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Abstract

Perturbation expansion of standard FEL equations is performed up to second nontrivial order in the Vlasov's equation. We found that the perturbation expansion can be characterized by a single parameter, $\Omega\tau$, the number of synchrotron oscillations in the wiggler. The validity of perturbation theory is discussed in this paper.

1. INTRODUCTION

In recent years there have been many experimental and theoretical works done on the free electron lasers. The physics of the FEL has been discussed by many authors[1-3]. These theoretical works have given a clear understanding of the basic properties of the FEL's.

In this paper we follow the classical theory discussed by C. Pellegrini[2], limiting ourself in the Compton regime where a single electron interacts with the existing electromagnetic radiation field and the wiggler field. We investigate the Maxwell-Vlasov pendulum equation and calculate the energy exchange between the electrons and the radiation field up to the second non-zero perturbation order. We study the criterion on using the perturbative method in solving the Vlasov's equation. The validity of the perturbation shall be discussed.

In what follows, section 2 briefly reviews the FEL model we are interested and the equations of motion; section 3 gives the first and the second order gain function and discusses the validity condition of the perturbation expansion. An example is used for demonstration. Some detailed calculation is left in appendix I and II.

2. EQUATIONS OF MOTION

In the simple FEL model, relativistic electrons move through a static magnetic wiggler, at the same time an existing plane electro-magnetic wave propagates parallel to the electron beam. The wiggler magnet gives the electrons a transverse velocity in the direction parallel to the electric vector of the wave, so that energy can be transfered between the electrons and the wave. The wiggler has a helical megnetic field as

$$B_w=-(B_w\cos k_0z,\ B_w\sin k_0z,\ B_{wz}),\qquad 0<z<L. \qquad (2.1)$$

The electromagnetic wave is assumed to be

$$E_L=(E_0\sin(kz-\omega t+\Phi_0),\ E_0\cos(kz-\omega t+\Phi_0),\Phi 0)$$

$$B_L=z\times E_L \qquad 0<z<L. \qquad (2.2)$$

To simplify the problem, we make the following assumptions:

(1) The radiation force is neglected. This means that the electron energy, γ, satisfies the condition,

$$\gamma < \left(\frac{m_0c}{eB_wr_0}\right), \qquad (2.3)$$

where $r_0=e^2/4\pi\varepsilon_0m_0c^2$ is the classical radius of the electron. Assuming the wiggler field of 1T, the right hand side of eq. (2.3) is of the order 8×10^5.

(2) The space charge force is small, i.e., the electron density satisfy the following condition,

$$n_e < \frac{eE_0}{m_0 c^2 \gamma \lambda r_0} \quad , \qquad (2.4)$$

where $\lambda=2\pi/k$ is the wavelength of the electromagnetic field;

(3) The electrons are highly relativistic, i.e.,

$\gamma \gg 1, \quad \beta_\perp/\beta_{//} \ll 1;$

(4) The wiggler strength parameter K is much larger than the E.M. wave parameter,

$$K \gg K_L, \qquad (2.5)$$

where $K=\frac{eB_w}{m_0 c k_0}$, $K_L=\frac{eE_0}{m_0 c^2 k}$;

Besides these conditions, our analysis is one dimensional, collision between electrons and quantum effects are disregarded. The single particle motion follows the pendulum equation[2],

$$\dot{\gamma}=-\frac{eE_0 K}{m_0 c \gamma} \sin\Phi \qquad (2.6.1)$$

$$\dot{\Phi}=ck_0\left(1-\frac{\gamma_r^2}{\gamma^2}\right)+\dot{\Phi}_0 \qquad (2.6.2)$$

where γ_r is the equilibrium value of γ and $\Phi=(k_0+k)z-\omega t+\Phi_0$.

Starting from the Maxwell's equation,

$$\Box A = -\mu_0 J \qquad (2.7)$$

with $J(x)=ec\sum_i \beta_i \delta(x-x_i)$, we find

$$\frac{\partial E_0}{\partial z}+\frac{\partial E_0}{c\partial t}-\frac{B_w c k_0 \sin\Phi}{2}=-\frac{c\mu_0 J_1}{2} \qquad (2.8.1)$$

$$E_0\left(\frac{\partial \Phi_0}{\partial z}+\frac{\partial \Phi_0}{c\partial t}\right)-\frac{B_w c k_0 \cos\Phi}{2}=\frac{c\mu_0 J_2}{2} \quad , \qquad (2.8.2)$$

where

$J_1=J_x\sin\alpha+J_y\cos\alpha, \quad J_2=-J_x\cos\alpha+J_y\sin\alpha, \quad \alpha=kz-\omega t+\Phi_0.$

The electron distribution function is described by the Vlasov's equation

$$\frac{\partial f}{\partial t}+\dot{\Phi}\frac{\partial f}{\partial \Phi}+\dot{\eta}\frac{\partial f}{\partial \eta}=0 \quad . \qquad (2.9)$$

Self-consistent solution of equations (2.6.1), (2.6.2), (2.8.1), (2.8.2) and (2.9) describes the evolusion of the particle-field system.

3. PERTURBATION CALCULATION

To find a simple analytical solution, we further assume that the system is within the small signal regime, i.e.,

$$\eta=(\gamma-\gamma_r)/\gamma<<1. \tag{3.1}$$

These self-consistent equations thus become

$$\dot{\eta}=-\frac{\omega_0}{2}\left(\frac{\Omega}{\omega_0}\right)^2\sin\phi \tag{3.2.1}$$

$$\dot{\phi}=2\omega_0\eta+\dot{\phi}_0 \tag{3.2.2}$$

$$\frac{\partial E_0}{\partial z}+\frac{\partial E_0}{c\partial t}-\frac{B_w ck_0\sin\phi}{2}=-\frac{c\mu_0 J_1}{2} \tag{3.2.3}$$

$$E_0\left(\frac{\partial\Phi_0}{\partial z}+\frac{\partial\Phi_0}{c\partial t}\right)-\frac{B_w ck_0\cos\phi}{2}=\frac{c\mu_0 J_2}{2} \tag{3.2.4}$$

$$\frac{\partial f}{\partial t}+\dot{\phi}\frac{\partial f}{\partial\Phi}+\dot{\eta}\frac{\partial f}{\partial\eta}=0 \tag{3.2.5}$$

where $\Omega=\left(\frac{4\pi eE_0K}{m_0\gamma_r^2\lambda_w}\right)^{1/2}$ is equivalent to the synchrotron oscillition frequency in a r.f. system. In the case when the net gain is not large, we can break up the self-consistent chain by assuming constant Ω, $\dot{\Phi}_0$ and solve eq.(3.2.1), (3.2.2) and (3.2.5). In general, we have $\Omega/\omega_0<<1$,

we can expand $f(\Phi, \eta, t)$ in the order of $(\Omega/\omega_0)^2$, i.e.,

$$f=\Sigma f_n(\Phi,\eta,t)(\Omega/\omega_0)^{2n}\ . \tag{3.3}$$

Following Pellegrini[2], we assume an initial electron distribution to be

$$f_0(\Phi,\eta)=\rho_e\lambda g(\eta)/2\pi, \tag{3.4}$$

i.e., the electron density is uniform in the length scale of λ, where ρ_e is the longitudinal electron density and $g(\eta)$ stands the initial energy distribution with

$$\int_{-\infty}^{\infty} d\eta\, g(\eta) = 1. \tag{3.5}$$

Substitute (3.2.1), (3.2.2) and (3.4) into (3.2.5), we obtain

$$\frac{\partial f_n}{\partial t} + 2\omega\eta \frac{\partial f_n}{\partial \phi} = \frac{\omega_0 \sin\phi}{2} \frac{\partial f_{n-1}}{\partial \eta}\ . \tag{3.6}$$

The averaged energy transfer is then given by

$$\langle \eta \rangle = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d\eta\, f(\phi,\eta,t)\ \eta$$
$$= \sum_0^{\infty} \langle \eta_n \rangle (\Omega/\omega_0)^{2n} \tag{3.7}$$

From the recurrence formula (3.6), (3.4) and (3.7) we observe that the average energy loss

$$\langle \eta_n \rangle = 0, \qquad \text{for all } n = \text{odd}.$$

3.1 FIRST ORDER PERTURBATION

Using the initial distribution in eq.(3.4), we obtain the first order distribution function as,

$$f_1 = \frac{\rho_e \lambda}{8\pi} \frac{1}{\eta} \frac{\partial g}{\partial \eta} \left(-\cos\phi + \cos(\phi - 2\omega_0 \eta t)\right). \tag{3.8}$$

the second order f_2 can be expanded into azimuth components as

$$f_2 = \sum_0^{\pm 2} f_{2m}(\eta, t)\ e^{im\phi} \tag{3.9}$$

one can easily obtain f_{20}, which gives the non-zero contribution to the average energy transfer,

$$f_{20} = \frac{\rho_e \lambda}{64\pi} \frac{\partial}{\partial \eta} \left[\frac{1}{\eta^2} \frac{\partial g}{\partial \eta} (1 - \cos 2\omega_0 \eta t) \right] \ . \tag{3.10}$$

$$\langle \eta \rangle = \langle \eta_0 \rangle + \langle \eta_2 \rangle .$$

$$\langle \eta_2 \rangle = (\Omega/\omega_0)^4 \int d\phi \int d\eta \, f_2(\phi,\eta,t)\, \eta$$

$$= \frac{\rho_e \lambda}{16} \left(\Omega/\omega_0 \right)^4 (2\omega_0 t_0)^3 \int d\eta \, g(\eta)\, F(2\omega_0 \eta t_0) \, , \qquad (3.11)$$

where $t_0 = N_w \lambda_w / (\beta_{//} c)$ and,

$$F(x) = (\cos x - 1 + x \sin x/2)/2. \qquad (3.12)$$

The energy loss by the electrons is equal to the energy gained by the electromagnetic field. By evaluating J_1, J_2 to the first order f_1, substituting it into eq.(3.2.3), (3.2.4) we find that the energy is exactly conserved to the first non-zero perturbation order.

3.2 SECOND ORDER PERTURBATION

To decide the appliable condition of the perturbation theory, we calculate the second non-zero order perturbation contribution. From eqs.(3.9),(3.10) and (3.7), we obtain

$$f_{21} = f_{2-1} = 0. \qquad (3.13)$$

$$\frac{\partial f_{22}}{\partial t} + 4i\omega_0 \eta f_{22} = \frac{\rho_e \lambda \omega_0}{64\pi i} \frac{\partial}{\partial \eta} \left(\frac{1}{\eta} \frac{\partial g}{\partial \eta} \left(e^{-2\omega_0 \eta t i} - 1 \right) \right) \qquad (3.14.1)$$

$$\frac{\partial f_{2-2}}{\partial t} - 4i\omega_0 \eta f_{2-2} = - \frac{\rho_e \lambda \omega_0}{64\pi i} \frac{\partial}{\partial \eta} \left(\frac{1}{\eta} \frac{\partial g}{\partial \eta} \left(e^{2\omega_0 \eta t i} - 1 \right) \right). \qquad (3.14.2)$$

f_{22} can be found by assuming $f_{22} = \frac{\rho_e \lambda}{64\pi} \left(A_2 e^{-4\omega_0 \eta t i} + \tilde{f}_{22} \right)$, where $\tilde{f}_{22}$ is a special solution of (3.14.1),

$$\tilde{f}_{22} = B_2 i \omega_0 t e^{-2\omega_0 \eta t i} + C_2 e^{-2\omega_0 \eta t i} + D_2 \qquad (3.15)$$

where

$$B_2 = \frac{1}{\eta^2} \frac{\partial g}{\partial \eta}$$

$$C_2 = -\frac{1}{2\eta^2}\frac{\partial^2 g}{\partial \eta^2}$$

$$D_2 = \frac{1}{4\eta}\frac{\partial}{\partial \eta}\left(\frac{1}{\eta}\frac{\partial g}{\partial \eta}\right) \quad . \qquad (3.16)$$

Similarly, f_{2-2} can be solved from eq.(3.14.2). The second order distribution function becomes,

$$f_2 = f_{20} + f_{22}e^{2i\Phi} + f_{2-2}e^{-2i\Phi}$$

$$= f_{20} + \frac{\rho_e \lambda}{64\pi}\left\{ \frac{1}{2\eta^3}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial g}{\partial \eta}\right)\cos(4\omega_0 \eta t - 2\Phi) \right.$$

$$+ 2\omega_0 t \frac{1}{\eta^2}\frac{\partial g}{\partial \eta}\sin(2\omega_0 \eta t - 2\Phi)$$

$$\left. - \frac{1}{\eta^2}\frac{\partial^2 g}{\partial \eta^2}\cos(2\omega_0 \eta t - 2\Phi) + \frac{1}{2\eta}\frac{\partial}{\partial \eta}\left(\frac{1}{\eta}\frac{\partial g}{\partial \eta}\right)\cos 2\Phi \right\} \quad . \qquad (3.17)$$

Using f_2, we can solve f_3 to be $f_{31} = \frac{\rho_e \lambda}{256\pi}(A_3 e^{-2\omega_0 \eta t i} + \tilde{f}_{31})$,

$$\tilde{f}_{31} = \frac{\rho_e \lambda}{256\pi}(B_3 \omega_0^2 t^2 e^{-\omega_0 \eta t i} + C_3 i\omega_0 t e^{-2\omega_0 \eta t i} + D_3 \omega_0^2 t^2 e^{2\omega_0 \eta t i}$$

$$+ E_3 i\omega_0 t e^{2\omega_0 \eta t i} + F_3 e^{2\omega_0 \eta t i} + G_3 i\omega_0 t e^{-4\omega_0 \eta t i} + H_3 e^{-4\omega_0 \eta t i} + I_3) \ ,$$

where

$$B_3 = -\frac{1}{\eta^3}\frac{\partial g}{\partial \eta}$$

$$C_3 = -\frac{\partial}{\partial \eta}\left(\frac{1}{\eta^3}\frac{\partial g}{\partial \eta}\right)$$

$$D_3 = -\frac{1}{2\eta^3}\frac{\partial g}{\partial \eta}$$

$$E_3 = -\frac{1}{4\eta^3}\left(5\frac{1}{\eta}\frac{\partial g}{\partial \eta} - 2\frac{\partial^2 g}{\partial \eta^2}\right)$$

$$F_3 = -\frac{1}{8\eta^3}\left(-\frac{\partial^3 g}{\partial \eta^3} + 5\frac{1}{\eta}\frac{\partial^2 g}{\partial \eta^2} - \frac{17}{2}\frac{1}{\eta^2}\frac{\partial g}{\partial \eta}\right)$$

$$G_3 = \frac{1}{2}\frac{1}{\eta^4}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial g}{\partial \eta}\right)$$

$$H_3=-\frac{\eta}{8}\frac{\partial}{\partial\eta}\left(\frac{1}{\eta^5}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial g}{\partial\eta}\right)\right)$$

and

$$I_3=-\frac{1}{2\eta}\frac{\partial}{\partial\eta}\left(\frac{3}{4}\frac{1}{\eta^2}\frac{\partial^2 g}{\partial\eta^2}-\frac{7}{4}\frac{1}{\eta^3}\frac{\partial g}{\partial\eta}\right) \quad . \qquad (3.18)$$

The initial condition also gives the following condition

$$A_3=-(F_3+H_3+I_3).$$

Hence, the third order f_3 is given by

$$\begin{aligned} f_3 &= f_{31}e^{i\Phi}+f_{3-1}e^{i\Phi}+f_{33}e^{3i\Phi}+f_{3-3}e^{-3i\Phi} \\ &= \frac{\rho_e\lambda}{128\pi}\{B_3\omega_0^2t^2\cos(2\omega_0\eta t-\Phi)+C_3\omega_0 t\sin(2\omega_0\eta t-\Phi) \\ &\quad +A_3\cos(2\omega_0\eta t-\Phi)+D_3\omega_0^2t^2\cos(2\omega_0\eta t+\Phi) \\ &\quad -E_3\omega_0 t\sin(2\omega_0\eta t+\Phi)+F_3\cos(2\omega_0\eta t+\Phi) \\ &\quad +G_3\omega_0 t\sin(4\omega_0\eta t-\Phi)+H_3\cos(4\omega_0\eta t-\Phi) \\ &\quad +I_3\cos\Phi\}+f_{33}e^{3i\Phi}+f_{3-3}e^{-3i\Phi} \ . \end{aligned} \qquad (3.19)$$

Since we only need to calculate f_{40} for the second order energy transfer, we need only $f_{3\pm1}$ components in the third order distribution function. The solution is given by

$$\begin{aligned} f_{40}=-\frac{\rho_e\lambda}{512\pi}\frac{\partial}{\partial\eta}\{ &(-B_3+D_3)[-\frac{1}{2\eta}\omega_0^2t_0^2\cos 2\omega_0\eta t_0 \\ &+\frac{1}{2\eta^2}\omega_0 t_0\sin 2\omega_0\eta t_0-\frac{1}{4\eta^3}(1-\cos 2\omega_0\eta t_0)] \\ &+(C_3+E_3)[\frac{1}{2\eta}\omega_0 t_0\sin 2\omega_0\eta t_0-\frac{1}{4\eta^2}(1-\cos 2\omega_0\eta t_0)] \\ &-(A_3-F_3)\frac{1}{2\eta}(1-\cos 2\omega_0\eta t_0) \\ &+G_3[\frac{1}{4\eta}\omega_0 t_0\sin 4\omega_0\eta t_0-\frac{1}{16\eta^2}(1-\cos 4\omega_0\eta t_0)] \\ &-H_3\frac{1}{4\eta}(1-\cos 4\omega_0\eta t_0)]\} \end{aligned} \qquad (3.20)$$

The averaged energy exchange is

$$\Delta\langle\eta\rangle \simeq \langle\eta_2\rangle + \langle\eta_4\rangle \,, \qquad (3.21)$$

where $\langle\eta_2\rangle$ is given by eq.(3.12) and

$$\langle\eta_4\rangle = \rho_e\lambda(\Omega/\omega_0)^8 \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d\eta \; f_{40}(\phi,\eta,t)\eta$$
$$= \rho_0\lambda(\Omega/\omega_0)^8(\omega_0 t_0)^7 \int g(\eta)F_2(\omega_0\eta t_0)d\eta \,, \qquad (3.22)$$

where (see appendix I) the second gain function is given by

$$F_2(x) = \frac{1}{x^4}\left\{ -\frac{1}{2}\sin 2x - \frac{13}{4}\frac{\cos 2x}{x} + \frac{53}{8}\frac{\sin 2x}{x^2} - 3\frac{(1-\cos 2x)}{x^3} - \frac{1}{2}\frac{\cos 4x}{x} + \frac{11}{8}\frac{\sin 4x}{x^2} - \frac{9}{8}\frac{(1-\cos 4x)}{x^3} \right\} \qquad (3.23)$$

If we define the first order gain function as

$$F_1(x) = \frac{1}{x^2}\left(\sin 2x - \frac{(1-\cos 2x)}{x} \right) , \qquad (3.24)$$

then the averaged energy exchange per electron is

$$\Delta\langle\eta\rangle/\rho_e\lambda \simeq \frac{1}{16}(\Omega/\omega_0)^4(\omega_0 t_0)^3\int d\eta\; g(\eta)F_1(\omega_0\eta t_0) + \frac{1}{256}(\Omega/\omega_0)^8(\omega_0 t_0)^7\int d\eta\; g(\eta)F_2(\omega_0\eta t_0) \,. \qquad (3.25)$$

The gain function $F_1(x)$ and $F_2(x)$ are plotted in Fig.1 and Fig.2 respectively. Note that (appendix I) $F_1(x=0)=0$, $F_2(x)=0$, which means that if initially all the particles are at the equilibrium energy, $g(\eta)=\delta(\eta)$, the net energy exchange during the whole process should be zero.

3.3 EXAMPLE

For the special case of $g(\eta)=\delta(\eta-\eta_0)$,

$$\langle\eta\rangle/\rho_e\lambda = \frac{1}{16}(\Omega/\omega_0)^4(\omega_0t_0)^3F_1(\omega_0t_0\eta_0) + \frac{1}{256}(\Omega/\omega_0)^8(\omega_0t_0)^7F_2(\omega_0t_0\eta_0). \qquad (3.26)$$

The higher order contribution can not be neglected when

$$\frac{\langle\eta_4\rangle}{\langle\eta_2\rangle} = \frac{1}{16}(\Omega/\omega_0)^4(\omega_0t_0)^4\left(\frac{F_2(\omega_0t_0\eta_0)}{F_1(\omega_0t_0\eta_0)}\right) \sim 1 . \qquad (3.27)$$

Even for a more general initial condition we can still convince ourself that the possible energy exchange intergration is of the following order:

$$\begin{aligned}
&\iint f_0\eta\, d\phi d\eta \sim \frac{1}{\omega_0t_0} \quad ;\\
&\iint f_1\eta\, d\phi d\eta \sim (\omega_0t_0)(\Omega/\omega_0)^2 \quad ;\\
&\iint f_2\eta\, d\phi d\eta \sim (\omega_0t_0)^3(\Omega/\omega_0)^4 \quad ;\\
&\iint f_3\eta\, d\phi d\eta \sim (\omega_0t_0)^5(\Omega/\omega_0)^6 \quad ;\\
&\iint f_4\eta\, d\phi d\eta \sim (\omega_0t_0)^7(\Omega/\omega_0)^8 \quad ;\\
&\quad\vdots \quad ,
\end{aligned} \qquad (3.28)$$

since the experimental set up is normally operating at $\omega_0t_0\eta_0 \sim 1$ to obtain the largest gain. We shall estimate the validity of perturbation method in that condition. For

$B_w=1$ (T), $E_0=10^6$ V/m, $\lambda_w=0.1$ m, $\gamma_r=100$, $N_w=50$,

$$(\Omega/\omega_0)^2 = \frac{e^2B_wE_0\lambda_w^2}{2\pi^2m_0^2c^3\gamma_r^2} \simeq 5.8 \times 10^{-5} ,$$

$$\omega_0t_0 = 2\pi N_w ,$$

$$\frac{1}{16}\frac{F_2(x)}{F_1(x)} \simeq 10^{-2} ,$$

hence

$$\frac{\langle\eta_4\rangle}{\langle\eta_2\rangle} \simeq 0.3 \quad ,$$

the higher order contribution may not negligible in this situation.

4. DISCUSSION

As pointed out by Morton[3], eq.(3.2.1) and (3.2.2) are the same standard rf equations used by accelerator physicists. When the parameters Ω and η_0 changes adiabatically we can draw the trajectories in phase space which correspond to the solution of eq.(3.2.1) and (3.2.2). The maximum stable phase curve of a single bucket is shown in Fig.3. When the electrons are injected into the FEL with the uniform distribution in phase, the condition[3]

$$\eta_0 < \eta_{max} \qquad (4.1)$$

with

$$\eta_{max} = (\Omega/\omega_0)\ \Gamma(\Phi_r) \qquad (4.2)$$

$$\Gamma(\Phi_r)=(\cos\Phi_r-(\frac{\pi}{2}\sin\Phi_r-\Phi_r)\sin\Phi_r)^{1/2},$$

must be satisfied to have the electrons partly trapped in the stable region. Under the influence of the E.M. wave, one bounce period of the electron motion inside the wiggler corresponds to

$$N_w\ (\Omega/\omega_0)=1. \qquad (4.3)$$

The energy transfer calculation shown on Fig.1 suggests that the maximum gain is achieved at

$$\omega_0 t_0 \eta_0 \simeq 1.3. \qquad (4.4)$$

At the maximum gain, the condition that the higher order perturbation contribution should not be negligible is $(\Omega/\omega_0)^4(\omega_0 t_0)^4 \times 10^{-2} \simeq 0.15$, i.e., the second order is 15% of the first order. The condition is equivalent to,

$$(\Omega/\omega_0)\, 2\pi N_w \simeq 2 \,. \qquad (4.5)$$

We observe that,

(1) To achieve a high gain, the intereaction should last for a time comparable to the bounce period, $N_w(\Omega/\omega_0) > 0.2$, otherwise either eq.(4.4) can not be satisfied or the electrons can not be trapped;

(2) When the intereaction goes on less than half bounce period, the perturbation method can be used and eq.(4.4) should be satisfied to achieve a high gain. The injection η_0 should be near the "top" of the closed region in Fig.3;

(3) When the intereaction goes on about half bounce period , the high gain can be achieved but higher order perturbation contribution should be included into consideration;

(4) When the intereaction goes on for more than one bounce period , the perturbation method can no longer be used. However it is implied by eq.(3.2.1) and (3.2.2) that on one bounce period the total energy exchange should be very small.

In most of the FEL experiment, the conditions are near to the marginal limit that perturbation can be applied. Our analysis indicates that in the usual case the first order perturbation on Vlasov's equation is useful to analyse the behavior of the FEL system. However in the situation when the criterion given in section 3 is not met, the perturbation calculation may not give meaningful result. Other kind of method, for example a numerical calculation may have to be used.

APPENDIX I: SECOND ORDER GAIN FUNCTION

From eq.(3.20) and (3.22),

$$\frac{\langle\eta_4\rangle}{\rho_e\lambda} = (\Omega/\omega_0)^8 2\pi \int d\eta\, f_{40}(\phi,\eta,t)\eta$$

$$= -(\Omega/\omega_0)^8 \frac{1}{256}\int \eta\, d\eta \frac{\partial}{\partial\eta} \{ \frac{1}{2\eta}\omega_0^2 t_0^2 \cos 2\omega_0\eta t_0 (B_3 - D_3)$$

$$+\omega_0 t_0 \sin 2\omega_0 \eta t_0 [-\frac{(B_3+D_3)}{2\eta^2} + \frac{(C_3+E_3)}{2\eta}]$$

$$+\frac{1}{4\eta}\omega_0 t_0 \sin 4\omega_0\eta t_0\, G_3$$

$$+(1-\cos 2\omega_0\eta t_0)[\frac{(B_3-D_3)}{4\eta^3} - \frac{(C_3+E_3)}{4\eta^2} - \frac{(A_3-F_3)}{2\eta}]$$

$$-(1-\cos 4\omega_0\eta t_0)(\frac{G_3}{16\eta^2} + \frac{H_3}{4\eta})\} . \qquad (A1.1)$$

The five terms given in eq.(A1.1) can be evaluated and is given respectively:

$$I=\frac{1}{256}\int g(\eta)d\eta\,\frac{(\omega_0 t_0)^3}{\eta^4}\left(-\frac{1}{2}\sin 2\omega_0\eta t_0-\frac{1}{\omega_0\eta t_0}\cos 2\omega_0\eta t_0\right)$$

$$II=\frac{1}{256}\int g(\eta)d\eta\,\frac{(\omega_0 t_0)^3}{\eta^4}\left(\sin 2\omega_0\eta t_0+\frac{7}{4}\frac{\cos 2\omega_0\eta t_0}{\omega_0\eta t_0}+\frac{5}{8}\frac{\sin 2\omega_0\eta t_0}{(\omega_0\eta t_0)^2}\right)$$

$$III=\frac{1}{256}\int g(\eta)d\eta\,\frac{(\omega_0 t_0)^3}{\eta^4}\left(-2\sin 4\omega_0\eta t_0-\frac{9}{2}\frac{\cos 4\omega_0\eta t_0}{\omega_0\eta t_0}+\frac{25}{8}\frac{\sin 4\omega_0\eta t_0}{(\omega_0\eta t_0)^2}\right)$$

$$IV=\frac{1}{256}\int g(\eta)d\eta\,\frac{(\omega_0 t_0)^3}{\eta^4}\left(-2\sin 4\omega_0\eta t_0-4\frac{\cos 2\omega_0\eta t_0}{\omega_0\eta t_0}+6\frac{\sin 2\omega_0\eta t_0}{(\omega_0\eta t_0)^2}-3\frac{(1-\cos 2\omega_0\eta t_0)}{(\omega_0\eta t_0)^3}\right)$$

$$V=\frac{1}{256}\int g(\eta)d\eta\,\frac{(\omega_0 t_0)^3}{\eta^4}\left(2\sin 4\omega_0\eta t_0+4\frac{\cos 4\omega_0\eta t_0}{\omega_0\eta t_0}-\frac{7}{4}\frac{\sin 4\omega_0\eta t_0}{(\omega_0\eta t_0)^2}-\frac{9}{8}\frac{(1-\cos 4\omega_0\eta t_0)}{(\omega_0\eta t_0)^3}\right). \quad (A1.2)$$

Hence,

$$\langle\eta_4\rangle=\frac{\rho_e\lambda}{256}\int F_2(\omega_0\eta t_0)\,g(\eta)\,d\eta\,, \quad (A1.3)$$

where $F_2(x)$ is given in eq.(3.26).

We expand $F_2(x)$ by the Taylor expansion and can easily find that the coefficient of the terms $\frac{1}{x^7}$, $\frac{1}{x^5}$, $\frac{1}{x^3}$ and $\frac{1}{x}$ is equal to zero. Thus

$$F_2(x=0)=0 \quad (A1.4)$$

and $F_2(x)$ is an odd funtion in x, as is $F_1(x)$.

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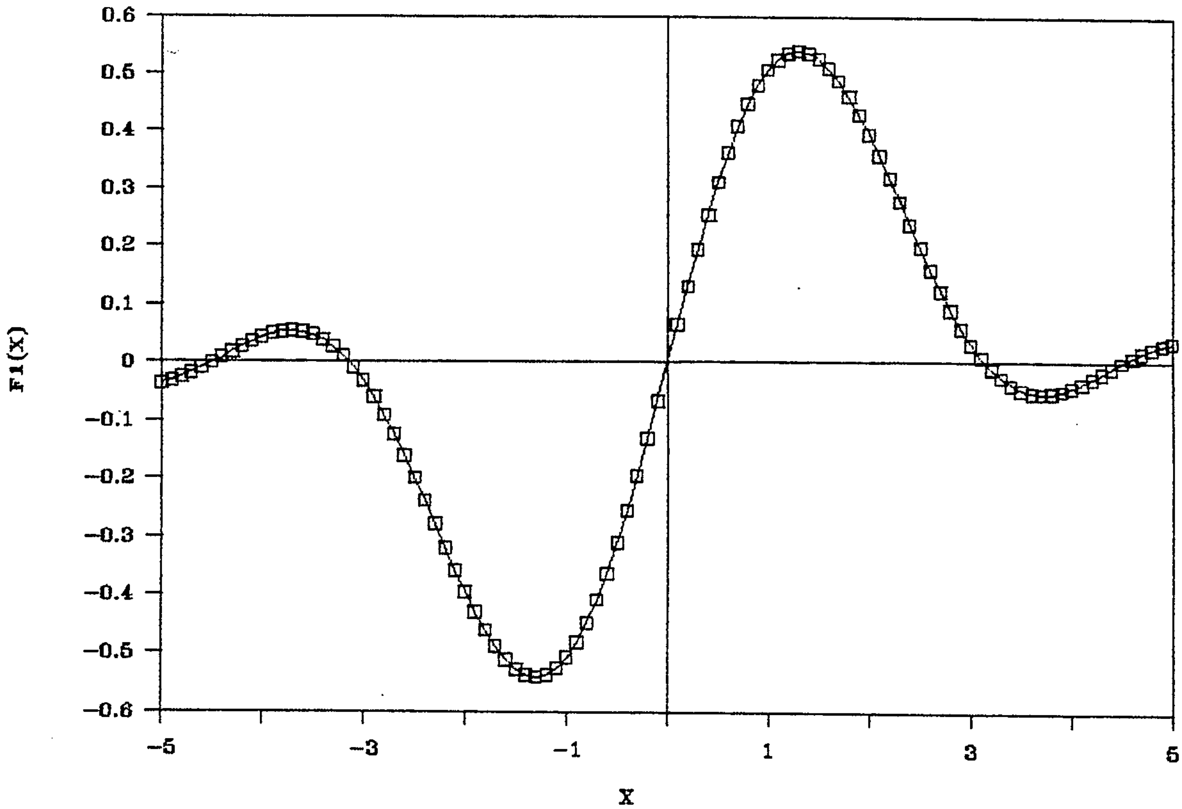
FIG.1 FIRST ORDER GAIN FUNCTION.

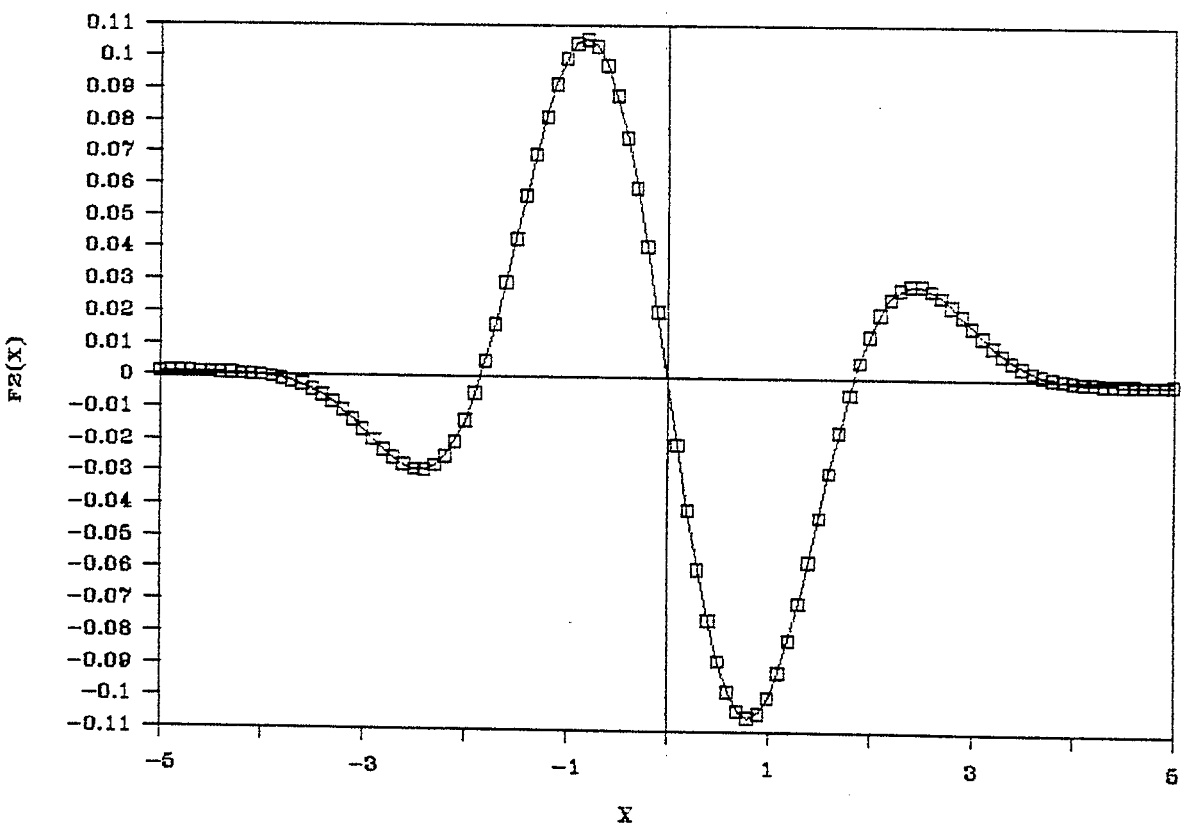
FIG.2 SECOND ORDER GAIN FUNCTION.

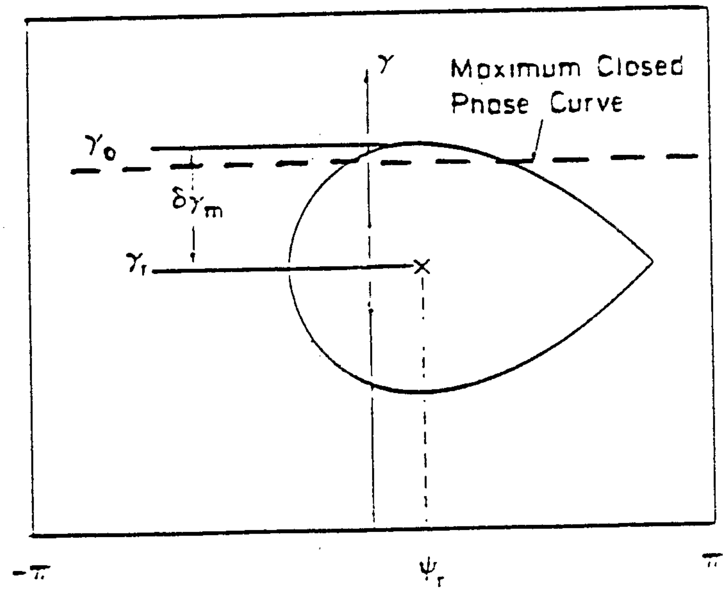
FIG.3 STABLE REGION IN PHASE SPACE.

FIRST ORDER GAIN FUNCTION



SECOND ORDER GAIN FUNCTION





Maximum Closed
Phase Curve
γ
γ_0
$\delta\gamma_m$
γ_r
$-\pi$
ψ_r
π

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