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Orbit Matrices For Helical Snakes

E. D. Courant

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Collider Accelerator Department

Brookhaven National Laboratory

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Helical snakes may be useful for preventing spin resonances in RHIC and the Tevatron. We must evaluate the impact they may have on orbit stability.

Blewett and Chasman[1] show that in a helical snake the fields are, up to terms quadratic in the displacements from the axis,

$$\begin{aligned} B_x &= -B_0\{[1+\frac{k^2}{8}(3x^2+y^2)]\sin kz - \frac{k^2}{4}xy\cos kz\} \\ B_y &= B_0\{[1+\frac{k^2}{8}(x^2+3y^2)]\cos kz - \frac{k^2}{4}xy\sin kz\} \\ B_z &= -kB_0(x\cos kz + y\sin kz)[1+\frac{k^2}{8}(x^2+y^2)] \end{aligned} \tag{1}$$

where $k = 2\pi/\lambda$ is the wave number of the helical field, B_o its value on the axis, and x and y the displacements from the axis, z being the distance along the longitudinal axis. This is in agreement with the field expressions obtained by Ptitsin [2].

The equations of motion for x and y are (to lowest order in x and y and their derivatives)

$$\begin{aligned} x'' &= (y'B_z - B_y)/B\rho \\ y'' &= (B_x - x'B_z)/B\rho \end{aligned} \tag{2}$$

A solution of these equations is the helical trajectory

$$\begin{aligned} x_0 &= r_0\cos kz \\ y_0 &= r_0\sin kz \end{aligned} \tag{3}$$

[1] J. P. Blewett and R. Chasman, J. App Phys. **48**, 2692 – 2698(1977)
[2] V. Ptitsin, Note RHIC/AP/41(Oct. 10, 1994)

where

$$r_0 = \frac{1}{k^2\rho} \tag{4}$$

is the radius of the helical orbit centered on the axis, and $\rho \equiv B\rho/B_0$ is the radius of curvature of the particle in a field B_0.

We may describe the actual motion of the particle as an oscillation about (3). Note that, if the helix is centered on the central orbit $x = y = 0$, the actual orbit will, in fact, not be the helical orbit (3) but an oscillation about it, with an amplitude of the order of r_0.

Using the fields (1), putting $x = x_0 + \xi$ and $y = y_0 + \eta$, linearizing in ξ and η, and neglecting terms of the order $(r_0/\rho)^2 = 1/(k\rho)^4$, we obtain the equations

$$\xi'' = -\frac{1}{2\rho^2}[(1 + \frac{3}{2}\cos 2kz)\xi - \frac{3}{2}\eta \sin 2kz] - \frac{1}{k\rho^2}\eta' \tag{5a}$$

$$\eta'' = -\frac{1}{2\rho^2}[(1 - \frac{3}{2}\cos 2kz)\eta - \frac{3}{2}\xi \sin 2kz] + \frac{1}{k\rho^2}\xi' \tag{5b}$$

Caution : the linear equations (5) are valid only in the immediate vicinity of the helical trajectory (3). Nonlinear terms in ξ and η in the fields B_x and B_y will be of the order $(k\xi)^2 B_0$, while the terms linear in ξ and η are of the order ξr_0. Since the displacements from the helix are in fact just of the order of magnitude of r_0, this means that the nonlinear terms are of the same order as the linear restoring forces, and therefore the results obtained below may be expected not to be very accurate. Nevertheless they should at least give a rough indication of the orbit-dynamical effect of the helix.

In RHIC (and most other applications) the length of the helix will be small compared to the radius ρ; therefore it should be a good approximation (equivalent to the thin lens approximation) to average the trigonometric functions in the coefficients in (5). Furthermore, if the length of the helix is a whole wavelength (360 degree twist) the trigonometric functions average to zero, and $kz \sin kz$ averages to 1. We shall assume this to be the case, so that (5) simplifies to

$$\xi\text{"} = -\epsilon^2\xi - 2\delta\eta' \tag{6a}$$

$$\eta\text{"} = -\epsilon^2\eta + 2\delta\xi' \tag{6b}$$

where

$$\epsilon^2 = \frac{1}{2\rho^2}; \delta = \frac{kr_0}{2\rho} = \frac{1}{2k\rho^2} \tag{7}$$

We look for a solution of the form

$$\xi = ae^{i\lambda z}; \eta = be^{i\lambda z}. \tag{8}$$

Substituting in (6) we find the determinantal equation

$$\begin{vmatrix} \epsilon^2 - \lambda^2 & 2\delta\lambda i \\ -2\delta\lambda i & \epsilon^2 - \lambda^2 \end{vmatrix} = (\epsilon^2 - \lambda^2)^2 - 4\delta^2\lambda^2 = 0 \tag{9}$$

the solutions of which are

$$\lambda_1 = \sqrt{\epsilon^2 + \delta^2} + \delta; \lambda_2 = \sqrt{\epsilon^2 + \delta^2} - \delta. \tag{10}$$

It may now be seen that the quantities

$$u = \lambda_2 \xi + \eta' \tag{11a}$$

$$v = \lambda_1 \eta + \xi' \tag{11b}$$

perform uncoupled harmonic oscillations, i.e.

$$u'' + \lambda_1^2 u = 0; v'' + \lambda_2^2 v = 0. \tag{12}$$

We can write (11) in matrix form. The transformation between u, u', v, v' and ξ, ξ', η, η' is

$$u = \mathbf{D}\xi; \mathbf{D} = \begin{pmatrix} \lambda_2 & & & 1 \\ & \lambda_1 & -\lambda_1\lambda_2 & \\ & 1 & \lambda_1 & \\ -\lambda_1\lambda_2 & & & \lambda_2 \end{pmatrix} \tag{13}$$

the inverse of which is

$$
\mathbf{D}^{-1} = \frac{1}{2\lambda_0} \begin{pmatrix} 1 & & & -1/\lambda_2 \\ & 1 & \lambda_2 & \\ & -1/\lambda_1 & 1 & \\ \lambda_1 & & & 1 \end{pmatrix}. \tag{14}
$$

Note $\lambda_1\lambda_2 = \epsilon^2$; we define $\lambda_0 = \sqrt{\epsilon^2 + \delta^2} = (\lambda_1 + \lambda_2)/2$.

To obtain the transfer matrix for the whole helix, we also have to take account of the end effects. If we assume the field of the helix steps abruptly from zero to the expressions (1) at $z = z_1$ and back to zero at z_2, we have to add delta-function singularities to satisfy Maxwell's equations at the ends.

If the fields (1) jump abruptly from zero at the beginning of the helix and back down at the end, one has to add delta function singularities at the ends. We note that the longitudinal field (1c) can be written

$$
B_z = -kB_0(r_0 + \xi \cos kz + \eta \sin kz) S(z) \tag{15}
$$

where $S(z)$ is the step function jumping from 0 to 1 at z_1 and back to zero at z_2. The Maxwell equation $\nabla \cdot B = 0$ then requires that we add terms

$$
\Delta B_x = kB_0 \frac{\xi}{2}(r_0 + \xi \cos kz)[\delta(z - z_1) - \delta(z - z_2)] \tag{16a}
$$

$$
\Delta B_y = kB_0 \frac{\eta}{2}(r_0 + \eta \sin kz)[\delta(z - z_1) - \delta(z - z_2)] \tag{16b}
$$

which leads, by (5), to jumps in ξ' and η' at the entrance and the exit: At the entrance

$$
\Delta\xi' = -\delta \cdot \eta; \Delta\eta' = \delta \cdot \xi; \tag{17}
$$

at the exit the jumps are reversed.

Thus the matrices $\mathbf{D}$ and $\mathbf{D}^{-1}$(eqs. 12 and 13) have to be modified:

$$
\mathbf{D} = \begin{pmatrix} \lambda_2 & & & 1 \\ & \lambda_1 & -\lambda_1\lambda_2 & \\ & 1 & \lambda_1 & \\ -\lambda_1\lambda_2 & & & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & -\delta & \\ & & 1 & \\ \delta & & & 1 \end{pmatrix} = \begin{pmatrix} \lambda_0 & & & 1 \\ & \lambda_1 & -\lambda_0\lambda_1 & \\ & 1 & \lambda_0 & \\ -\lambda_0\lambda_2 & & & \lambda_2 \end{pmatrix} \tag{18}
$$

$$\mathbf{D}^{-1} = \frac{1}{2\lambda_0}\begin{pmatrix} 1 & & & -1/\lambda_2 \\ & \lambda_0/\lambda_1 & \lambda_0 & \\ & -1/\lambda_1 & 1 & \\ \lambda_0 & & & \lambda_0/\lambda_2 \end{pmatrix} \tag{19}$$

We now construct a transfer matrix for the whole helix, using the matrices (18) and (19), and the standard 2×2 matrix transformations for the uncoupled variables u and v. We assume the length of the helix is $L = 2\pi/k$. The matrix for the whole helix is

$$\mathbf{M} = \mathbf{D}^{-1}\begin{pmatrix} c_1 & s_1/\lambda_1 & & \\ -\lambda_1 s_1 & c_1 & & \\ & & c_2 & s_2/\lambda_2 \\ & & -\lambda_2 s_2 & c_2 \end{pmatrix}\mathbf{D} \tag{20a}$$

$$= \frac{1}{2}\begin{pmatrix} c_1+c_2 & (s_1+s_2)/\lambda_0 & s_2-s_1 & (c_1-c_2)/\lambda_0 \\ -\lambda_0(s_1+s_2) & c_1+c_2 & -\lambda_0(c_1-c_2) & s_2-s_1 \\ s_1-s_2 & (c_2-c_1)/\lambda_0 & c_1+c_2 & (s_1+s_2)/\lambda_0 \\ -\lambda_0(c_2-c_1) & s_1-s_2 & -\lambda_0(s_1+s_2) & c_1+c_2 \end{pmatrix} \tag{20b}$$

$$= \begin{pmatrix} c_\delta & & -s_\delta & \\ & c_\delta & & -s_\delta \\ s_\delta & & c_\delta & \\ & s_\delta & & c_\delta \end{pmatrix}\begin{pmatrix} c_0 & s_0/\lambda_0 & & \\ -\lambda_0 s_0 & c_0 & & \\ & & c_0 & s_0/\lambda_0 \\ & & -\lambda_0 s_0 & c_0 \end{pmatrix}. \tag{20c}$$

Here c_0, c_1, c_2, c_δ stand for $\cos\lambda_0 L, \cos\lambda_1 L, \cos\lambda_2 L, \cos\delta L$, and similarly for s_0 etc.

In RHIC, it is proposed to install two snakes in each ring, in the straight sections between magnets $Q7$ and $Q8$ upstream of the 2 o'clock and 8 o'clock intersection point. Each snake consists of four 360° helices. The helix modules are all 2.4 m long, with fields of approximately $4T$ in two of the modules and $1.5T$ in the other two. With the RHIC injection energy of $\gamma = 27$ we then have

$$B\rho = 84\mathrm{T-m}; \rho = 21\mathrm{m}; \mathrm{k} = 2\pi/2.4 = 2.62\mathrm{m}^{-1}$$

so that $k\rho = 55; r_0 = 6.95\mathrm{mm}$

and neglecting terms of the order $(kr_0)^2 \approx .0003$ is justifiable.

An addition has been incorporated in the SYNCH orbit program to define an element type "HELIX", characterized by length, twist angle, field, field orientation, and magnetic rigidity. (The same can, of course, be done in MAD or TEAPOT or other dynamics programs). Matrices are computed by eqs. (20), and included in beam lines for a storage ring. Doing this for RHIC at $\gamma = 27$ we find shifts in the tunes, as well as coupling between the two transverse modes. A run with the lattice at $\beta* = 10$ m gives

$$\text{No snakes: } \nu_x = 28.196999; \nu_y = 29.192962$$

snakes:

$$\gamma = 27 : \nu_x = 28.221027; \nu_y = 29.216518; \mid \Delta\nu_{\text{min}} \mid = .00143$$

$$\gamma = 268 : \nu_x = 28.197254; \nu_y = 29.193220; \mid \Delta\nu_{\text{min}} \mid = .000017.$$

Here $\mid \Delta\nu_{\text{min}} \mid$ is obtained from the off-diagonal submatrices of the 4×4 matrix of the whole ring. Following Edwards and Teng [3] and Peggs[4] we have

$$\mid \Delta\nu_{\text{min}} \mid = \sqrt{\text{Det}(\mathbf{m} + \mathbf{n}^{+})}/2\pi \tag{21}$$

where $\mathbf{m}$ and $\mathbf{n}$ are the off-diagonal submatrices of the $4x4$ matrix, and $\mathbf{n}^{+}$ is the symplectic conjugate of $\mathbf{n}$.

We see that the tune shift is of the order of .025 units at injection energy, while the coupling due to the snakes is no larger than the coupling one may expect from ordinary orbit errors. Both the tune shift and the coupling decrease with the square of the energy, so that they are minuscule at storage energy.

Numerical results by Luccio, using numerical integration in his program SNIG[5], give matrices comparable to those of the matrix (20c). His coupling strengths are of the same order of magnitude as obtained here. The main result is firm: A pair of helical snakes have a modest (though not entirely negligible) effect on the orbit dynamics of the ring, and small tune corrections are probably desirable.

I have benefited from discussions with A. Luccio, F. Pilat, and V. Ptitsin.

[3] D. Edwards and L. Teng, IEEE Trans. Nucl Sci **NS–20**, No. 3, 1973
[4] S. Peggs, *Ibid.*, **NS–30**, No. 4, 1983
[5] A. Luccio, private communication