BNL-101917-2014-TECH
AD/RHIC/5;BNL-101917-2013-IR

Beam Loss Due To The Aperture Limitation Resulting From Intrabeam Scattering

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September 1984

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USDOE Office of Science (SC)

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RHIC Technical Note No. 5

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September 12, 1984

ABSTRACT

Diffusion equation is used to evaluate the beam loss in the presence of aperture limitation resulting from the intrabeam scattering. We discuss the effect of different boundary conditions. Satisfactory beam intensity can be maintained within the proposed RHIC operation time.

I. Introduction

Intrabeam scattering (IBS)[1] has become one of the important topics in the heavy ion collider design study. Detailed calculation of the emittance ε_N, momentum spread σ_E and bunch length σ_ℓ blow up due to IBS has been performed by G. Parzen.[2] Fig. 1 summarizes these results, where we have plotted σ_E, ε_N and σ_ℓ vs. $t^{\frac{1}{2}}$. It is tempting to argue that σ_E, ε_N and σ_ℓ satisfies certain statistical random diffusion process. In fact, the situation is not very simple. First, the Einstein relation of Brownian motion is not satisfied at all. However Fig. 1 shows that within this limited region of time scale, the Einstein relation is not a very bad approximation after all.

With the diffusion equation in mind, we would like to ask the question of beam survival rate in these diffusion processes. Section 2 reviews briefly the Fokker-Planck equation and Section 3 discusses the transversed and longitudinal losses and the Conclusion is given in Section 4.

2. Fokker-Planck diffusion equation

For Markovian Processes, the distribution function $\rho(y,t)$ satisfies the Fokker-Planck equation.[3]

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial y}(A_1\rho) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(A_2\rho) \tag{1}$$

where

$$A_1 = \lim_{\Delta t\to 0}\left[\langle\Delta y\rangle/\Delta t\right] \tag{2}$$

$$A_2 = \lim_{\Delta t\to 0}\left[\langle(\Delta y)^2\rangle/\Delta t\right] \tag{3}$$

Higher order contributions in eq. (1) are neglected. For dynamical system where a Hamiltonian governs the motion of dynamical variables,[4] one has

$$A_1 = \frac{1}{2} \frac{\partial}{\partial y} A_2 \ . \tag{4}$$

With this relation, eq. (1) becomes

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial y} \left(\frac{A_2}{2} \frac{\partial \rho}{\partial y}\right) \ . \tag{5}$$

A special interesting situation is $A_1 = \langle \Delta y \rangle / \Delta t = \text{constant} = D$, then

$$A_2 = 2D(t)\ y \tag{6}$$

Eq. (5) then becomes

$$\frac{\partial \rho}{\partial t} = D \frac{\partial}{\partial y}\, y\, \frac{\partial}{\partial y}\, \rho \tag{7}$$

The fundamental solution of eq. (7) is

$$\rho(y,t) = \frac{1}{\langle \Delta y(t) \rangle}\, e^{-y/\langle \Delta y(t) \rangle} \tag{8}$$

On the other hand, another interesting situation is that $A_1 = 0$. We have then $A_2 = \langle (\Delta y)^2 \rangle / \Delta t = D(t)$. The fundamental solution in this case is

$$\rho(y,t) = \frac{1}{\sqrt{4\pi \langle \Delta y^2(t) \rangle}}\, e^{-y^2/2\langle \Delta y^2(t) \rangle} \tag{9}$$

3. Applications

3.1. Transverse phase space

The Fokker-Planck equation for the transverse phase space is given

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \varepsilon} \left(D\varepsilon \frac{\partial \rho}{\partial \varepsilon}\right)$$

where ε is the emittance of the beam and

$$D = \frac{d<\varepsilon>}{dt} = D(t)$$

Let us define new variable τ and y as

$$\tau = \frac{1}{\varepsilon_a} \int D\, dt = \frac{1}{\varepsilon_o} (<\varepsilon_f>-<\varepsilon_i>)$$

$$y = \varepsilon/\varepsilon_o$$

The diffusion equation becomes then

$$\frac{\partial\rho}{\partial\tau} = \frac{\partial}{\partial y} (y \frac{\partial\rho}{\partial y}) \tag{10}$$

Besides the fundamental solution of equation (8), there are alternative solutions to equation (10). Teng,[5] MacLachlan[6] and Ruggiero[7] used the sink boundary condition

$$\rho\ (y=1,\tau) = 0 \tag{11a}$$

and initial condition

$$\rho\ (y,\tau=0) = \rho_o(y)\ , \tag{11b}$$

where $\rho_o(y)$ is the initial distribution.

The solution is given by

$$\rho\ (y,t) = \sum_{n=1}^{\infty} \frac{J_o(\lambda_n\sqrt{y})\ e^{-\frac{\lambda_n^2}{4}\tau}}{J_1^{\ 2}(\lambda_n)} \int_o^1 \rho_o(y')J_o(\lambda_n\sqrt{y'})\ dy' \tag{12}$$

where J_o and J_1 are Bessel functions and λ_n's are given by

$$J_o(\lambda_n) = 0$$

Due to the aperture limitation, the number of particles confined is given by

$$N(\tau) = \int_0^1 \rho(y,\tau)\, dy = 2 \sum \frac{e^{-\lambda_n^2 \tau/4}}{\lambda_n J_1(\lambda_n)} \int_0^1 \rho_o(y)\, J_o(\lambda_n \sqrt{y})\, dy \qquad (13)$$

Table 1 summarizes the result calculated with boundary conditions (11a) and (11b). While the loss rate calculated from the fundamental solution in eq. (8) is also shown for comparison. With the Courant-Snyder invariant, the fundamental solution of eq. (8) is a Gaussian distribution of the phase space variables x and x'. The loss rate shown on the last row of Table 1 is therefore determined by the available phase space, $\sqrt{6}\sigma_a/\sigma_x$.

Since the aperture acts only as a beam scraper, which gives no dynamical action to the diffusion process, I believe that eq. (11a) is incorrect. The similarity between the diffusion process here and heat conduction is inappropriate, because the heat loss is proportional to the temperature gradient. In the present situation, the diffusion process comes from the interaction between beam particles alone.

Table 1

	γ	5 (.65hr)	7 (2hr)	12^{*}	30^{*}	100^{*}
	$\delta\ (10^{-3})$	0.93	1.15	1.56	1.98	1.10
	ε_i^N	10π	10π	10π	10π	10π
	ε_f^N	37.2π	52.1π	44π	33.2π	27.8π
(±26mm good field)	ε_a	$6\text{x}9.42\pi$	$6\text{x}8.79\pi$	$6\text{x}7.68\pi$	$6\text{x}6.62\pi$	$6\text{x}8.93\pi$

Analysis with boundary conditions in eqs. (11a) and (11b)[+]

τ	.096	.114	.061	.019	.0033
loss rate(%)	<1	<1	<1	<1	0

Analysis with the fundamental solution of eq. (8).

$\varepsilon_a/(\varepsilon_f^N/\gamma)$	7.60	7.09	12.6	36	19.3
$\sqrt{6}\sigma_a/\sigma_x$	2.76	2.66	3.5	6.	14.
loss rate(%)	2.2	2.9	.2	0	0

δ is the rms momentum spread

ε_a is the admittance space available for betatron oscillation

$\varepsilon_i^N, \varepsilon_f^N$ are the normalized initial and final emittances respectively

$\tau = (\varepsilon_f^N - \varepsilon_i^N)/\gamma\varepsilon_a$

$\sqrt{6}\sigma_a/\sigma_x$ is the ratio of aperture to the beam rms transverse physical size

* 10 hr operation time is assumed

\+ see ref. 7 for details.

3.2. Diffusion in the longitudinal phase space

Let us define the momentum spread as

$$\delta = \frac{\Delta p}{p} \tag{14}$$

The Fokker-Planck equation becomes

$$\frac{\partial\rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial\delta}\left(A_2 \frac{\partial\rho}{\partial\delta}\right) \tag{15}$$

where

$$A_2 = \frac{d}{dt}\langle\delta^2\rangle ,$$

is assumed to be a function of time only. Let Δ_a be the aperture limit. Defining Z and τ as

$$Z \equiv \frac{\delta}{\Delta_a} \tag{16}$$

$$\tau = \frac{1}{2\Delta_a^{\,2}} \int_o^t A_2(t')dt' = \frac{1}{2\Delta_a^{\,2}} \left(\delta^2(t) - \delta^2(0)\right) , \tag{17}$$

we obtain the following Fokker-Planck equation

$$\frac{\partial\rho}{\partial\tau} = \left(\frac{\partial^2}{\partial Z^2}\rho\right) \tag{18}$$

Using the similar condition as that of eqs. (11a) and (11b), we obtain the solution

$$N(\tau) = 2 \sum_{n=0}^{\infty} \frac{(-)^n}{\beta_n} e^{-\beta_n{}^2\tau} \simeq \frac{4}{\pi} e^{-\frac{\pi^2}{4}\tau} \tag{19}$$

with

$$\beta_n = \frac{\pi}{2}(2n+1)$$

for a delta function initial distribution.

Table 2 summarizes the loss rate calculated from eq. (19) and that calculated from the fundamental Gaussian distribution of eq. (9). We found that the loss rate obtained in these two analysis are about the same. However in reality they are very different. Fig. 2 shows the survival number as a function of parameter τ in equation (17) for these two cases. Curve (A) obtained from the boundary conditions (11a) and (11b), while (B) is equal to $\mathrm{erf}(1/2\sqrt{\tau})$ obtained from the fundamental solution of eq. (9). We feel that boundary conditions (11a) and (11b) are inappropriate in this case too.

Table 2

γ	5	7	12	30	100
$\Delta_a = (\Delta p/p)$bucket (10^{-3})	2.28	2.72	3.82	9.95	2.68
$\delta(t=0)$	.818	.751	.678	1.261	.359
$\delta_F = \delta(t=t_F)$	.93	1.14	1.563	1.985	1.099
Analysis with boundary conditions (11a) and (11b)					
τ_o	.064	.038	.016	.008	.009
τ_t	.083	.088	.084	.024	.085
loss(%)	4	5	4	0	4
Analysis with the fundamental solution of eq. (9).					
Δ_a/δ_F	2.45	2.39	2.44	5.01	2.44
loss(%)	3	3	3	0	3

4. Conclusion

In conclusion, we have applied the Fokker-Planck diffusion equation to estimate the beam loss due to the intrabeam scattering. Due to many particle dynamics, diffusion equation should be appropriate in describing this random process. We discuss however the implication of boundary conditions on the beam loss. We argue to prefer one special model to the other. The effect of the intrabeam scattering process is found to contribute only about 5% of total loss of beam as a whole. Details are however summarized in Tables 1 and 2 respectively for transverse and longitudinal phases.

References

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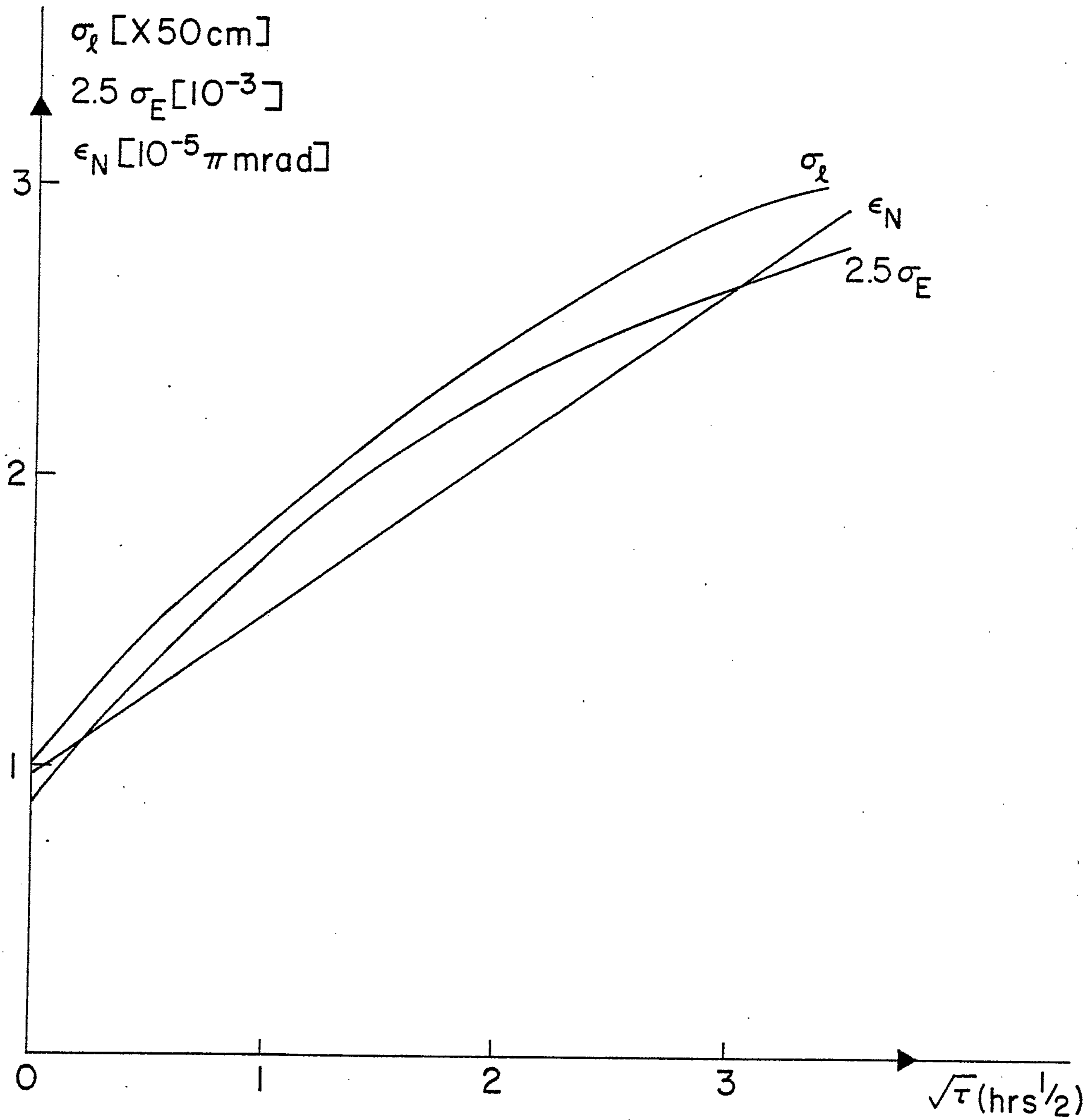

Fig. 1 Bunch length, energy spread and the normalized emittance of the beam particles are plotted as a function of storage time (see ref. 2)

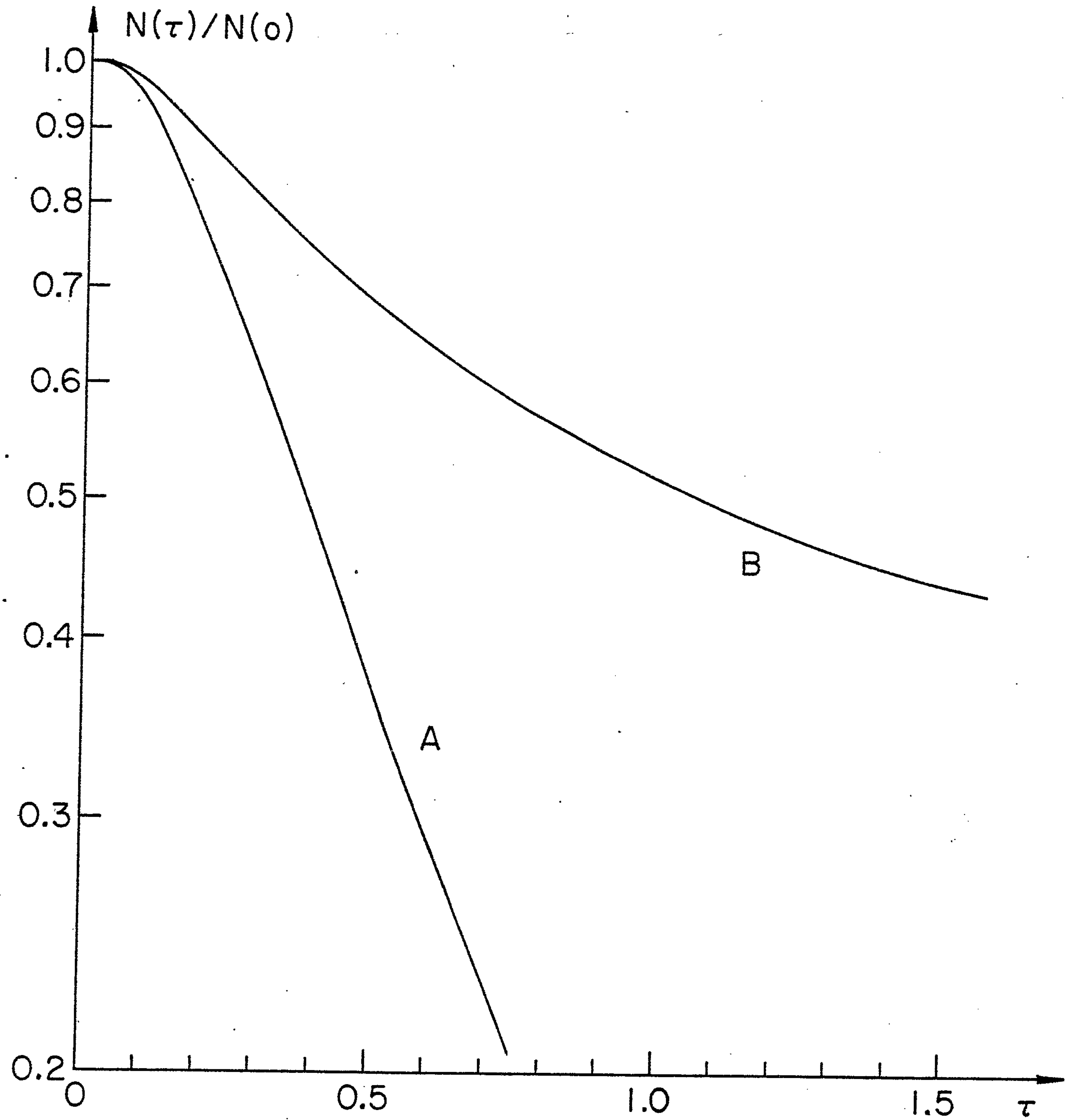

Fig. 2 The particle survival ratio is plotted as a function of τ parameter defined in eq. 17. for longitudinal phase space.