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**Estimate of Diffusion Rate Due to
Periodic Crossing of Non-Linear Resonances**

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ESTIMATE OF DIFFUSION RATE DUE TO PERIODIC CROSSING OF NON-LINEAR RESONANCES

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The purpose of this calculation is to estimate the emittance growth when crossing a high order resonance.

1. General Analysis for Several Multipoles

We start with the equations

$$x'' + Q_x^2 x = \sum_{k,\ell,h} \mu_{k,\ell,h} k x^{k-1} y^\ell \cos h\theta, \quad (1)$$

$$y'' + Q_y^2 y = \sum_{k,\ell,h} \mu_{k,\ell,h} \ell x^k y^{\ell-1} \cos h\theta, \quad (2)$$

where prime means $\frac{d}{d\theta}$ and where

$$Q_x = \nu_x - \Delta\nu \cos \nu_s \theta, \quad (3)$$

$$Q_y = \nu_y - \Delta\nu \cos \nu_s \theta. \quad (4)$$

We are interested in the single resonance at

$$s\nu_x + t\nu_y = h + \delta, \quad (5)$$

where s, t, h are integers, and $\delta \ll 1$.

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We use the phase amplitude method and write

$$\begin{aligned} x &= A \cos \psi_x, & y &= B \cos \psi_y, \\ x' &= -Q_x A \sin \psi_x, & y' &= -Q_y B \sin \psi_y, \end{aligned} \quad (6)$$

where

$$\psi_x = \int_0^\theta Q_x d\theta + \alpha, \quad \psi_y = \int_0^\theta Q_y d\theta + \beta. \quad (7)$$

It is easy to show that

$$A' = - \sum_{k,\ell,h} \mu_{k,\ell,h} \frac{k}{Q_x} A^{k-1} B^\ell \cos^{k-1} \psi_x \sin \psi_x \cos^\ell \psi_y \cos h\theta, \quad (8)$$

$$B' = - \sum_{k,\ell,h} \mu_{k,\ell,h} \frac{\ell}{Q_y} A^k B^{\ell-1} \cos^k \psi_x \cos^{\ell-1} \psi_y \sin \psi_y \cos h\theta. \quad (9)$$

We now use

$$\cos^k \psi_x = \frac{1}{2^{k-1}} \sum_{\substack{s=0,1 \\ \Delta s=2}} \frac{\cos s\psi_x}{\left(\frac{k-s}{2}\right)! \left(\frac{k+s}{2}\right)!} \frac{k!}{\left(\frac{\ell-t}{2}\right)! \left(\frac{\ell+t}{2}\right)!} \quad (10)$$

($\times \frac{1}{2}$ if there is a term with $s = 0$ when k is even) and average over all rapidly varying oscillations except for the resonance in (5) to obtain

$$AA' \cong - \sum_{k,\ell,h} \frac{k}{Q_x} \frac{A^k B^\ell}{2^{k+\ell}} \frac{k!}{\left(\frac{k-s}{2}\right)! \left(\frac{k+s}{2}\right)!} \frac{\ell!}{\left(\frac{\ell-t}{2}\right)! \left(\frac{\ell+t}{2}\right)!} \cos \chi, \quad (11)$$

$$BB' = - \sum_{k,\ell,h} \frac{\ell}{Q_y} \frac{A^k B^\ell}{2^{k+\ell}} \frac{k!}{\left(\frac{k-s}{2}\right)! \left(\frac{k+s}{2}\right)!} \frac{\ell!}{\left(\frac{\ell-t}{2}\right)! \left(\frac{\ell+t}{2}\right)!} \cos \chi, \quad (12)$$

where

$$\chi = s\psi_x + t\psi_y - h\theta. \quad (13)$$

Since

$$\psi_x = \alpha + \int_0^\theta Q_x d\theta = \nu_x \theta - \frac{\Delta\nu}{\nu_s} \sin \nu_s \theta + \alpha, \quad (14)$$

$$\psi_y = \beta + \int_0^\theta Q_y d\theta = \nu_y \theta - \frac{\Delta\nu}{\nu_s} \sin \nu_s \theta + \beta, \quad (15)$$

we have

$$\chi(\theta) = (s\nu_x + t\nu_y - h)\theta + s\alpha + t\beta - (s+t) \frac{\Delta\nu}{\nu_s} \sin \nu_s \theta. \quad (16)$$

The resonance is crossed at θ_0 , where

$$\chi'(\theta_0) = s\nu_x + t\nu_y - h = (s+t) \Delta\nu \cos \nu_s \theta_0. \quad (17)$$

The change in A in crossing the resonance is determined by

$$\int_{-\infty}^{\infty} d\theta \cos \chi(\theta) \cong \int_{-\infty}^{\infty} d\theta \cos \left[\chi(\theta_0) + \frac{(\theta - \theta_0)^2}{2} \chi''(\theta_0) \right] = \sqrt{\frac{2\pi}{|\chi''(\theta_0)|}} \cos \left(\chi_0 \pm \frac{\pi}{4} \right), \quad (18)$$

where $\chi_0 \equiv \chi(\theta_0)$ and where the \pm sign has to do with the sign of $\chi''(\theta_0)$. With δ defined in (5) we have

$$|\chi''(\theta_0)| = (s+t) \Delta\nu \nu_s |\sin \nu_s \theta_0| = \nu_s \sqrt{(s+t)^2 (\Delta\nu)^2 - \delta^2}. \quad (19)$$

Finally, we have from (11), (12), assuming A and B do not change much in the crossing:

$$A\Delta A = -\frac{\sqrt{2\pi} \cos(\chi_0 \pm \frac{\pi}{4})}{\sqrt{(s+t) \Delta\nu \nu_s |\sin \nu_s \theta_0|}} \sum_{k,\ell,h} \frac{k}{Q_x} G(s,t), \quad (20)$$

$$B\Delta B = -\frac{\sqrt{2\pi} \cos(\chi_0 \pm \frac{\pi}{4})}{\sqrt{(s+t) \Delta\nu \nu_s |\sin \nu_s \theta_0|}} \sum_{k,\ell,h} \frac{\ell}{Q_y} G(s,t), \quad (21)$$

where

$$G(s,t) = \mu_{k,\ell,h} \frac{A^k B^\ell}{2^{k+\ell}} \frac{k!}{(\frac{k-s}{2})! (\frac{k+s}{2})!} \frac{\ell!}{(\frac{\ell-t}{2})! (\frac{\ell+t}{2})!}. \quad (22)$$

2. Diffusion Approximation

If we assume that many betatron oscillations occur between resonance crossings, it is reasonable to assume that the betatron phase at the crossing (χ_0) is uncorrelated from one crossing to the next. In this case we can average A^2 to obtain

$$A^2(\theta) = A_0^2 + \frac{2\pi \times \frac{1}{2}}{(s+t) \Delta\nu \nu_s |\sin \nu_s \theta_0|} \left[\sum_{k,\ell,h} \frac{k}{Q_x} G \right]^2 \times \frac{2\theta}{\frac{2\pi}{\nu_s}}, \quad (23)$$

where $2\pi/\nu_s$ is the change in θ between pairs of crossings. This leads to the diffusion result (random walk)

$$A^2(\theta) = A_0^2 + K_x \theta, \quad B^2(\theta) = B_0^2 + K_y \theta, \quad (24)$$

where

$$K_x = \frac{1}{(s+t) \Delta\nu |\sin \nu_s \theta_0|} \left[\sum_{k,\ell,h} \frac{k}{Q_x} G \right]^2, \quad (25)$$

$$K_y = \frac{1}{(s+t) \Delta\nu |\sin \nu_s \theta_0|} \left[\sum_{k,\ell,h} \frac{\ell}{Q_y} G \right]^2. \quad (26)$$

Note that the diffusion constant is more or less independent of ν_s . This is because each change in A, B is greater for small ν_s , but the number of crossings per turn is correspondingly smaller for small ν_s . Moreover, the diffusion (amplitude growth) rate increases as $\Delta\nu$ decreases.

To proceed further in the general case, we need all the values of $\mu_{k,\ell,h}$ so that the sums in (25) and (26) can be performed and/or estimated.

3. Single Multipole Term

If we assume a single multipole term and assume that only the $s = k, t = \ell$ term is important, (11) and (12) can be rewritten as

$$AA' = -\frac{s}{Q_x} \mu \frac{A^s B^t}{2^{s+t}} \cos \chi, \quad (27)$$

$$BB' = -\frac{t}{Q_y} \mu A^s B^t \cos \chi, \quad (28)$$

from which we can write the invariant integral of motion

$$\frac{Q_s A^2}{s} - \frac{Q_y B^2}{t} = C^2. \quad (29)$$

Once again we have

$$\Delta A = -\frac{s}{Q_x} \mu \frac{A^{s-1} B^t}{2^{s+t}} \sqrt{\frac{2\pi}{|\chi_0''|}} \cos \left(\chi_0 + \frac{\pi}{4} \right). \quad (30)$$

Assuming crossings with uncorrelated phases, we can write

$$A^2 - A_0^2 = \frac{s^2 \mu^2 A^{2s-2} B^{2t}}{Q_x^2 2^{2s+2t}} \frac{\pi}{|\chi_0''|} \frac{2\theta}{2\pi/\nu_s}. \quad (31)$$

Averaging over crossings leads to

$$\frac{dA^2}{d\theta} = \frac{s^2 \mu^2}{Q_x^2 2^{2s+2t}} \frac{A^{2s-2} B^{2t}}{(s+t) \Delta\nu |\sin \nu_s \theta_0|}. \quad (32)$$

Using (29), we find

$$\frac{dA^2}{d\theta} = \frac{s^2 \mu^2}{Q_x^2 2^{2s+2t}} \frac{A^{2s-2} \left(\frac{t}{Q_y}\right)^t \left[\frac{Q_x A^2}{s} - C^2\right]^t}{(s+t) \Delta \nu |\sin \nu_s \theta_0|}. \quad (33)$$

which leads to a slow increase of A^2 with θ as the result of many crossings in the form

$$\int_{A_0}^A \frac{dA^2}{(A^2)^{s-1} \left(A^2 - \frac{sC^2}{Q_x}\right)^t} = \frac{s^2 \mu^2 \left(\frac{tQ_x}{sQ_y}\right)^t}{Q_x^2 2^{2s+2t} (s+t) \Delta \nu |\sin \nu_s \theta_0|} \theta. \quad (34)$$

For $s+t \geq 3$, the left side remains finite even if $A \rightarrow \infty$, implying more rapid than exponential buildup, including infinite A even for finite θ . Clearly our assumption of a single multipole term is no longer valid, and amplitude dependence of the tunes must be included from terms with $h=0$ and even k and ℓ .

4. Many Resonances in the Same Vicinity

Let us now consider all multipole terms of a given order, such as 10^{th} , where the corresponding terms in the Hamiltonian will be of the form

$$H_{10} = \mu_{10} (x^{10} - 45x^8y^2 + 210x^6y^4 - 210x^4y^6 + 45x^2y^8 - y^{10}) \cos h\theta \quad (35)$$

constructed so as to satisfy Laplace's equation for the magnetic field. In this case, resonances can occur in 10^{th} order if

$$10\nu_x = h$$

$$8\nu_x + 2\nu_y = h$$

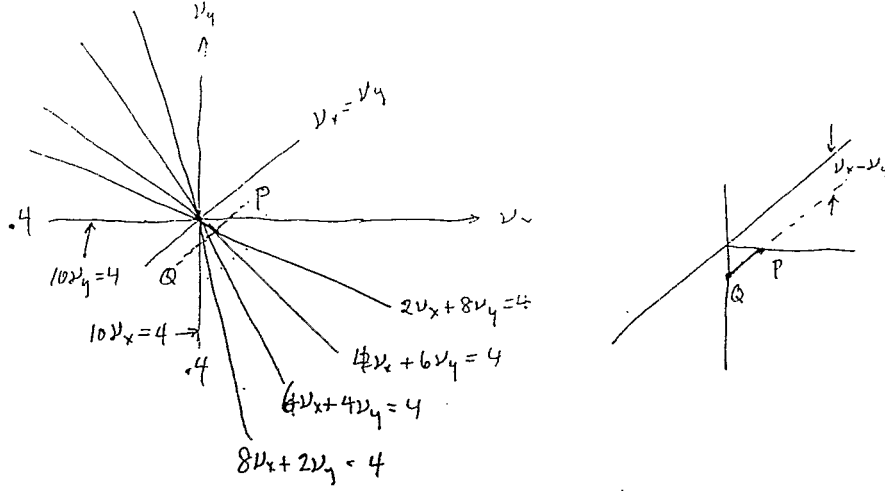
$$6\nu_x + 4\nu_y = h$$

$$4\nu_x + 6\nu_y = h$$

$$2\nu_x + 8\nu_y = h$$

$$10\nu_y = h$$

(and in 8^{th} , 6^{th} , 4^{th} , 2^{th} order as well). For the case $h=4$ (modulus 10) and a tune near .4, .4 (modulus 1), 6 resonant lines will cluster near .4, .4 as shown in the Figure, and even a small excursion of the trajectory in the tune space can cross several resonances. For example



with $QP = 2\Delta\nu \geq \sqrt{2}(\nu_x - \nu_y)$, we may cross 5 or 6 resonances on one half swing. Thus the diffusion constant in (25), (26), (32) may be increased by a factor of order 5 ($\frac{s+t}{2}$) if the phases are uncorrelated. A more precise estimate requires knowing the values of A_0 and B_0 , as well as the overall multipole coefficient.

5. Conclusion

It appears that the diffusion constant due to resonance crossing is 1) independent of ν_s and 2) inversely proportional to $\Delta\nu$. For small $|\nu_x - \nu_y|$ many high order resonances can be crossed with a small $\Delta\nu$, further enhancing the diffusion rate. The seriousness of the growth depends on the specific values of μ , A_0 , B_0 .