

The Tune Shift Due to Linear Coupling

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August 1991

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USDOE Office of Science (SC)

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AD/AP-25

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Accelerator Physics Technical Note No. 25

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1. Introduction

In RHIC the coupling between X and Y degrees of freedom is expected from various sources.^{1,2} We shall specifically examine the machine tune shift produced by skew-quadrupoles randomly distributed around the ring. In this case the X - Y coupling is linear, and may be calculated exactly within a model in which skew-quadrupole magnets are treated as point objects of strengths q_k and locations s_k , $k = 1, \dots, N^3$. This approximation is justified by comparing the length $\ell = 0.6$ m of a quadrupole magnet and the length $C = 3833.852$ m of RHIC's circumference, $\ell/C \sim 10^{-4}$.

A transfer matrix $\overset{\circ}{T}_{SQ}$ of a single skew-quadrupole magnet of length ℓ in the thin lense approximation (in the circular representation denoted with \circ above T_{SQ}) is given by

$$\overset{\circ}{T}_{SQ} = BT_{SQ}B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & 1 \end{bmatrix}, \quad (1.1)$$

where

$$q = (\beta_x \beta_y)^{1/2} \frac{\ell}{\rho} a_1, \quad (1.2)$$

represents the strength of a skew-quadrupole field, and B contains lattice functions of a perfect ring

$$B = \begin{bmatrix} B_x & 0 \\ 0 & B_y \end{bmatrix}, \quad (1.3)$$

$$B = \begin{bmatrix} \beta_x^{-1/2} & 0 \\ \alpha_x \beta_x^{-1/2} & \beta_x^{1/2} \end{bmatrix}, \quad \text{similar for } B_y. \quad (1.4)$$

The perfect lattice has the following transfer matrix:

$$\overset{\circ}{T}^{(0)} = \begin{bmatrix} R(\mu_x) & 0 \\ 0 & R(\mu_y) \end{bmatrix}, \quad (1.5)$$

where $R(\mu_x)$ and $R(\mu_y)$ are rotations, and μ_x, μ_y are tunes ($\mu = 2\pi\nu$),

$$R(\varphi) = \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix}. \quad (1.6)$$

For RHIC, we have $\nu_x = 28.826$ and $\nu_y = 28.821$.

2. Calculation of the Tune Shift

The full transfer matrix can be written as a polynomial in the q 's

$$\overset{\circ}{T} = \begin{bmatrix} \overset{\circ}{M} & \overset{\circ}{n} \\ \overset{\circ}{m} & \overset{\circ}{N} \end{bmatrix} = \sum_{k=0}^N \overset{\circ}{T}^{(k)}, \quad (2.1)$$

where submatrices $\overset{\circ}{M}^{(k)}, \overset{\circ}{n}^{(k)}$ etc. are given by k -th order in the q 's driving terms. For the purpose of this note it will be sufficient to display $\overset{\circ}{M}, \overset{\circ}{N}$ submatrices up to the second order in the q 's only.

$$\overset{\circ}{M} = R(\mu_x) + \overset{\circ}{M}^{(2)} + \text{higher terms of even order}, \quad (2.2)$$

where

$$\begin{aligned} \overset{\circ}{M}^{(2)} = \frac{1}{4} \sum_{r < s} q_r q_s \{ & R(\mu_x + \mu_x^r + \mu_y^r - \mu_x^s - \mu_y^s) - R(\mu_x + \mu_x^r - \mu_y^r - \mu_x^s + \mu_y^s) + \\ & + [R(\mu_x - \mu_x^r + \mu_y^r - \mu_x^s - \mu_y^s) - R(\mu_x - \mu_x^r - \mu_y^r - \mu_x^s + \mu_y^s)] J \}, \end{aligned} \quad (2.3)$$

and J is one of the fundamental Pauli matrices

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (2.4)$$

and μ_x^r, μ_y^r denote phase advances between the point of observations $s = 0$ and the location of s_r of the r -th skew-quadrupole

$$\mu_x^r = \int_0^{s_r} \frac{ds}{\beta_x}, \text{ and similar for } \mu_y^r. \quad (2.5)$$

Using symmetry arguments a corresponding expansion for the $\overset{\circ}{N}$ submatrix is obtained as

$$\overset{\circ}{N} = \overset{\circ}{M} \big|_{x \leftrightarrow y}. \quad (2.6)$$

The presence of skew-quadrupole fields produces the differences

$$\begin{aligned}\frac{1}{2}Tr \overset{\circ}{M} - \cos \mu_x &\neq 0, \\ \frac{1}{2}Tr \overset{\circ}{N} - \cos \mu_y &\neq 0,\end{aligned}\tag{2.7}$$

where this means that actual machine tunes are shifted relative to the tunes of a perfect machine. Substituting relevant traces of $\overset{\circ}{M}$ and $\overset{\circ}{N}$ one finds the final expression for the tune shifts in terms of the second order driving terms,

$$\begin{aligned}\Delta\mu_x &= -\frac{1}{2} \sum_{r < s} q_r q_s \cos(\mu_x^s - \mu_x^r) \sin(\mu_y^s - \mu_y^r) + \\ &= \frac{1}{2} \cot \mu_x \sum_{r < s} q_r q_s \sin(\mu_x^s - \mu_x^r) \sin(\mu_y^s - \mu_y^r) + 0(q^4),\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}\Delta\mu_y &= -\frac{1}{2} \sum_{r < s} q_r q_s \sin(\mu_x^s - \mu_x^r) \cos(\mu_y^s - \mu_y^r) + \\ &+ \frac{1}{2} \cot \mu_y \sum_{r < s} q_r q_s \sin(\mu_x^s - \mu_x^r) \sin(\mu_y^s - \mu_y^r) + 0(q^4) .\end{aligned}\tag{2.9}$$

The second order driving terms are defined as follows

$$\begin{bmatrix} d_{ss}^{(2)} \\ d_{sc}^{(2)} \\ d_{cs}^{(2)} \\ d_{cc}^{(2)} \end{bmatrix} = \sum_{1 \leq r < s \leq N} q_r q_s \sin(\mu_y^s - \mu_y^r) \begin{bmatrix} \sin \mu_x^s & \sin \mu_x^r \\ \sin \mu_x^s & \cos \mu_x^r \\ \cos \mu_x^s & \sin \mu_x^r \\ \cos \mu_x^s & \cos \mu_x^r \end{bmatrix} .\tag{2.10}$$

Additional sets of the second order driving terms denoted $\check{d}_{ss}^{(2)}$, $\check{d}_{sc}^{(2)}$ etc. are obtained from the above definitions by simply exchanging x and y .

Let us notice that the tune shift vanishes when the tune splitting is corrected. This is most easily seen by first writing $\Delta\mu_x$ and $\Delta\mu_y$ as

$$\Delta\mu_x = -\frac{1}{2} (d_{cc}^{(2)} + d_{ss}^{(2)}) - \frac{1}{2} (d_{cs}^{(2)} - d_{sc}^{(2)}) \cot \mu_x + 0(q^4),\tag{2.11}$$

$$\Delta\mu_y = -\frac{1}{2} (\check{d}_{cc}^{(2)} + \check{d}_{ss}^{(2)}) - \frac{1}{2} (\check{d}_{cs}^{(2)} - \check{d}_{sc}^{(2)}) \cot \mu_y + 0(q^4).\tag{2.12}$$

On correction of the tune splitting one requires that the following conditions hold

$$\begin{aligned}d_{sc}^{(2)} - d_{cs}^{(2)} &= 0, \\ \check{d}_{sc}^{(2)} - \check{d}_{cs}^{(2)} &= 0, \\ d_{cc}^{(2)} + d_{ss}^{(2)} &= 0, \\ \check{d}_{cc}^{(2)} + \check{d}_{ss}^{(2)} &= 0.\end{aligned}\tag{2.13}$$

Clearly, the tune shifts $\Delta\mu_x$ and $\Delta\mu_y$ given by Eqs. (2.11) and (2.12) vanish under these conditions. This conclusion was recently obtained, using different methods by G. Parzen.⁴

3. References

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