BNL-101511-2014-TECH

AD/AP 21;BNL-101511-2013-IR

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July 1990

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U.S. Department of Energy

USDOE Office of Science (SC)

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AD/AP–21

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Accelerator Physics Technical Note No. 21

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July 18, 1990

A parameterization of 4x4 matrices describing linear beam transport systems has been obtained by Edwards and Teng[1]. Here we extend their formalism to include dispersive effects, and give prescriptions for incorporating it in the program SYNCH[2].

A period of a beam transport system, or an element or segment of such a system (periodic or not) is characterized by a 6x6 transfer matrix, which we write in the form

$$\mathbb{T} = \begin{pmatrix} \mathbf{M} & \mathbf{n} & \mathbf{d1} \\ \mathbf{m} & \mathbf{N} & \mathbf{d2} \\ \mathbf{e1} & \mathbf{e2} & \mathbf{F} \end{pmatrix} \qquad (1)$$

Here we have written the 6x6 matrix $\mathbb{T}$ in terms of 2x2 submatrices **M**, **m**, etc. The dynamic variables are taken to be x, $x' \equiv dx/ds$, y, y', $-\Delta s$, $\Delta p/p$ in that order. We consider only transport systems without acceleration or damping; then the elements in the fifth column and sixth row of $\mathbb{T}$ vanish except for $T_{55} = T_{66} = 1$, and the matrix $\mathbb{T}$ is symplectic, which means that the inverse of $\mathbb{T}$ is given by

$$\mathbb{T}^{-1} = \overline{\mathbb{T}} \equiv \begin{pmatrix} \overline{\mathbf{M}} & \overline{\mathbf{m}} & \overline{\mathbf{e1}} \\ \overline{\mathbf{n}} & \overline{\mathbf{N}} & \overline{\mathbf{e2}} \\ \overline{\mathbf{d1}} & \overline{\mathbf{d2}} & \overline{\mathbf{F}} \end{pmatrix} \qquad (2)$$

where the "symplectic conjugate" $\overline{\mathbf{a}}$ of any 2x2 matrix **a** is defined as

$$\overline{\mathbf{a}} \equiv \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \qquad (3)$$

and the symplectic conjugate of a 4x4 or 6x6 matrix is given by (2).

1. Parameterization of 4x4 matrices for a complete period.

When $\mathbb{T}$ is a matrix describing a complete period (a circular accelerator or storage ring, or a cell of a periodic system), Edwards and Teng find a similarity transformation that transform the x-y 4x4 submatrix of $\mathbb{T}$ (which we also designate by $\mathbb{T}$) into uncoupled form:

$$\mathbb{T} = \mathbb{R}\,\mathbb{U}\,\overline{\mathbb{R}} \qquad \text{with } \mathbb{U} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \tag{4}$$

where $\mathbb{R}$ has the form

$$\mathbb{R} = \begin{pmatrix} \mathbf{I}\cos\varphi & \overline{\mathbf{D}}\sin\varphi \\ -\mathbf{D}\sin\varphi & \mathbf{I}\cos\varphi \end{pmatrix} \tag{5}$$

where **A**, **B** and **D** are 2x2 unimodular (symplectic) matrices, **I** and **0** are the 2x2 unit and null matrices, and φ is an equivalent rotation angle. The eigenvalues of $\mathbb{T}$ and the matrix **D** and the angle φ are determined as follows[3]:

Eigenvalues are $\exp(\pm i\mu_1)$, $\exp(\pm i\mu_2)$, (μ_1 and μ_2 are the *phase advances* of the normal modes).

$$\cos\mu_1 + \cos\mu_2 = \tfrac{1}{2}\,\mathrm{Tr}\,\mathbb{T} = \tfrac{1}{2}\,\mathrm{Tr}\,(\mathbf{M} + \mathbf{N}) \tag{6}$$

Define

$$t = \tfrac{1}{2}\,\mathrm{Tr}\,(\mathbf{M} - \mathbf{N});\ \Delta = \det\,(\mathbf{m} + \overline{\mathbf{n}}) \tag{7}$$

then

$$\cos\mu_1 - \cos\mu_2 = \delta \equiv t\left\{ 1 + \frac{\Delta}{t^2} \right\} = \sqrt{t^2 + \Delta}\ \mathrm{sign}(t) \tag{8}$$

and

$$\mathbf{D} = \frac{-(\mathbf{m}+\overline{\mathbf{n}})}{\sqrt{\Delta}}\ ;\ \mathbf{D}\sin\varphi = \frac{-(\mathbf{m}+\overline{\mathbf{n}})}{\sqrt{2\delta(\delta+t)}}\ ;$$
$$\cos\varphi = \sqrt{\frac{\delta+t}{2\delta}}\ ;\ \sin\varphi = \sqrt{\frac{\delta-t}{2\delta}} \tag{9}$$

We confine ourselves to the case where the phase advances are real, i.e. the motion is stable; then $t^2+\Delta$ must be positive so that δ is real. Note that we resolve the ambiguity of the sign of square roots by requiring δ to have the same sign as t. (If Δ is negative then $\sin\varphi$ and **D** are imaginary, but $\mathbf{D}\sin\varphi$ is still real, which is all that really matters).

In Reference 1 it is shown that the transfer matrices A and B for the uncoupled normal modes are given by

$$\mathbf{A} = \mathbf{M} + \frac{\mathbf{n}(\mathbf{m}+\overline{\mathbf{n}})}{\delta+t};\quad \mathbf{B} = \mathbf{N} - \frac{\mathbf{m}(\mathbf{n}+\overline{\mathbf{m}})}{\delta+t}\ . \tag{10}$$

These are 2x2 unimodular matrices, and can be parameterized in terms of phase advances and Twiss (Courant-Snyder) parameters in the usual way:

$$A = \begin{pmatrix} \cos\mu_x + \alpha_x \sin\mu_x & \beta_x \sin\mu_x \\ -\gamma_x \sin\mu_x & \cos\mu_x + \alpha_x \sin\mu_x \end{pmatrix} \quad (11)$$

$$B = \begin{pmatrix} \cos\mu_y + \alpha_y \sin\mu_y & \beta_y \sin\mu_y \\ -\gamma_y \sin\mu_y & \cos\mu_y + \alpha_y \sin\mu_y \end{pmatrix} \quad (12)$$

The parameters α, β, γ, μ in (11) and (12) may be taken as the definition of the generalized Twiss parameters of the matrix $\mathbb{T}$.

2. Parameterization of Elements of a Periodic System.

The parameterization just found applies to the matrix for a complete period. It does not apply to the individual elements or components of the period, since when the beam traverses an element the α and β functions are generally different at the beginning and the end.

Consider a periodic system $\mathbb{G}$. The matrix elements of $\mathbb{G}$ are periodic in s, as are the parameters α, β, γ. At each azimuth s the parameters can be determined as detailed above. Now suppose the matrix for going from azimuth s_1 to s_2 is $\mathbb{T}$, so that

$$\mathbb{G}_2 = \mathbb{T}\, \mathbb{G}_1\, \mathbb{T}^{-1}. \quad (13)$$

We reduce $\mathbb{G}_1$ and $\mathbb{G}_2$ to semi-diagonal form by the methods of the previous section:

$$\mathbb{G}_1 = \mathbb{R}_1 \mathbb{U}_1 \mathbb{R}_1^{-1} \;; \quad \mathbb{G}_2 = \mathbb{R}_2 \mathbb{U}_2 \mathbb{R}_2^{-1} \quad (14)$$

Then $\mathbb{T}$ may be written as

$$\mathbb{T} = \mathbb{R}_2 \mathbb{V}\, \mathbb{R}_1^{-1} \quad (15)$$

so that

$$\mathbb{V} = \mathbb{R}_2^{-1}\, \mathbb{T}\, \mathbb{R}_1 \quad (16)$$

which, with (13) and (14), gives

$$\mathbb{U}_2 = \mathbb{V}\mathbb{U}_1 \mathbb{V}^{-1} \quad \text{or} \quad \mathbb{U}_2 \mathbb{V} = \mathbb{V}\mathbb{U}_1 \quad (17)$$

Since $\mathbb{U}_1$ and $\mathbb{U}_2$ are semi-diagonal, so is $\mathbb{V}$, i.e $\mathbb{V}$ may be regarded as the semi-diagonalization of the component matrix $\mathbb{T}$ *in the context of* $\mathbb{T}$ *as an element of* $\mathbb{G}$.

To find $\mathbb{V}$ explicitly we write $\mathbb{R}_1$ and $\mathbb{R}_2$ in the form (5), and use (15) in the form $\mathbb{R}_2\mathbb{V} = \mathbb{T}\mathbb{R}_1$ with $\mathbb{T}$ in the form (1):

$$\begin{aligned} \mathbf{A}\cos\varphi_2 &= \mathbf{M}\cos\varphi_1 - \mathbf{n}\mathbf{D}_1\sin\varphi_1 \\ \mathbf{B}\cos\varphi_2 &= \mathbf{N}\cos\varphi_1 + \mathbf{m}\bar{\mathbf{D}}_1\sin\varphi_1 \end{aligned} \tag{18}$$

Thus the element matrix $\mathbb{T}$ is semi-diagonalized, with the help of the semi-diagonalization parameters of $\mathbb{G}_1$ and $\mathbb{G}_2$.

For computational purposes it would be preferable if one did not first have to carry out the procedure for both the matrices $\mathbb{G}_1$ and $\mathbb{G}_2$. In fact the explicit computation of $\cos\varphi_2$ can be avoided: We note that the uncoupled matrices **A** and **B** must be unimodular. Therefore we may simply compute the right-hand sides of (18), and then normalize by dividing by the square root of the determinant of the resulting matrices. Using (9) we have

$$\mathbf{M}\cos\varphi_1 - \mathbf{n}\mathbf{D}_1\sin\varphi_1 = \left[\mathbf{M} + \frac{\mathbf{n}(\mathbf{m}_1+\bar{\mathbf{n}}_1)}{\delta+t_1}\right]\sqrt{\frac{\delta+t_1}{2\delta}} \tag{19}$$

and similarly for the second line of (18). Here the subscript 1 refers to the global matrix $\mathbb{G}_1$ and its components, while **M**, **N**, **m**, **n** without subscripts are the 2x2 submatrices of the matrix $\mathbb{T}$.Thus the uncoupled transfer matrices **A** and **B** for the two normal modes for the matrix $\mathbb{T}$ are found as follows:

Find t_1 and δ for the global matrix $\mathbb{G}_1$. From the 2x2 submatrices of $\mathbb{G}_1$ and $\mathbb{T}$ form the matrices

$$\mathbf{A}' = \mathbf{M} + \mathbf{n}(\mathbf{m}_1+\bar{\mathbf{n}}_1)/(\delta+t_1)$$

$$\mathbf{B}' = \mathbf{N} - \mathbf{m}(\mathbf{n}_1+\bar{\mathbf{m}}_1)/(\delta+t_1)$$

Find the determinants of these matrices (they should be equal). The uncoupled matrices **A** and **B** are the unimodular 2x2 matrices

$$\mathbf{A} = \mathbf{A}'/\sqrt{\det(\mathbf{A}')} \qquad \mathbf{B} = \mathbf{B}'/\sqrt{\det(\mathbf{B}')} \tag{20}$$

The phase advances for going through $\mathbb{T}$ can be found using the parameterization

$$A = \begin{pmatrix} (\beta_{x2}/\beta_{x1})^{\frac{1}{2}} (\cos\psi_x + \alpha_{x1}\sin\psi_x) & (\beta_{x1}\beta_{x2})^{\frac{1}{2}}\sin\psi_x \\ \ldots\;\ldots & (\beta_{x1}/\beta_{x2})^{\frac{1}{2}} (\cos\psi_x - \alpha_{x2}\sin\psi_x) \end{pmatrix} \qquad (21)$$

where the 21 element is obtained by requiring **A** to be unimodular; the B matrix has the same form with the *y* parameters. The phase advances are therefore

$$\psi_x = \tan^{-1}[A_{12}/(\beta_{x1}A_{11} - \alpha_{x1}A_{12})]$$

$$\psi_y = \tan^{-1}[B_{12}/(\beta_{y1}B_{11} - \alpha_{y1}B_{12})] \qquad (21)$$

which are again expressed in terms of the parameters of the previous global matrix $\mathbb{G}_1$ and of $\mathbb{T}$.

3. Dispersion.

The fifth and sixth rows and columns of the full matrices refer to the change in path length $-\Delta s$ and to relative momentum deviation $\Delta p/p$. In the uncoupled case, the x-variables still depend on momentum; this is customarily described (as in the SYNCH program), by augmenting the 2x2 matrix with a third column, where the elements A_{13} and A_{23} describe the dependence of excursion and slope on momentum. The corresponding elements in the decoupled matrices developed here can be obtained by augmenting the matrices $\mathbb{R}$ effecting the similarity transformations with fifth and sixth rows and columns, all zero except for $R_{55}=R_{66}=1$.

4. Modifications of the SYNCH Program.

In the SYNCH program matrices with *x*-*y* coupling are generally formulated in a 7x7 format (the seventh column describes perturbations, and need not concern us here). However, in the CYC and FXPT operations, which calculate the Twiss parameters at the end of each element of a lattice or transport line, all 7x7 matrices are truncated to two 2x3 matrices each, with coupling information lost. We have attempted to remedy this truncation algorithm in a new version of SYNCH. In the FXPT operation in the new version, the results of the previous sections are used to generate α and β functions, dispersion functions, and phase advances pertaining to the normal oscillation modes of the coupled system (called *x* and *y* but not necessarily horizontal and vertical in space), while the closed orbit FXPT produces should still be in space coordinates. The transformations between normal modes and space coordinates (e.g. the matrix D) are not exhibited in the SYNCH output, but the coupling angle φ is printed out at each point of the lattice, together with the other orbit functions.

[1]D. A Edwards and L. C. Teng, IEE Transactions in Nuclear Science, v. NS-20, 885-888 (1973)

[2] A. A. Garren, A. S Kenney, E. D. Courant and M. J. Syphers, Report FN-420, FNAL, 1985 (to be revised 1990)

[3] E. D. Courant and H. S. Snyder, *Ann. Phys.* 3, 1 (1958)