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Helical Siberian Snakes

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To eliminate spin resonances in circular accelerators "Siberian Snakes" may be inserted at one or more azimuths in such a way that the overall spin precession tune ν_s equals 1/2.[1] A snake is a sequence of horizontal and vertical deflection magnets whose overall effect is to rotate the spin by π about an axis in the plane of the orbit, either longitudinal or transverse or any angle in between. At the same time the magnets of the snake should be arranged so as to produce zero net deflection and displacement of the particle orbit.

We adopt a coordinate system (s,x,y) where s is the direction of the orbit, x the horizontal displacement from it, and y the vertical displacement; these coordinates are also denoted as 1,2,3. The spinor equation of motion in this rotating coordinate system is

$$\frac{d\psi}{ds} = \frac{i}{2}\vec{\sigma}\cdot\vec{b}\ \psi \tag{1}$$

where

$$\vec{b} = [(1+G)\vec{B}_{\parallel} + \gamma G\cdot\vec{B}_{\perp}]/B\rho \tag{2}$$

with G (=1.793 for protons) the anomalous magnetic moment coefficient, $B\rho$ the magnetic rigidity, and $\vec{B}_{\parallel}$ and $\vec{B}_{\perp}$ the parts of the magnetic field parallel and perpendicular to the reference orbit. If we assume that the transverse field is *helical* with

$$B_2 = B\cos ks;\quad B_3 = B\sin ks \tag{3}$$

and define

$$\rho = B\rho/B;\ \varkappa = \gamma G/\rho, \tag{4}$$

we obtain

$$\frac{d\psi}{ds} = \frac{i}{2}\varkappa(\sigma_2\cos ks + \sigma_3\sin ks)\psi = \frac{i}{2}\varkappa\sigma_2 e^{iks\sigma_1}\ \psi. \tag{5}$$

Now change to the variable

$$\varphi = e^{\frac{i}{2}ks\sigma_1}\ \psi \tag{6}$$

so that the spin equation becomes

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* Work performed under the auspices of the US Department of Energy.

[1] Ya. S. Derbenev and A. M. Kondratenko, Proceedings of 10th International Conference on High Energy Accelerators, Protvino, USSR, 1977.

$$\varphi' = \frac{i}{2}(k\sigma_1 + \varkappa\sigma_2)\varphi \qquad (7)$$

which can be solved explicitly. For a helix extending from $s = -a$ to $s = a$ we obtain

$$\varphi(a) = e^{i\vartheta(\sigma_1 + \varepsilon\sigma_2)}\ \varphi(-a) \qquad (8)$$

where

$$\vartheta = ka; \quad \varepsilon = \varkappa/k. \qquad (9)$$

In terms of the original spinor variable ψ the spinor transfer matrix for the helix is therefore

$$\mathbb{M}_H = e^{-\frac{i}{2}\vartheta\sigma_1}\ \exp(i\lambda\vartheta\sigma_1 e^{i\delta\sigma_3})e^{-\frac{i}{2}\vartheta\sigma_1}. \qquad (10)$$

where

$$\lambda = (1+\varepsilon^2)^{1/2}; \qquad \delta = \tan^{-1}\varepsilon. \qquad (11)$$

The trajectory in the helical magnet is determined by the equations

$$x'' = -B_3/B\rho = -\frac{1}{\rho}\sin ks\ ; \quad y'' = \frac{1}{\rho}\cos ks.$$

One solution is

$$x = r\sin ks; \qquad y = -r\cos ks \qquad (12)$$

which is a helix with radius $r = 1/\rho k^2$ and the same pitch as the magnet helix. The general solution is (12) plus any straight line orbit.

To minimize orbit excursions we must add dipole bending magnets at each end of the helix such that a centered incoming beam is matched into the helix (12) at $s = -a$ and is deflected back to zero excursion and slope after going through the helix from $-a$ to a.

To see the requirements for these orbit compensating magnets, let us consider the ones at the output end of the helix, and describe each deflector magnet by it x and y deflection angles at the bending center. The magnets at the input end will be in the opposite sequence, with the signs of x deflections reversed and y deflections unchanged.

Assume there are two magnets at location $s = a+b$ and $a+b+c$, with x and y deflections at each one. Then, with the orbit (12) in the helix, the excursions at the end of the helix and the centers of the magnets will be at α: $\quad x/r = \sin\vartheta \qquad y/r = -\cos\vartheta$

$$x'/kr = \cos\vartheta \qquad y'/kr = \sin\vartheta \tag{13}$$

at $\alpha+b$

$$x/r = \sin\vartheta + kb\cos\vartheta \qquad y/r = -\cos\vartheta + kb\sin\vartheta$$
$$x'/kr = \cos\vartheta + \alpha \qquad y'/kr = \sin\vartheta + \beta \tag{14}$$

at $\alpha+b+c$

$$x/r = \sin\vartheta + kb\cos\vartheta + kc(\cos\vartheta+\alpha) \qquad y/r = -\cos\vartheta + kb\sin\vartheta + kc(\sin\vartheta+\beta) \tag{15}$$
$$x'/kr = \cos\vartheta + \alpha + \alpha_2 \qquad y'/kr = \sin\vartheta + \beta + \beta_2$$

if the magnet at $\alpha+b$ has horizontal deflection angle αkr and vertical deflection angle βkr, and the one at $\alpha+b+c$ has deflection angles $\alpha_2 kr$ and $\beta_2 kr$.

We require that x, y, x', y' all be zero after the second magnet, i.e. at $\alpha+b+c$. This imposes four conditions on the six quantities kb, kc, α, β, α_2, β_2. We may therefore impose two more conditions. Let us suppose, for example, that ϑ lies in the fourth quadrant, i.e. between $2\pi(n - 1/4)$ and $2\pi n$, so that $\sin\vartheta < 0$ and $\cos\vartheta > 0$. We may then require that the maximum excursion within the two magnets be not more than r, and that the first bending magnet be a pure vertical bend:

$$\alpha = 0, \; y(\alpha+b) = -1. \tag{16}$$

This gives

$$kb = \frac{1-\cos\vartheta}{-\sin\vartheta} \; ; \; kc = \frac{1-\cos\vartheta}{-\sin\vartheta\,\cos\vartheta}$$
$$\beta = \frac{-\sin\vartheta}{1-\cos\vartheta} \; ; \; \beta_2 = \frac{\sin\vartheta\,\cos\vartheta}{1-\cos\vartheta} \; ; \; \alpha_2 = -\cos\vartheta. \tag{17}$$

The overall spin transfer matrix is

$$\mathbb{M} = \mathbb{E}\,\mathbb{M}_H\,\bar{\mathbb{E}} \tag{18}$$

where $\mathbb{M}_H$ is given by (10) and

$$\mathbb{E} = \exp\left[\tfrac{i}{2}\varepsilon(\alpha_2\sigma_3 - \beta_2\sigma_2)\right] \exp(-\tfrac{i}{2}\varepsilon\beta\sigma_2) \tag{19}$$

and $\bar{\mathbb{E}}$ is obtained from $\mathbb{E}$ by reversing the order of factors and changing the sign of the coefficients of σ_3.

It is tedious and not particularly instructive to multiply (18) out and obtain a formula for the overall matrix. But one can easily do it numerically. A FORTRAN program named HELIX has been written to do this. For a given helix twist angle ϑ one can then find the value of ε for which the trace of $\mathbb{M}$ is zero, i.e. $\mathbb{M}$ represents a 180 degree spin rotation about an axis. It turns out that this will always be the longitudinal axis to within a few degrees. (Because of the assumed symmetry of the configuration it may be seen that the rotation axis of $\mathbb{M}$ is always horizontal, i.e. contains no y component).

For a given field B and energy $E = \gamma mc^2$ one may derive

Helix radius

$$r = 1/(\rho k^2) = 1.038\ \varepsilon^2/(BE) \tag{20}$$

with r in meters, B in Tesla, E in GeV

Helix length

$$L_H = 3.49\ \varepsilon\vartheta/B \text{ meters} \tag{21}$$

Distance from end of helix to center of first compensation magnet

$$b = 1.745\ \varepsilon\left|\tan(\vartheta/2)\right|/B \tag{22}$$

Distance from end of helix to center of second magnet:

$$b + c = 1.745\ \varepsilon\ \left|\tan\vartheta\right|/B \tag{23}$$

Length of first (vertical deflector) magnet

$$L_{Mag1} = 1.745\ \varepsilon\left|\cot(\vartheta/2)\right|/B \tag{24}$$

Length of second magnet

$$L_{Mag2} = 1.745\ \varepsilon\left|\frac{\cos\vartheta}{\sin\ \vartheta/2}\right|/B \tag{25}$$

Tilt angle of second magnet from vertical: $\vartheta/2$ (mod π).

If ϑ lies in a different quadrant than the fourth, the detailed expressions are somewhat different; for first and third quadrants it turns out that the first bending magnet should deflect in the horizontal rather than the vertical plane. The value of ϑ has to be chosen so that the magnets do not overlap; it turns out that this means that ϑ (mod 90°) $< 19^\circ$.

In Table I we show some of the quantities computed for ϑ = n × 90 + 18 degrees, with ε computed so as to make $\nu_s = 1/2$. The helix radius is, of course, inversely proportional to magnetic field and to proton energy; the length of the helix, the other magnets, and the drifts are independent of energy. If we assume B = 2 Tesla and E = 10 GeV, we see that with ϑ = 288 degrees (helix just over 1 1/2 twists) the radius of the orbit in the helix is 3.4 cm; with $\vartheta = 558^\circ$ (just over 3 whole twists) the excursion radius is down to 1.7 cm. The overall length of the system is not very different in the two cases, namely 11.8 m in the first case and 13.15 in the second. One could even go to four twists (2×738 degrees of twist in 11 meters of helix) with only another meter of penalty in overall length, and the helix radius reduced to 1.3 cm.

Note that phi, the angle of orientation of the effective rotation axis, is just a few degrees in all cases, i.e. the helical snake always produces a *longitudinal* rotation. Since no helical snakes with *transverse* rotation axes have been devised (at least so far), they do not seem to be useful for the "two-snake" scheme where the spin axis is vertical in the arcs, but only for the "one-snake" scheme where the spin precesses in the horizontal plane, and is always longitudinal at 180° from the snake. Fortunately this scheme is best suited to relatively low energy rings, where the resonances are not too strong.

Thus a single helical snake may be suitable for use in a ring such as the proposed European Hadron Facility, with injection at, say, γ=10 and acceleration to 30 GeV. If helical magnets with 2 Tesla transverse fields can be made, one could have a single helix with compensators in one straight section; the length of the helix plus compensators would be around 14 meters.

TABLE I

ϑ	ε	ν_s	$r \cdot B \cdot E$	BL_{helix}	BL_{total}	Bb	BL_{Mag1}	Bc	BL_{Mag2}	φ
deg			m·T·GeV	T–m	T–m	T–m	T–m	T–m	T–m	deg
288.00	.8091	.5000	.6835	14.19	23.63	1.026	1.943	3.320	.742	−2.5
378.00	.7032	.5000	.5162	16.19	24.39	.891	1.689	2.885	.645	−4.3
468.00	.6308	.5000	.4154	17.98	25.33	.800	1.515	2.588	.579	1.4
558.00	.5763	.5000	.3467	19.59	26.30	.731	1.384	2.364	.529	2.5
648.00	.5340	.5000	.2977	21.08	27.30	.677	1.283	2.191	.490	−1.0
738.00	.4997	.5000	.2607	22.46	28.29	.633	1.200	2.050	.458	−1.7
828.00	.4713	.5000	.2319	23.77	29.26	.598	1.132	1.934	.432	.7