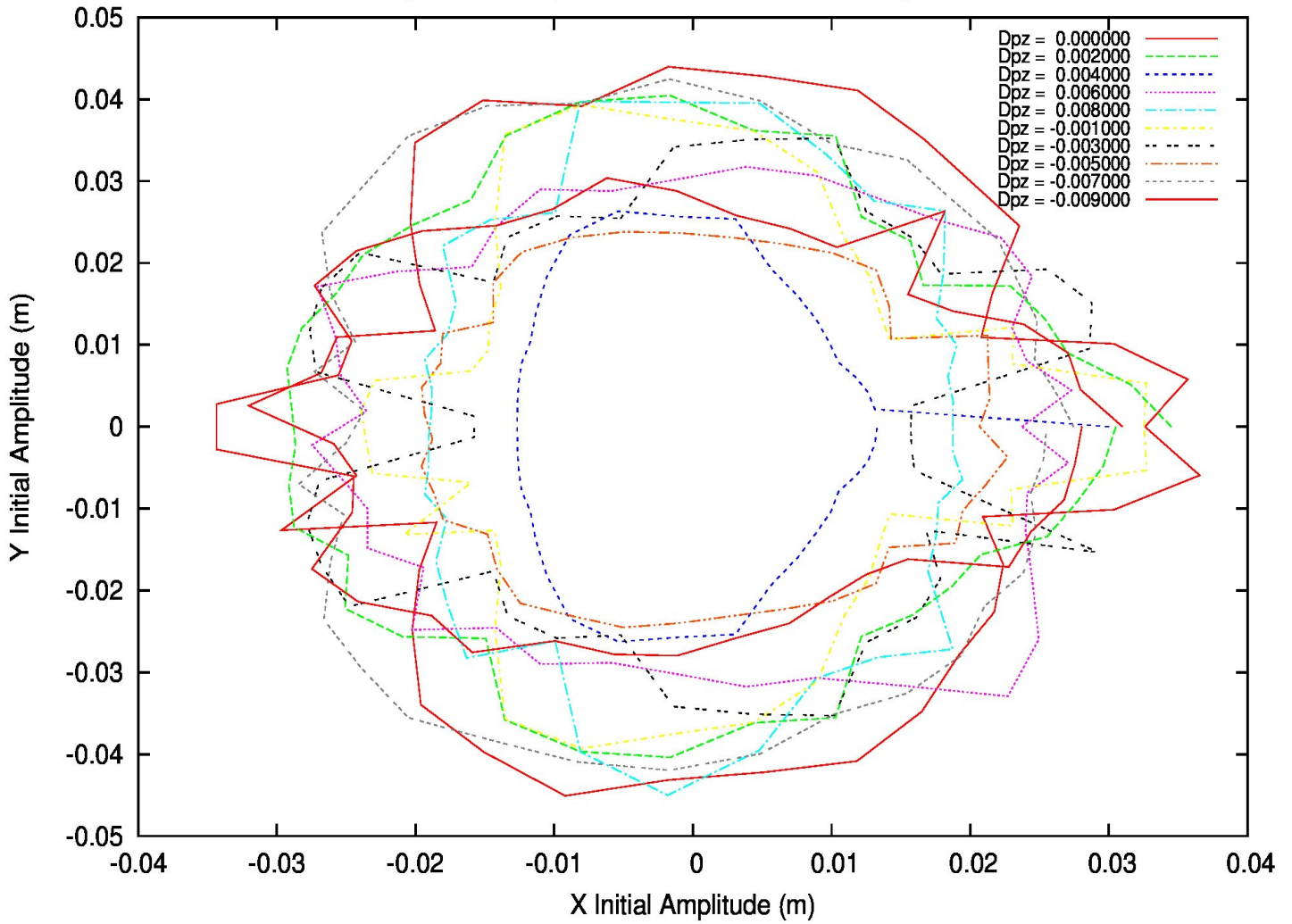
Conclusion

A comparison between BMAD and MADX simulations is useful. The power of BMAD in creating, modeling, and particle tracking through a lattice configuration was demonstrated. The close approximation in the results for lattice modeling show that BMAD may be implented as a stand alone program in future lattice designs.The percent error in the measurement of the horizontal beta amplitudes at the two IR, 6 o'clock and 8 o'clock, at store energy for protons are $1.62 * 10^{-3}$ % and $2.36 * 10^{-3}$ %, respectively. For gold, the percent error of the horizontal beta amplitudes at the two IR are less than 2%, 1.777% and 1.632%. The increased percent error for the beta amplitudes IR is due to the -3.419% error horizontally. If the tunes were matched completely as in the proton lattice, then the percent error would be less than 1%. The single particle tracking through specific elements in a lattice allows for the study of the development of the beam envelope over a given amount of turns. Amplitude plots can be created using BMAD for protons with particle survival being the point generator.

Acknowledgements

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BMAD

Lattice Parameter	Protons injection	Protons store	Gold injection	Gold store
γ	25.37862	271.6349	10.52046	107.3960
ν_h	28.69700	28.68878	28.24416	28.24203
ν_v	29.68700	29.684020	29.22492	29.21889
ξ_h	2.08977	2.01335	2.0146	2.0057
ξ_v	2.084395	2.00643	2.0259	2.0074
ξ_h^{nat}	-80.581	-70.697	-84.563	-101.662
ξ_v^{nat}	-15.953	-63.413	-8.273	-80.435
β_h^{max}	142.4171	894.9579	147.0132	1895.4639
β_v^{max}	142.1800	916.5739	150.5372	1874.1010
η_h^{max}	1.8991	1.7505	1.8194	1.8237
η_v^{max}	0	0	0	0
Rev. frequency	78135.5588*	78195.7569	77842.2325	78192.8973

*circumference lengthening not accounted for in calculation

MADX

Lattice Parameter	Protons injection	Protons store	Gold injection	Gold store
γ	25.37862	271.6349	10.52046	107.3960
ν_h	28.69700108	28.68878855	28.29902955	28.2339528
ν_v	29.68700054	29.68402021	29.16344748	29.22804679
ξ_h	2.246452883	2.248794477	-7.885572288	2.285092098
ξ_v	2.083578948	2.09939207	-6.015872636	2.11034623
β_h^{max}	142.4170621	894.9569606	162.6747942	1928.242576
β_v^{max}	142.1800042	916.5740045	153.4966541	1900.295911
η_h^{max}	1.900555941	1.750497005	1.829091458	1.795521833
η_v^{max}	0	0	0	0
Rev. frequency*	78135.474	78195.757	77842.232	78192.897

* Logview Application

<div align="center">

Appendix

Derivation Courant-Snyder invariant using phasor notation

$$F(z_1, z_2) \quad \text{one degree of freedom}$$

$$\langle F \rangle = \frac{(F \circ m + F \circ m \circ m + ... + F \circ m^N)}{N} \quad \text{where} \quad N \to \infty$$

$$m \equiv \text{one turn map}$$

$$m = a \circ r \circ a^{-1}$$

$$r\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{bmatrix} \cos\mu(J) & \sin\mu(J) \\ \sin\mu(J) & \cos\mu(J) \end{bmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{where} \quad 2J = z_1^2 + z_2^2$$

$$\langle F \rangle = \frac{(F \circ a \circ r^0 \circ a^{-1} + F \circ a \circ r \circ a^{-1} + F \circ a \circ r^2 \circ a^{-1} + ... + F \circ \{a \circ r \circ a^{-1}\})^N}{N} \quad \text{where} \quad N \to \infty$$

$$\langle F \rangle = \frac{(F \circ a \circ r^0 + F \circ a \circ r + F \circ a \circ r^2 + ... + F \circ a \circ r^N)}{N} \circ a^{-1} \quad \text{where} \quad N \to \infty$$

$$\langle \bar{F} \rangle = \frac{(\bar{F} \circ a \circ r^0 + \bar{F} \circ a \circ r + \bar{F} \circ a \circ r^2 + ... + \bar{F} \circ a \circ r^N)}{N} \circ a^{-1} \quad \text{where} \quad N \to \infty$$

$$\bar{F} = F \circ a$$

$$\bar{F} \circ r^k \circ c = \{\bar{F} \circ c\} \circ \{c^{-1} \circ r^k \circ c\} = F^r \circ \Lambda$$

$$\Lambda \equiv \text{diagonalized rotation}$$

</div>

The resonance basis is

<div align="center">

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = c\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{-i}{2} & \dfrac{-i}{2} \end{vmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$c^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$F^r = \sum F_{mn}^r r_1^m r_2^n$$

$$F^r \circ \Lambda$$

</div>

Example:

<div align="center">

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{vmatrix} \sqrt{\beta} & 0 \\ \dfrac{-\alpha}{\sqrt{\beta}} & \dfrac{1}{\sqrt{\beta}} \end{vmatrix}\begin{pmatrix} x \\ p \end{pmatrix} = a\begin{pmatrix} x \\ p \end{pmatrix}$$

$$a^{-1}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{vmatrix} \dfrac{1}{\sqrt{\beta}} & 0 \\ \dfrac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{vmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$$

$$\bar{F} \circ r^k \circ c = \{\bar{F} \circ c\} \circ \{c^{-1} \circ r^k \circ c\} = F^r \circ \Lambda$$

$$F^r = \bar{F} \circ c\sqrt{\beta} \circ c = \beta\frac{(r_1^2 + r_2^2 + 2r_1 r_2)}{4}$$

</div>

For the average,

<div align="center">

$$\langle F^r \rangle = \beta\frac{(2r_1 r_2)}{4} = \frac{\beta}{2}(r_1 r_2)$$

$$\langle \bar{F} \rangle = \frac{\beta}{2}(r_1 r_2) \circ c^{-1} = \frac{\beta}{2}(z_1^2 + z_2^2) = \beta J$$

$$\text{where} \quad r_1 = z_1 + z_2 i \quad \text{and} \quad r_1 = z_1 + z_2 i$$

$$\langle F \rangle = \beta J \circ a^{-1} = \frac{\beta}{2}(z_1^2 + z_2^2) \circ a^{-1}$$

</div>

$$\langle F \rangle = \frac{\beta}{2}\left(\frac{1}{\beta} + \frac{\alpha^2}{\beta}x^2 + 2\alpha\, xp + \beta\, p^2\right)$$

$$\langle F \rangle = \frac{\beta}{2}\left(\gamma\, x^2 + 2\alpha\, xp + \beta\, p^2\right)$$

Let,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{where} \quad ad - bc = 1$$

$$M = SRS^{-1}$$

$$R = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \qquad S = \begin{pmatrix} S_1 & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{pmatrix}$$

$$\det S = S_{11}S_{22} - S_{12}S_{21} = 1$$

$$M = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{pmatrix}$$

$$M = \begin{pmatrix} S_{11}S_{22}\cos\mu - S_{11}S_{21}\sin\mu - S_{12}S_{22}\sin\mu - S_{12}S_{21}\cos\mu & -S_{11}S_{12}\cos\mu + S_{11}^2\sin\mu + S_{12}^2\sin\mu + S_{12}S_{11}\cos\mu \\ S_{21}S_{22}\cos\mu - S_{21}^2\sin\mu - S_{22}^2\sin\mu - S_{21}S_{22}\cos\mu & -S_{12}S_{21}\cos\mu + S_{21}S_{11}\sin\mu + S_{22}S_{12}\sin\mu + S_{11}S_{22}\cos\mu \end{pmatrix}$$

$$M = \left(S_{11}S_{22} - S_{12}S_{21}\right)\cos\mu\, \bar{I} + \sin\mu \begin{pmatrix} -S_{11}S_{21} - S_{12}S_{22} & S_{11}^2 + S_{12}^2 \\ -S_{21}^2 - S_{22}^2 & S_{21}S_{11} + S_{22}S_{12} \end{pmatrix}$$

$$M = \cos\mu\, \bar{I} + \sin\mu \begin{pmatrix} -S_{11}S_{21} - S_{12}S_{22} & S_{11}^2 + S_{12}^2 \\ -S_{21}^2 - S_{22}^2 & S_{21}S_{11} + S_{22}S_{12} \end{pmatrix}$$

$$M = \cos\mu\, \bar{I} + \sin\mu \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$
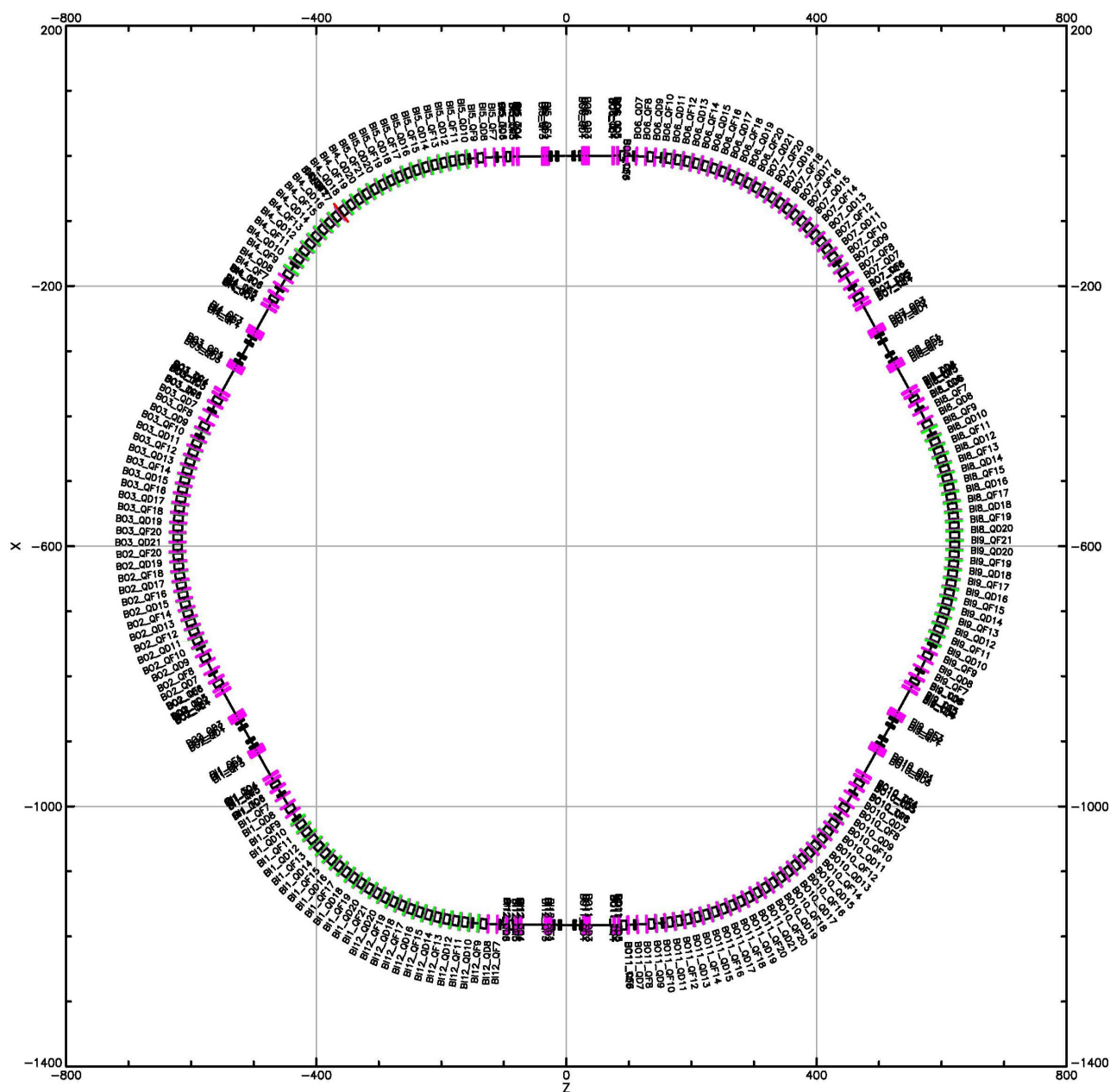
$$r^2 = z_1^2 + z_2^2$$

$$\epsilon(x,p) = r^2(z(x,p)) = (s_1^{-1}z(x,p))^2 + (s_2^{-1}z(x,p))^2$$

$$\epsilon = (S_{22}x - S_{12}p)^2 + (-S_{21}x + S_{11}p)^2$$

$$\epsilon = (S_{22}^2 + S_{21}^2)x^2 + (S_{12}^2 + S_{11}^2)p^2 + 2(-S_{22}^2 S_{12} - S_{21}S_{11})xp$$

RHIC LATTICE

Typical Print out of BMAD program

Beam Particles are: Au79
Particle mass (GeV):   183.43333704400001
Particle charge: 79
Particle momentum: 1921066659.9370658
Lattice length: 3833.8451693137899   Revolution Frequency: 77842.232479744911
Unstable factor: 0.0000000000000000
Lattice is stable: T
Energy loss per turn: 3.2683323063642604E-013 MeV
Tune A: 28.244158457777463
Tune B: 29.224921868926973
Chromaticity A and B: 2.0145921230787835   ,   2.0259229001796117
Emittance: 8.4999999999999994E-008 ,   8.3000000000000002E-008
SigmaE/E :   0.58222714288160404
Sigma_Z :   10.792642016351699
Momentum compaction factor :   1.9049909967975222E-003
Gamma factor: 10.520466997421192   Transition gamma factor: 22.911500695093000   Slip factor:
-7.1300465844504213E-003
Kinetic energy: 1746.3710315542403   MeV Beta(v/c): 0.99547223086269565   Brho: 81.113780030030810
dbeta: 4.5277691373043494E-003

!Note!
Emittances, SigmaE/E, and Sigma_Z are given as parameters to the program not calculated.