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ESTIMATE OF DIFFUSION RATE DUE TO PERIODIC CROSSING OF NON-LINEAR RESONANCES

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The purpose of this calculation is to estimate the emittance growth when crossing a high order resonance.

1. General Analysis for Several Multipoles

We start with the equations

$$x'' + Q_x^2\ x = \sum_{k,\ell,h} \mu_{k,\ell,h}\ k\ x^{k-1}\ y^{\ell} \cos h\theta, \tag{1}$$

$$y'' + Q_y^2\ y = \sum_{k,\ell,h} \mu_{k,\ell,h}\ \ell\ x^{k}\ y^{\ell-1} \cos h\theta, \tag{2}$$

where prime means $\frac{d}{d\theta}$ and where

$$Q_x = \nu_x - \Delta\nu \cos \nu_s\theta, \tag{3}$$

$$Q_y = \nu_y - \Delta\nu \cos \nu_s\theta. \tag{4}$$

We are interested in the single resonance at

$$s\nu_x + t\nu_y = h + \delta, \tag{5}$$

where s, t, h are integers, and $\delta << 1$.

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We use the phase amplitude method and write

$$\begin{aligned} x &= A\cos\psi_x, & y &= B\cos\psi_y, \\ x' &= -Q_x A\sin\psi_x, & y' &= -Q_y B\sin\psi_y, \end{aligned} \tag{6}$$

where

$$\psi_x = \int_0^\theta Q_x d\theta + \alpha\ ,\ \psi_y = \int_0^\theta Q_y d\theta + \beta. \tag{7}$$

It is easy to show that

$$A' = -\sum_{k,\ell,h} \mu_{k,\ell,h} \frac{k}{Q_x} A^{k-1} B^\ell \cos^{k-1}\psi_x \sin\psi_x \cos^\ell\psi_y \cos h\theta, \tag{8}$$

$$B' = -\sum_{k,\ell,h} \mu_{k,\ell,h} \frac{\ell}{Q_y} A^k B^{\ell-1} \cos^k\psi_x \cos^{\ell-1}\psi_y \sin\psi_y \cos h\theta. \tag{9}$$

We now use

$$\cos^k\psi_x = \frac{1}{2^{k-1}} \sum_{\substack{s=0,1 \\ \Delta s=2}} \frac{\cos s\psi_x \quad k!}{\left(\frac{k-s}{2}\right)!\left(\frac{k+s}{2}\right)!} \tag{10}$$

($\times\frac{1}{2}$ if there is a term with $s = 0$ when k is even) and average over all rapidly varying oscillations except for the resonance in (5) to obtain

$$AA' \cong -\sum_{k,\ell,h} \frac{k}{Q_x} \frac{A^k B^\ell}{2^{k+\ell}} \frac{k!}{\left(\frac{k-s}{2}\right)!\left(\frac{k+s}{2}\right)!} \frac{\ell!}{\left(\frac{\ell-t}{2}\right)!\left(\frac{\ell+t}{2}\right)!} \cos\chi, \tag{11}$$

$$BB' = -\sum_{k,\ell,h} \frac{\ell}{Q_y} \frac{A^k B^\ell}{2^{k+\ell}} \frac{k!}{\left(\frac{k-s}{2}\right)!\left(\frac{k+s}{2}\right)!} \frac{\ell!}{\left(\frac{\ell-t}{2}\right)!\left(\frac{\ell+t}{2}\right)!} \cos\chi, \tag{12}$$

where

$$\chi = s\psi_x + t\psi_y - h\theta. \tag{13}$$

Since

$$\psi_x = \alpha + \int_0^\theta Q_x d\theta = \nu_x\theta - \frac{\Delta\nu}{\nu_s}\sin\nu_s\theta + \alpha, \tag{14}$$

$$\psi_y = \beta + \int_0^\theta Q_y d\theta = \nu_y\theta - \frac{\Delta\nu}{\nu_s}\sin\nu_s\theta + \beta, \tag{15}$$

we have

$$\chi(\theta) = (s\nu_x + t\nu_y - h)\theta + s\alpha + t\beta - (s+t)\frac{\Delta\nu}{\nu_s}\sin\nu_s\theta. \tag{16}$$

The resonance is crossed at θ_0, where

$$\chi'(\theta_0) = s\nu_x + t\nu_y - h = (s+t)\Delta\nu\cos\nu_s\theta. \tag{17}$$

The change in A in crossing the resonance is determined by

$$\int_{-\infty}^{\infty} d\theta\cos\chi(\theta) \cong \int_{-\infty}^{\infty} d\theta\cos\left[\chi(\theta_0) + \frac{(\theta-\theta_0)^2}{2}\chi''(\theta_0)\right] = \sqrt{\frac{2\pi}{|\chi''(\theta_0)|}}\cos\left(\chi_0 \pm \frac{\pi}{4}\right), \tag{18}$$

where $\chi_0 \equiv \chi(\theta_0)$ and where the $\pm$ sign has to do with the sign of $\chi''(\theta_0)$. With δ defined in (5) we have

$$|\chi''(\theta_0)| = (s+t)\Delta\nu\ \nu_s|\sin\nu_s\theta_0| = \nu_s\sqrt{(s+t)^2(\Delta\nu)^2 - \delta^2}. \tag{19}$$

Finally, we have from (11), (12), assuming A and B do not change much in the crossing:

$$A\Delta A = -\frac{\sqrt{2\pi}\cos\left(\chi_0 \pm \frac{\pi}{4}\right)}{\sqrt{(s+t)\Delta\nu\ \nu_s|\sin\nu_s\theta_0|}}\sum_{k,\ell,h}\frac{k}{Q_x}G(s,t), \tag{20}$$

$$B\Delta B = -\frac{\sqrt{2\pi}\cos\left(\chi_0 \pm \frac{\pi}{4}\right)}{\sqrt{(s+t)\Delta\nu\ \nu_s|\sin\nu_s\theta_0|}}\sum_{k,\ell,h}\frac{\ell}{Q_y}G(s,t), \tag{21}$$

where

$$G(s,t) = \mu_{k,\ell,h}\frac{A^kB^\ell}{2^{k+\ell}}\frac{k!}{\left(\frac{k-s}{2}\right)!\left(\frac{k+s}{2}\right)!}\frac{\ell!}{\left(\frac{\ell-t}{2}\right)!\left(\frac{\ell+t}{2}\right)!}. \tag{22}$$

2. Diffusion Approximation

If we assume that many betatron oscillations occur between resonance crossings, it is reasonable to assume that the betatron phase at the crossing (χ_0) is uncorrelated from one crossing to the next. In this case we can average A^2 to obtain

$$A^2(\theta) = A_0^2 + \frac{2\pi\times\frac{1}{2}}{(s+t)\Delta\nu\ \nu_s|\sin\nu_s\theta_0|}\left[\sum_{k,\ell,h}\frac{k}{Q_x}G\right]^2\times\frac{2\theta}{\frac{2\pi}{\nu_s}}, \tag{23}$$

where $2\pi/\nu_s$ is the change in θ between pairs of crossings. This leads to the diffusion result (random walk)

$$A^2(\theta) = A_0^2 + K_x\theta,\ B^2(\theta) = B_0^2 + K_y\theta, \tag{24}$$

where

$$K_x = \frac{1}{(s+t)\,\Delta\nu\,|\sin\nu_s\theta_0|}\left[\sum_{k,\ell,h}\frac{k}{Q_x}G\right]^2, \tag{25}$$

$$K_y = \frac{1}{(s+t)\,\Delta\nu\,|\sin\nu_s\theta_0|}\left[\sum_{k,\ell,h}\frac{\ell}{Q_y}G\right]^2. \tag{26}$$

Note that the diffusion constant is more or less independent of ν_s. This is because each change in A, B is greater for small ν_s, but the number of crossings per turn is correspondingly smaller for small ν_s. Moreover, the diffusion (amplitude growth) rate increases as $\Delta\nu$ decreases.

To proceed further in the general case, we need all the values of $\mu_{k,\ell,h}$ so that the sums in (25) and (26) can be performed and/or estimated.

3. Single Multipole Term

If we assume a single multipole term and assume that only the $s = k$, $t = \ell$ term is important, (11) and (12) can be rewritten as

$$AA' = -\frac{s}{Q_x}\mu\frac{A^sB^t}{2^{s+t}}\cos\chi, \tag{27}$$

$$BB' = -\frac{t}{Q_y}\mu A^sB^t\cos\chi, \tag{28}$$

from which we can write the invariant integral of motion

$$\frac{Q_sA^2}{s} - \frac{Q_yB^2}{t} = C^2. \tag{29}$$

Once again we have

$$\Delta A = -\frac{s}{Q_x}\mu\frac{A^{s-1}B^t}{2^{s+t}}\sqrt{\frac{2\pi}{|\chi_0''|}}\cos\left(\chi_0+\frac{\pi}{4}\right). \tag{30}$$

Assuming crossings with uncorrelated phases, we can write

$$A^2 - A_0^2 = \frac{s^2\mu^2A^{2s-2}B^{2t}}{Q_x^2 2^{2s+2t}}\frac{\pi}{|\chi_0''|}\frac{2\theta}{2\pi/\nu_s}. \tag{31}$$

Averaging over crossings leads to

$$\frac{dA^2}{d\theta} = \frac{s^2\mu^2}{Q_x^2 2^{2s+2t}}\frac{A^{2s-2}B^{2t}}{(s+t)\,\Delta\nu\,|\sin\nu_s\theta_0|}. \tag{32}$$

Using (29), we find

$$\frac{dA^2}{d\theta} = \frac{s^2\mu^2}{Q_x^2 2^{2s+2t}} \frac{A^{2s-2}\left(\frac{t}{Q_y}\right)^t \left[\frac{Q_x A^2}{s} - C^2\right]^t}{(s+t)\,\Delta\nu\,|\sin\nu_s\theta_0|}. \tag{33}$$

which leads to a slow increase of A^2 with θ as the result of many crossings in the form

$$\int_{A_0}^{A} \frac{dA^2}{(A^2)^{s-1}\left(A^2 - \frac{sC^2}{Q_x}\right)^t} = \frac{s^2\mu^2\left(\frac{tQ_x}{sQ_y}\right)^t}{Q_x^2 2^{2s+2t}(s+t)\,\Delta\nu\,|\sin\nu_s\theta_0|}\theta. \tag{34}$$

For $s+t \geq 3$, the left side remains finite even if $A \to \infty$, implying more rapid than exponential buildup, including infinite A even for finite θ. Clearly our assumption of a single multipole term is no longer valid, and amplitude dependence of the tunes must be included from terms with $h = 0$ and even k and ℓ.

4. Many Resonances in the Same Vicinity

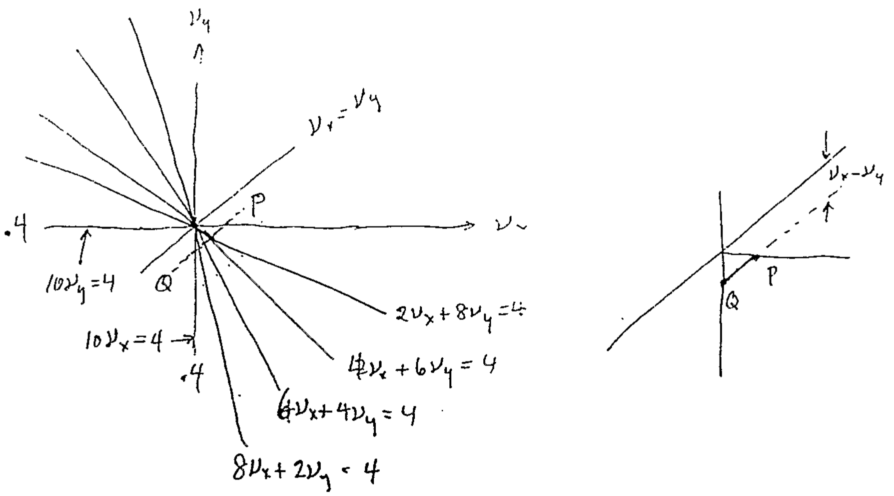
Let us now consider all multipole terms of a given order, such as 10^{th}, where the corresponding terms in the Hamiltonian will be of the form

$$H_{10} = \mu_{10}\left(x^{10} - 45x^8y^2 + 210x^6y^4 - 210x^4y^6 + 45x^2y^8 - y^{10}\right)\cos h\theta \tag{35}$$

constructed so as to satisfy Laplace's equation for the magnetic field. In this case, resonances can occur in 10^{th} order if

$$\begin{aligned} 10\nu_x &= h \\ 8\nu_x + 2\nu_y &= h \\ 6\nu_x + 4\nu_y &= h \\ 4\nu_x + 6\nu_y &= h \\ 2\nu_x + 8\nu_y &= h \\ 10\nu_y &= h \end{aligned}$$

(and in 8^{th}, 6^{th}, 4^{th}, 2^{th} order as well). For the case $h = 4$ (modulus 10) and a tune near .4, .4 (modulus 1), 6 resonant lines will cluster near .4, .4 as shown in the Figure, and even a small excursion of the trajectory in the tune space can cross several resonances. For example



with $QP = 2\Delta\nu \geq \sqrt{2}(\nu_x - \nu_y)$, we may cross 5 or 6 resonances on one half swing. Thus the diffusion constant in (25), (26), (32) may be increased by a factor of order 5 $\left(\frac{s+t}{2}\right)$ if the phases are uncorrelated. A more precise estimate requires knowing the values of A_0 and B_0, as well as the overall multipole coefficient.

5. Conclusion

It appears that the diffusion constant due to resonance crossing is 1) independent of ν_s and 2) inversely proportional to $\Delta\nu$. For small $|\nu_x - \nu_y|$ many high order resonances can be crossed with a small $\Delta\nu$, further enhancing the diffusion rate. The seriousness of the growth depends on the specific values of μ, A_0, B_0.