BNL-101522-2014-TECH

AD/AP 32;BNL-101522-2013-IR

# The Maximum and Maximum in the Presence of Linear Coupling

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November 1991

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**U.S. Department of Energy**

USDOE Office of Science (SC)

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AD/AP-32

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**Accelerator Physics Technical Note No. 32**

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# THE MAXIMUM $x$ and MAXIMUM $y$ IN THE PRESENCE OF LINEAR COUPLING

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## 1. Introduction

This note provides analytical results for the following 2 problems:

1. In the presence of linear coupling, given the initial coordinates $x_0$, $x'_0$, $y_0$, $y'_0$ at the longitudinal position $s_0$, what is the maximum $x$ and the maximum $y$ at the position $s$?
2. In the presence of linear coupling, given the initial total emittance $\epsilon_t$. What is the maximum $x$ and the maximum $y$ at the position $s$?

## 2. The Maximum $x$ and $y$ for a given $x_0$, $x'_0$, $y_0$, $y'_0$

I assume that the motion is linearly coupled by a skew quadrupole field. The normal modes and the parameters $\beta$, $\alpha$, $\gamma$ of each normal mode can be found using the results of Edwards and Teng.[1] The uncoupled normal mode coordinates $v$, $p_v$, $u$, $p_u$ are related to $x$, $x'$, $y$, $y'$ by a $4 \times 4$ matrix $R$

$$x = R\, v \tag{2.1}$$

The $R$ matrix can be computed from the one turn transfer matrix.[1] The $R$ matrix can be written as[1]

$$R = \begin{pmatrix} I\cos\varphi & \overline{D}\sin\varphi \\ -D\sin\varphi & I\cos\varphi \end{pmatrix}, \tag{2.2}$$

where $D$, $\overline{D}$ are $2 \times 2$ matrices, $\overline{D} = D^{-1}$, $|D| = 1$ and $I$ is the $2 \times 2$ identity matrix. $D$ and $\varphi$ can be found[1] from the one turn transfer matrix.

For the normal mode $v$, $p_v$ the parameters $\beta_1$, $\alpha_1$, $\gamma_1$, ca be found,[1] such that

$$\epsilon_1 = \gamma_1 v^2 + 2\alpha, v p_v + \beta_1 p_v^2, \tag{2.3a}$$

is an invariant. For the $u$, $p_u$ mode, one finds the parameters $\beta_2$, $\alpha_2$, $\gamma_2$ such that

$$\epsilon_2 = \gamma_2 \mu^2 + 2\alpha_2 \mu p_\mu + \beta^2 p_\mu^2 \tag{2.3b}$$

is an invariant.

For the given initial coordinates $x_0$, $x'_0$, $y_0$, $y'_0$ at $s_0$, one can find $\epsilon_1$ and $\epsilon_2$ by using Eq. (2.1) to find the corresponding normal mode coordinates, $v_0$, $p_{v0}$, $\mu_0$, $p_{\mu_0}$ and then using Eqs. (2.3) to find the invariants $\epsilon_1$, $\epsilon_2$.

To find $x$ as a function of $s$, one notes that the $v$ and $u$ motion is given by[1]

$$\begin{aligned}
v &= (\beta_1 \epsilon_1)^{\frac{1}{2}} \cos \psi_1 \\
p_v &= (\gamma_1 \epsilon_1)^{\frac{1}{2}} \cos (\psi_1 - \delta_1) \\
u &= (\beta_2 \epsilon_2)^{\frac{1}{2}} \cos \psi_2 \\
p_u &= (\gamma_2 \epsilon_2)^{\frac{1}{2}} \cos (\psi_2 - \delta_2) \\
\delta_1 &= \arctan (1/\alpha_1), \quad \delta_2 = \arctan (1/\alpha_2).
\end{aligned} \tag{2.4}$$

$\psi_1$, $\psi_2$ are the betatron phases of the normal modes.

The $x$ motion can then be found using Eqs. (2.1) and (2.2),

$$x = v \cos \varphi + \left(\overline{D}_{11} u + \overline{D}_{12} p_u\right) \sin \varphi \tag{2.5}$$

$$\begin{aligned}
x = &(\beta_1 \epsilon_1)^{\frac{1}{2}} \cos \varphi \quad \cos \psi_1 \\
&+ \overline{D}_{11} (\beta_2 \epsilon_2)^{\frac{1}{2}} \sin \varphi \quad \cos \psi_2 \\
&+ \overline{D}_{12} (\gamma_2 \epsilon_2)^{\frac{1}{2}} \sin \varphi \quad \cos (\psi_2 - \delta_2),
\end{aligned}$$

$$\begin{aligned}
x = &(\beta_1 \epsilon_1)^{\frac{1}{2}} \cos \varphi \quad \cos \psi_1 \\
&+ \left(\overline{D}_{11} (\beta_2 \epsilon_2)^{\frac{1}{2}} + \overline{D}_{12} (\gamma_2 \epsilon_2)^{\frac{1}{2}} \cos \delta_2\right) \sin \varphi \quad \cos \psi_2 \\
&+ \overline{D}_{12} (\gamma_2 \epsilon_2)^{\frac{1}{2}} \sin \delta_2 \sin \varphi \quad \sin \psi_2
\end{aligned}$$

The maximum $x$ from these three oscillating terms may be found by adding the coefficients of $\cos\psi_2$ and $\sin\psi_2$ quadratically, and then adding this result to the coefficient of $\cos\psi$. This gives

$$\begin{aligned} x_{\max} &= (\beta_1\epsilon_1)^{\frac{1}{2}}\cos\varphi + (\beta_{x,2}\epsilon_2)^{\frac{1}{2}}\sin\varphi \\ \beta_{x,2} &= \overline{D}_{11}^2\beta_2 + \overline{D}_{12}^2\gamma_2 + 2\overline{D}_{11}\overline{D}_{12}\alpha_2 \end{aligned} \tag{2.6a}$$

where $u$ was made of the relationship

$$\cos\delta_2 = \alpha_2/\,(\beta_2\gamma_2)^{\frac{1}{2}}\,.$$

Equation (2.6a) shows that $x_{\max}(s)$ may be increased by the presence of linear coupling for reasons that include the following

1. the coupling may increase the beta functions $\beta_1$, $\beta_2$;
2. the coupling may cause an emittance mismatch. A particle which has the emittances $\epsilon_x$, $\epsilon_y$ in the absence of coupling may have emittances $\epsilon_1$, $\epsilon_2$ in the presence of coupling which are larger than $\epsilon_x$, $\epsilon_y$.

In a similar way, one can find $y_{\max}$ using

$$y = u\cos\varphi - (D_{11}\;v + D_{12}p_v)\sin\varphi$$

and one finds

$$\begin{aligned} y_{\max} &= (\beta_2\epsilon_2)^{\frac{1}{2}}\cos\varphi + (\beta_{y,1},\epsilon_1)^{\frac{1}{2}}\sin\varphi \\ \beta_{y,1} &= D_{11}^2\beta_1 + D_{12}^2\gamma_1 + 2D_{11}D_{12}\alpha_1 \end{aligned} \tag{2.6b}$$

One can also find expressions for $x'_{\max}$ and $y'_{\max}$. Using

$$\begin{aligned} x' &= p_v\cos\varphi + \left(\overline{D}_{21}u + \overline{D}_{22}p_u\right)\sin\varphi \\ &= (\gamma_1\epsilon_1)^{\frac{1}{2}}\cos(\psi_1 - \delta_1) \\ &\quad + \overline{D}_{21}\sin\varphi\,(\beta_2\epsilon_2)^{\frac{1}{2}}\cos\psi_2 \\ &\quad + \overline{D}_{22}\sin\varphi\,(\gamma_2\epsilon_2)^{\frac{1}{2}}\cos(\psi_2 - \delta_2)\,. \end{aligned}$$

one finds

$$\begin{aligned} x'_{\max} &= (\gamma_1\epsilon_1)^{\frac{1}{2}}\cos\varphi + (\gamma_{x,2}\epsilon_2)^{\frac{1}{2}}\sin\varphi \\ \gamma_{x,2} &= \overline{D}_{21}^2\beta_2 + \overline{D}_{22}^2\gamma_2 + 2\overline{D}_{21}\overline{D}_{22}\alpha_2 \end{aligned} \tag{2.6c}$$

Using

$$y' = p_u\cos\varphi - (D_{21}v + D_{22}p_v)\sin\varphi,$$

one finds

$$y'_{\max} = (\gamma_2\epsilon_2)^{\frac{1}{2}}\cos\varphi + (\gamma_{y,1}\epsilon_1)^{\frac{1}{2}}\sin\varphi$$
$$\gamma_{y,1} = D_{21}^2\beta_1 + D_{22}^2\gamma_1 + 2D_{21}D_{22}\alpha_1 \tag{2.6d}$$

## 3. $x_{\max}$ and $y_{\max}$ for a given $\epsilon_t$

Consider all the particles that lie on the 4-dimensional surface in $x$, $x'$, $y$, $y'$ given by $\epsilon_t = \epsilon_1 + \epsilon_2 =$ constant. The question answered here is what is the largest $x$ and the largest $y$ reached by all particle lying on the surface $\epsilon_t =$ constant. Note that $\epsilon_t$ is the total emittance in the presence of linear coupling and $\epsilon_t$ is an invariant of the motion.

Each particle lying on the surface of constant $\epsilon_t$ will have emittances $\epsilon_1$, $\epsilon_2$ and its $x_{\max}$ and $y_{\max}$ may be computed from Eq. (2.6)

$$x_{\max} = \sqrt{\beta_1\epsilon_1}\cos\varphi + \sqrt{\beta_{x,2}\epsilon_2}\sin\varphi \tag{3.1a}$$

$$y_{\max} = \sqrt{\beta_2\epsilon_2}\cos\varphi + \sqrt{\beta_{y,1}\epsilon_1}\sin\varphi \tag{3.1b}$$

On the surface of constant $\epsilon_t$, $\epsilon_1+\epsilon_2 = \epsilon_t$ and there is a choice of $\epsilon_1$, $\epsilon_2$ which maximizes $x_{\max}$ and another choice that maximizes $y_{\max}$. For example, for $x_{\max}$ one can replace $\epsilon_2$ with $\epsilon_2 = \epsilon_t - \epsilon_1$ in Eq. (3.1) and find the maximum $x$, $\hat{x}_{\max}(s)$

$$\hat{x}_{\max} = \left(\hat{\beta}_x\epsilon_t\right)^{\frac{1}{2}}$$
$$\hat{\beta}_x = \left(\beta_1\cos^2\varphi + \beta_{x,2}\sin^2\varphi\right)^{\frac{1}{2}} \tag{3.2a}$$
$$\beta_{x,2} = \overline{D}_{11}^2\beta_2 + \overline{D}_{12}^2\gamma_2 + 2\overline{D}_{11}\overline{D}_{12}\alpha_2$$

For $\hat{y}_{\max}$, one finds

$$\hat{y}_{\max} = \left(\hat{\beta}_y\epsilon_t\right)^{\frac{1}{2}}$$
$$\hat{\beta}_y = \left(\beta_2\cos^2\varphi + \beta_{y,1}\sin^2\varphi\right)^{\frac{1}{2}} \tag{3.2b}$$
$$\beta_{y,1} = D_{11}^2\beta_1 + D_{12}^2\gamma_1 + 2D_{11}D_{12}\alpha_1$$

# 4. Examples of Using $x_{\max}$, $y_{\max}$ Results

## 4.1 Dynamic Aperture and Linear Coupling

Often the dynamic aperture is computed by finding the largest initial $x$, $A_{SL}$, which is stable for a specified number of turns and with the initial conditions $x_0$, $y_0$, $x'_o = y'_0 = 0$ and $\epsilon_{x0} = \epsilon_{y0}$. The tracking run is often done starting at a QF in a normal cell. It has been found[2] in RHIC that in the presence of linear coupling, $A_{SL}$ can depend strongly on which QF the tracking run is started at. This can be understood, and estimated, by computing $x_{\max}$, using Eq. (2.6) at the high-$\beta$ quadrupoles in the insertion region. This $x_{\max}$ depends on which QF the particle is started at, and thus explains the dependence of $A_{SL}$ on the choice of QF at which the particle is started.

## 4.2 Aperture Requirements at Injection

The presence of linear coupling may increase the aperture requirements at injection. Let us assume that the injected beam in the absence of linear coupling, has maximum emittances $\epsilon_x$ and $\epsilon_y$. The particles when injected in the presence of linear coupling may have a maximum total emittance $\epsilon_t$ such that $\epsilon_t$ is larger than $\epsilon_x + \epsilon_y$. This emittance mismatch is one source of the possible increase in aperture requirements. Another source is the increase in the beta functions due to linear coupling. The possible increase in aperture due to this source may be computed using Eq. (3.2a) $\hat{x}_{\max} = \left(\hat{\beta}_x \epsilon_t\right)^{1/2}$.

# 5. Effect of Solenoidal Fields

The previous results were obtained assuming no solenoidal fields were present. If solenoidal fields are present, the expressions for $x_{\max}$, $y_{\max}$ are unchanged. However $\varphi$ and $D$ which appear in the expressions for $x_{\max}$ and $y_{\max}$, will be changed by the presence of solenoids. The results for $x'_{\max}$ and $y'_{\max}$ are unchanged outside the solenoids.

If solenoids are present, then $p_x = x'$ and $p_y = y'$ are replaced by

$$\begin{aligned} p_x &= x' + Ly \\ p_y &= y' - Lx, \end{aligned} \tag{5.1}$$

where $L = B_s/2B\rho$, and $B_s$ is the longitudinal field in the solenoidal. The results for $x'_{\rm max}$ and $y'_{\rm max}$ are still valid if $x'_{\rm max}$ is replaced by $p_{x,{\rm max}}$, and $y'_{\rm max}$ by $p_{y,{\rm max}}$. This shows that outside the solenoids, when $L = 0$, the results for $x'_{\rm max}$ and $y'_{\rm max}$ are unchanged.

## References

1. D. Edwards and L. Teng, IEEE 1973 PAC, p. 885 (1973).
2. G. Parzen, BNL AD/RHIC-94 (1991); and Proc. IEEE 1991 PAC (to be published).