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Comparing and Defining Magnetic Multipoles: I. MAD and TEAPOT And II. RHIC Measurements and TEAPOT

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1. Introduction

The definitions of the magnetic field in MAD and in the standard multipole expansion (SME) used internally by Teapot are different. This note, perhaps for the umpTeenth time, defines the relationship between the two field definitions, assuming that they are in fact referring to the same physical quantity. After this discussion we define the relationship between RHIC's measured multipole coefficients and those of Teapot.

2. B Field in MAD

The magnetic field in MAD is defined by a Taylor series expansion along the x axis as [1]

$$B_y^{MAD}(x,0) = \sum_{n=0}^{N_{max}} \frac{B_n^{MAD} x^n}{n!} \tag{2.1}$$

The strength of a multipole, K_n, is defined to be

$$K_n = \frac{B_n^{MAD}}{p_o/e}, \tag{2.2}$$

and thus B_n^{MAD} can be computed as

$$\left(\frac{\partial^n B_y^{MAD}}{\partial x^n}\right)_{x=y=0}. \qquad 2b_2$$

3. B field according to the Standard Multipole Expansion

The true magnetic field of a physcial magnet can be described by a field strength $B_x^{True}(x,y,z), B_y^{True}(x,y,z)$. The thin element model then expresses this true field strength in terms of nominal field strengths $B_x(x,y), B_y(x,y)$ as

$$\int [B_y^{True}(x,y,z) + iB_x^{True}(x,y,z)]dz = L[B_y(x,y) + iB_x(x,y)] \tag{3.1}$$

where the integral is taken over the length of the magnet. The standard multipole expansion for B_y and B_x is then given by [2]

$$(LB_y) + i(LB_x) = (LB_o) \sum_{n=0}^{N_{max}} (b_n + ia_n)(x+iy)^n . \tag{3.2}$$

where we specialize to the case of a dipole. So B_o is the dipole field strength at the origin. N_{max} is the highest order of multipole in the series expansion. Defining $R_n + iI_n = (x+iy)^n$ we can re-write the field components in the SM expansion as

$$\tilde{B}_y = \frac{LB_y}{p_o/e} = \sum_{n=0}^{N_{max}} \tilde{b}_n R_n - \tilde{a}_n I_n \tag{3.3}$$

$$\tilde{B}_x = \frac{LB_x}{p_o/e} = \sum_{n=0}^{N_{max}} \tilde{b}_n I_n + \tilde{a}_n R_n \tag{3.4}$$

where

$$\tilde{a}_n = \frac{LB_o}{p_o/e} a_n, \quad \tilde{b}_n = \frac{LB_o}{p_o/e} b_n. \tag{3.5}$$

The scaling factors above are conventional, but the magnetic fields $(\tilde{B}_x, \tilde{B}_y)$ so defined are just the deflections that particles experience passing through the field, and the coefficients $(\tilde{a}_n, \tilde{b}_n)$ are the ones used directly by Teapot. The factor of p_o/e is often referred to as "$B\rho$". The following table [2] gives some explicit examples of the multipole expansion. Also note the traditional jargon used there, e.g., $\Delta\theta$ is the "bend angle", f is the "focal length" and S is the "sextupole strength", etc.

	n	R_n	I_n	$\tilde{b}_n$	$\tilde{a}_n$	$\Delta x' = -\tilde{B}_y$	$\Delta y' = \tilde{B}_x$
Horizontal bend	0	1	0	$\Delta\theta_x$	0	$-\Delta\theta_x$	0
Vertical bend				0	$\Delta\theta_y$	0	$\Delta\theta_y$
Erect quadrupole	1	x	y	$q = 1/f$	0	$-qx$	qy
Skew quadrupole				0	$q_s = 1/f_s$	$q_s y$	$q_s x$
Erect sextupole	2	$x^2 - y^2$	$2xy$	$S/2$	0	$-\frac{S}{2}(x^2 - y^2)$	$\frac{S}{2}2xy$
Skew sextupole				0	$S_s/2$	$\frac{S_s}{2}2xy$	$\frac{S_s}{2}(x^2 - y^2)$
Erect octupole	3	$x^3 - 3xy^2$	$3x^2y - y^3$	$O/6$	0	$-\frac{O}{6}(x^3 - 3xy^2)$	$\frac{O}{6}(3x^2y - y^3)$
Skew octupole				0	$O_s/6$	$\frac{O_s}{6}(3x^2y - y^3)$	$\frac{O_s}{6}(x^3 - 3xy^2)$
Erect decapole	4	$x^4 - 6x^2y^2$	$4xy(x^2 - y^2)$	$D/24$	0	$-\frac{D}{24}(x^4 - 6x^2y^2 + y^4)$	$\frac{D}{24}4xy(x^2 - y^2)$
Skew decapole		$+y^4$		0	$D_s/24$	$\frac{D_s}{24}4xy(x^2 - y^2)$	$\frac{D_s}{24}(x^4 - 6x^2y^2 + y^4)$

Table 3.1: Deflections, $\Delta x', \Delta y'$, caused by standard magnets and notation for their strengths

4. Relation between MAD K_n and SME b_n

Assuming that the field strength B_y is the same physical quantity in either representation, we want to find how K_n is related to b_n. Using the definition in terms of partial derivatives and noticing that

$$(\frac{\partial^n I_n}{\partial x^n})_{x=y=0} = 0$$

$$(\frac{\partial^n R_n}{\partial x^n})_{x=y=0} = n!$$

we see that

$$K_n \equiv (\frac{B_n^{MAD}}{p_o/e}) = L^{-1}n!\tilde{b}_n = \frac{B_o}{p_o/e}n!b_n. \tag{4.1}$$

The integrated strength is

$$K_n \cdot L = \frac{B_o L}{p_o/e}n!b_n = \frac{B_o L}{B\rho}n!b_n \tag{4.2}$$

and in terms of $\tilde{b}_n$,

$$K_n \cdot L = n!\tilde{b}_n. \tag{4.3}$$

The above K_n apply to the case of standard elements of length L. If one is instead talking about MAD's multipole element which is defined to have zero length, replace $K_n \cdot L$ in Eq. (4.3) by Kl_n to get the corresponding MAD multipole notation.

5. Skew components in MAD and SME

The definition of the magnetic field from the MAD documentation in Eq. (2.1) explicitly excludes skew multipole moments, so it is not possible to derive a relation between MAD's way of defining skew multipole elements and SME's a_n directly unless one looks into the Teapot code itself. We have done this (with the help of R. Talman). However, one can perhaps see the result on physical grounds if one considers only a single multipole. One can 'convert' an erect multipole of order n into its corresponding skew element by a rotation of the erect element around the longitudinal direction by its natural symmetry angle of $\pi/(2n+2)$. In this way a pure erect multipole of order n becomes a pure skew multipole of the same order. So one can conclude that a the strength of a skew element which is defined in MAD by specifying a multipole of strength Kl_n = *value* and T_n (without a value) is related to Teapot's a_n as

$$(Kl_n)^{skew} = \frac{B_oL}{p_o/e} n! a_n = \frac{B_oL}{B\rho} n! a_n, \tag{5.1}$$

and

$$(Kl_n)^{skew} = n! \tilde{a}_n. \tag{5.2}$$

(If one is instead interested in skew magnets of finite length, then replace Kl_n above by $K_n \cdot L$ as discussed earlier for the non-skew case.)

5.1. Aside: How *does* Teapot do it?

For those who might want to know precisely how Teapot transforms the MAD input specification of multipole strength into the SME form, here is an example of the (old) Fortran code that was used to do this transformation.(The C++ version of Teapot uses an equivalent formulation.) The example shown below is for MAD's octupole and general multipole. The relevant lines to focus on have been indicated with arrow marks. The explanation follows the listing.

```
                 ELSEIF (itype .EQ. 7) THEN
C                   ---- "octupole"
                    nmax(ikelem) = 3
  -->               el = pdata(idp)
                    thklen(ikelem) = el
  -->               val = pdata(idp + 1)*el/6.
 *-->*              ang = pdata(idp + 2)*4.
  -->               btw(3, ikelem) = val*cos(ang)
  -->               atw(3, ikelem) = val*sin(ang)
                    typeaper(ikelem) = pdata(idp + 3)
                    xapsize(ikelem) = pdata(idp + 4)
                    yapsize(ikelem) = pdata(idp + 5)
                    xoffset(ikelem) = pdata(idp + 6)
                    yoffset(ikelem) = pdata(idp + 7)
                    mxstreng(ikelem) = pdata(idp + 8)
```

```
            ELSEIF (itype .EQ. 8) THEN
C              ---- "multipol"
               DO i = 1, 9
 -->              val = pdata(idp)/fact(i)
*-->*             ang = pdata(idp + 1)*(i + 1)
                  idp = idp + 2
 -->              btw(i, ikelem) = val*cos(ang)
 -->              atw(i, ikelem) = val*sin(ang)
                  IF (iptyp(idp - 2) .NE. -1) nmax(ikelem) = i
               ENDDO
               typeaper(ikelem) = 0
```

Note: The variables $atw(n,-), btw(n,-)$ in the above code have exactly the same meaning as $\tilde{a}_n, \tilde{b}_n$ in Eq. (3.5).

First we discuss the octupole case. In the above code *pdata* is the array containing all the information gleaned by MAD's parser from the original standard input file. *el* is the length L of the octupole. *val* is a local variable which equals $K_n * L/3!$. *ang* is another local variable which in the case where one just specifies $TILT$ without an argument is the default roll angle of $\pi/8$ times 4. (This is the mysterious point. We only show how Teapot does this transformation, we don't explain the basis for it). One can see that $4*\pi/8 = \pi/2$, and thus $btw(3,-) = 0, atw(3,-) = val = K_n * L/3!$.

Now that we've done the octupole case, consider the thin multipole case that follows it. The meaning of the code variables is the same as in the octupole case. Since thin multipole have no length, the code is simpler in some respects. The variable *ang* specifies how the roll angle determines the strength of the skew element. For example, in the case where one takes the default roll angle of $\pi/2i+2$ for a skew multipole of strength Kl_i, one states T_i without an argument in the input file. Then the value of $ang = pdata(idp+1) * (i+1) = (\pi/2i+2)*(i+1) = \pi/2$. This leads to values of $btw(i,-), atw(i,-)$ of 0 and $pdata(idp)/fact(i) = Kl_i/i!$, respectively, in agreement with Eq. (5.2).

6. Relating RHIC's Measured Multipole Coefficients to those of Teapot

The field expansion in Eq. (3.2) actually applies in general only to dipoles since the central field value B_o vanishes (or at least ought to) for other types of magnets such as quadrupoles, sextupoles, etc. So one must adopt a different but analogous convention for other magnet types. In chapter two of Ref. 2 there is a clear discussion of one way to do this for the case of quadrupoles, and we can compare that with the way RHIC describes a general magnet. The multipole expansion for a quadrupole magnet from Ref. 2 is

$$\left(LB_y^Q\right) + i\left(LB_x^Q\right) = (L\frac{\partial B_y^Q}{\partial x})[x + iy + 10^{-4}\sum_{n=2}^{N_{max}}\left(b_n^Q + ia_n^Q\right)\frac{(x+iy)^n}{R_r^{n-1}}] \tag{6.1}$$

where R_r is the reference radius where the measurement is made, and along with R_r, the factor of 10^{-4} is chosen so that a_n^Q, b_n^Q are of order 1 for "bad", low order multipoles. The prefactor, in this case the field gradient, $(\partial B_y^Q/\partial x)_{x=y=0}$, serves the same purpose as B_o in Eq. (3.2).

In general for every type of magnet, there is a formula of this type. The prefactor like $B_o(\partial B_y^Q/\partial x)$ in the case of dipoles(quadrupoles) sets the scale so that the coefficients $a_n, b_n(a_n^Q, b_n^Q)$ represent fractional deviations from the measured field strength. A similar analysis can be done for the other types of magnets. To summarize, the normalization of multipole coefficients via Eq. (6.1) requires knowing the behaviour of the field at the origin.

In contrast to Eq. (6.1) RHIC has used a slightly different form to represent the multipole expansion for a general magnet. According to our sources [3–4] and references [5–6], there is uniform strategy for every type of magnet that is representative of the way the magnets are actually measured. In the following we will assume that the local coordinate system of Teapot and the magnetic measurement system are the same. If this is not true, for example, if the magnet is oriented differently in the lattice compared to the way it was measured, appropriate modifications to the sign of the coefficients will need to be made [6]. With this caveat, the RHIC convention for a magnet's multipole expansion is [5]

$$(LB_y) + i\,(LB_x) = L\,B(R_r)[10^{-4} \sum_{n=0}^{N_{max}} \left(b_n^M + ia_n^M\right) \frac{(x+iy)^n}{R_r^n}] \tag{6.2}$$

where the superscript M in a_n^M, b_n^M denotes the fact that these are *measured* multipole coefficients. $B(R_r)$ is a normalization factor. This normalization is chosen so that the magnitude of the term of order k in the expansion, $|b_k^M + ia_k^M| = 10^4$ for a magnet with multipolarity $2(k+1)$. Consequently the multipole coefficient, b_k^M, for a "normal" or "upright" magnet of order k is 10^4. I.e., b_0^M for dipoles is 10^4, b_1^M for quadrupoles is 10^4, and similarly for skew magnets so that for a skew quadrupole a_1^M would be 10^4.

Since RHIC normalizes its multipole coefficients in this way, comparison with an expression like Eq. (6.1) for a specfic kind of magnet can be obtained by evaluating Eq. (6.2) along the x axis near the origin. We will do this exercise in the appendix, but it is not actually necessary. Teapot only requires that the magnetic field be brought to a form like Eq. (3.2). Eq. (6.2) is already in this form, so making the correspondence with Teapot is straightforward up to possible reversals in sign that are discussed in RHIC/AP/95 [6] and summarized in the next section.

The factor $L\,B(R_r)$ on the right hand side of Eq. (6.2) is measured at a fixed current by the magnetic measurement group of RHIC and quoted as the Integral Transfer Function or ITF, i.e., $ITF\cdot I = L\,B(R_r)$, where I is the current in kA at which the measurement was made. The reference radius, R_r, is also given for each measurement.

The $\tilde{a}_n, \tilde{b}_n$ of Teapot are recovered from the above *measured* expansion coefficients in analogy to Eq. (3.5) by

$$\tilde{b}_n = (\frac{ITF\cdot I}{p_o/e})\frac{10^{-4}}{R_r^n}b_n^M \tag{6.3}$$

$$\tilde{a}_n = (\frac{ITF\cdot I}{p_o/e})\frac{10^{-4}}{R_r^n}a_n^M \tag{6.4}$$

where the b_n^M, a_n^M are the measured multipole coefficients, p_o/e is $B\rho$, and I is the current at which the measurement was made in kA.

In some cases, particularly for dipoles, the RHIC magnetic measurements group does more detailed measurements of the magnetic multipoles. They measure them at the body center as well as the return and lead ends of the magnet. If this group of measurements is available, a different form of the Teapot coefficients is needed since the physical dimension of the measured multipole coefficients are different for the body and end data.

If Body measurements exist, we specify body $\tilde{b}_n^{Body}, \tilde{a}_n^{Body}$ for Teapot as

$$\tilde{b}_n^{Body} = (\frac{BTF \cdot I}{p_o/e})(\frac{ITF}{BTF})\frac{10^{-4}}{R_r^n} b_n^{M-Body} = (\frac{ITF \cdot I}{p_o/e})\frac{10^{-4}}{R_r^n} b_n^{M-Body} \tag{6.5}$$

$$\tilde{a}_n^{Body} = (\frac{BTF \cdot I}{p_o/e})(\frac{ITF}{BTF})\frac{10^{-4}}{R_r^n} a_n^{M-Body} = (\frac{ITF \cdot I}{p_o/e})\frac{10^{-4}}{R_r^n} a_n^{M-Body} \tag{6.6}$$

where BTF is the body transfer function with dimension $Tesla/kA$, and the superscript $M-Body$ refers to "measured Body". The factor of ITF/BTF has dimension of length in $meters$ and is needed to scale BTF so that it has the dimensions of an integral transfer function since $b_n^{M-Body}, a_n^{M-Body}$ are dimensionless.

If End measurements exist, then the lead and return end coefficients for Teapot are given by:

$$\tilde{b}_n^{End} = (\frac{BTF \cdot I}{p_o/e})\frac{10^{-4}}{R_r^n} b_n^{M-End} \tag{6.7}$$

$$\tilde{a}_n^{End} = (\frac{BTF \cdot I}{p_o/e})\frac{10^{-4}}{R_r^n} a_n^{M-End} \tag{6.8}$$

where BTF is again the body transfer function referred to above, and note that in this case since the dimension of b_n^{M-End}, a_n^{M-End} is in $meters$, only BTF rather than ITF is needed.

7. Afterward on Sign Conventions for Multipole Coefficients

RHIC magnets are measured in a standard way, i.e., the lead end of each magnet is oriented with respect to a local magnet coordinate system in the same way during the measurement process. Thus the measured multipole coefficients are directly tied to the local measurement coordinate system's orientation.

During installation in the tunnel a magnet may need to be rotated by π radians around the Y axis relative to the coordinate system in which it was measured either for physics or mechanical/installation reasons. In these cases the sign of some multipole coefficients used in Teapot will need to change (relative to their signs in the measurement database) to properly model the dynamics in the global coordinate system used by Teapot. The nature of these sign changes has been explained in Ref. 6, and we will not reproduce their detailed analysis here. However, for purposes of keeping the definitions of Teapot multipole coefficients in terms of RHIC's measured values all in one place, we include the necessary rules here. We thank Fritz Dell for the following formulation of these rules.

The rules require an understanding of a magnet's "orientation". A magnet's orientation is defined to be positive if a positive displacement relative to the horizontal closed orbit corresponds to a positive horizontal displacement with respect to the *magnet local* *coordinate system* discussed above. Otherwise the orientation is negative. See Ref. 6 for a clear statement of the definition of the local magnet measurement coordinate system and its relation to the lead and non-lead ends of the magnet.

1. For Normal magnets whose main multipole is even (dipoles, sextupoles, etc.), or for Skew magnets whose main multipole is odd (quadrupoles, octupoles, etc.)
   - Positive orientation: use b_n^M, a_n^M as is.
   - Negative orientation: change sign of b_n^M with *odd* n, and change sign of a_n^M with *even* n.
2. For Normal magnets whose main multipole is odd (quadrupoles, octupoles, etc.), or for Skew magnets whose main multipole is even (dipoles, sextupoles, etc.)
   - Positive orientation: use b_n^M, a_n^M as is.
   - Negative orientation: change sign of b_n^M with *even* n, and change sign of a_n^M with *odd* n.

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A. Relating $B(R_r)$ to the Field at the Origin

The normalizing factor $B(R_r)$ can be related theoretically to the value of the field at the origin in the following way. From Eq. (6.2) the value of the integrated field at $y = 0$ is

$$(LB_y + iLB_x)|_{y=0} = L\,B(R_r)[10^{-4} \sum_{n=0}^{N_{max}} \left(b_n^M + i a_n^M\right) \frac{x^n}{R_r^n}] \qquad (A.1)$$

For a "normal" magnet of order $k, b_k^M = 10^4, a_k^M = 0$. Taking partial derivatives k times, we have

$$\frac{\partial^k (LB_y)}{\partial x^k}|_{x=y=0} = k!\,(LB\,(R_r))\,10^{-4}\frac{b_k^M}{R_r^k}. \qquad (A.2)$$

Noting that $b_k^M = 10^4$ in the case of a normal magnet of order k, we find

$$B\,(R_r) = (\frac{\partial^k B_y}{\partial x^k})|_{x=y=0}\frac{R_r^k}{k!}. \qquad (A.3)$$

For the case of a skew magnet of order k a similar analysis yields

$$B\,(R_r) = (\frac{\partial^k B_x}{\partial x^k})|_{x=y=0}\frac{R_r^k}{k!}. \qquad (A.4)$$

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