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Particle Losses Due To Diffusion Processes In Presence of an Aperture Limitation

A. G. Ruggiero

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Collider Accelerator Department

Brookhaven National Laboratory

U.S. Department of Energy

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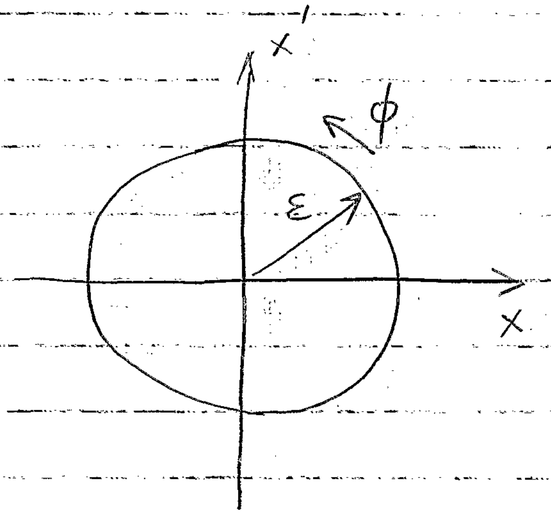
Particle Distribution Function

$$f = f(x, x', t)$$

$N(t)$, number of particles

$$N(t) = \iint f(x, x', t)\, dx\, dx'$$

Transform the variables from x, x' to ε, ϕ



$$f = f(\varepsilon, \phi, t)$$

ε is the amplitude-emittance

What is the transport equation that f satisfies?

Assume there is no diffusion. The motion preserves energy. Liouville theorem applies

total time derivative $\equiv \frac{df}{ds} = 0$

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + x'' \frac{\partial f}{\partial x'} = 0$$

or

$$\frac{\partial f}{\partial s} + \phi' \frac{\partial f}{\partial \phi} + \varepsilon' \frac{\partial f}{\partial \varepsilon} = 0$$

In a storage ring the time $\underline{t}$ is replaced by the longitudinal coordinate s.

ε is an invariant of motion, $\varepsilon' = 0$ then

$$\frac{\partial f}{\partial s} + \nu \frac{\partial f}{\partial \phi} = 0$$

where $\phi' = \nu$ is about the betatron tune

We can assume that

$$\frac{\partial f}{\partial \phi} = 0$$

and

$$f = f(\varepsilon, t)$$

For a stationary distribution $\partial f / \partial t = 0$ and

$$f = f(\varepsilon) \quad \text{of } \varepsilon \text{ alone.}$$

In conclusion

$$\boxed{\frac{\partial f}{\partial t} = 0}$$

If one introduce diffusion D the equation changes to

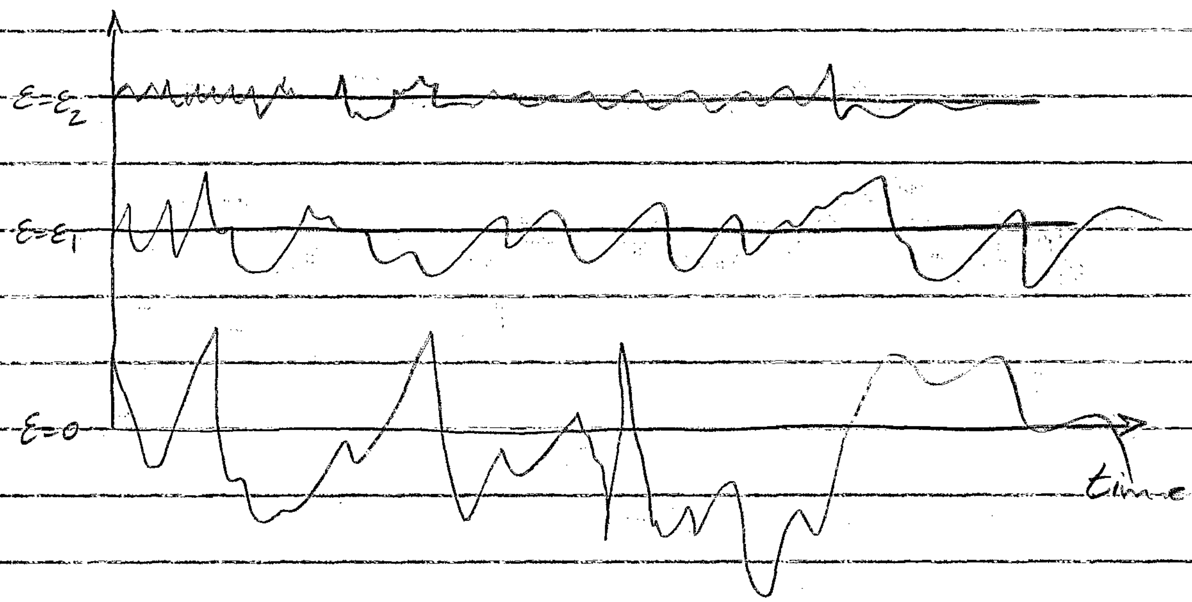
$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial}{\partial \varepsilon}\left(\varepsilon D \frac{\partial f}{\partial \varepsilon}\right)}$$

Fokker-Planck equation

D is the average rms increase of the emittance per unit of time

$$D = \frac{d\langle\varepsilon\rangle}{dt}$$

It is a strong function of the amplitude-emittance ε



We will assume here

$$D = D(t)$$

Let us change variables

$$\tau = \frac{1}{\varepsilon_a} \int_0^t D(t')\,dt'$$

$$y = \frac{\varepsilon}{\varepsilon_a}$$

where ε_a is an aperture limit

Then the equation is

$$\boxed{\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial y}\left(y \frac{\partial f}{\partial y} \right)}$$

This is a partial differential equation. To solve it one requires knowledge of two boundary conditions

1.) $f(y, \tau = 0) = f_0(y)$ initial distribution

in particular $f_0(y=1) = 0$

2.) $f(y=1, \tau) = 0$ that is there are no particles at the wall (sink) at any time τ

Let us solve the equation

$$f = \sum_n c_n f_n(y)\, e^{-\beta_n \tau}$$

By insertion one can find

$$f_n(y) = J_0(\lambda_n \sqrt{y})$$

and

$$\beta_n = \frac{\lambda_n^2}{4}$$

λ_n are the zeros of the Bessel function $J_0(z)$.
Boundary condition 2) is automatically satisfied since

$$J_0(\lambda_n) = 0$$

The coefficient c_n are to be calculated from the boundary condition 1).

$$\sum_n c_n J_0(\lambda_n \sqrt{y}) = f_0(y)$$

or by inversion

$$c_n = \frac{1}{J_1^2(\lambda_n)} \int_0^1 f_0(y) J_0(\lambda_n \sqrt{y})\, dy$$

We are interested in the number of particles $N(\tau)$ surviving after time τ

$$\underline{N(\tau)} = \int_0^1 f(y,\tau)\, dy$$

$$= 2 \sum_n \frac{e^{-\lambda_n^2 \tau/4}}{\lambda_n J_1(\lambda_n)} \int_0^1 f_0(y) J_0(\lambda_n \sqrt{y})\, dy$$

Observe that at no time a gaussian distribution is possible

The results for some interesting cases:

(1) $$f_0(y) = \delta(y)$$

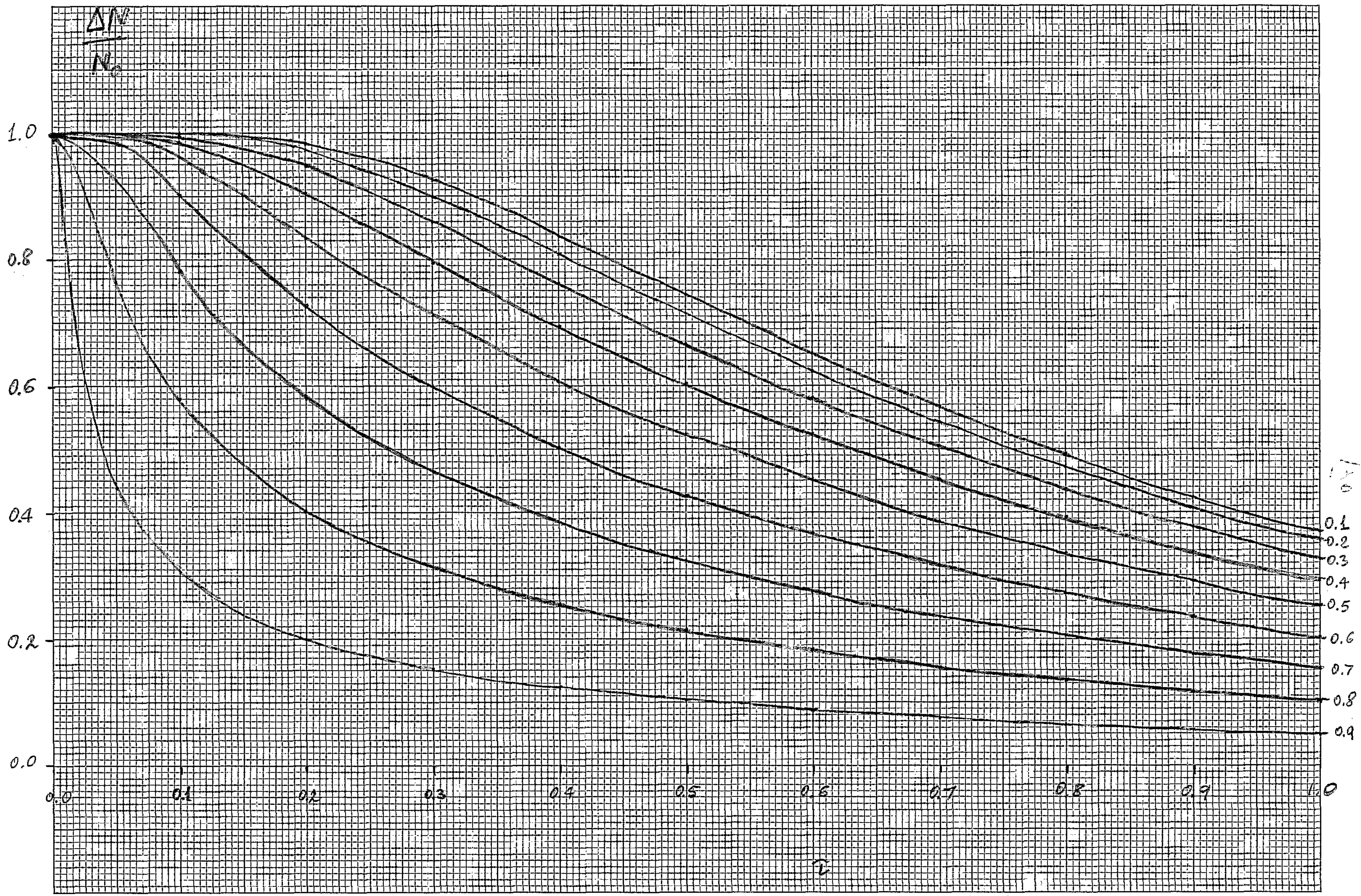
$$N(\tau) = 2 \sum_n e^{-\lambda_n^2 \tau/4} \Big/ \lambda_n J_1(\lambda_n)$$

(2) $$f_0(y) = 1 \qquad 0 \le y < 1$$

$$N(\tau) = 4 \sum_n e^{\lambda_n^2 \tau/4} \Big/ \lambda_n^2$$

(3) $$f_0(y) = \lambda_1 J_0(\lambda_1 \sqrt{y}) \Big/ 2 J_1(\lambda_1)$$

$$N(\tau) = e^{-\lambda_1^2 \tau/4}$$



ΔN / N₀
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

The good field aperture is

± 25.2 mm

We take $\beta_{max} = 51.6$ m
$\eta_{max} = 1.385$ m

For $\Delta p/p = 0$ $\varepsilon_a = 12.3\ \pi$ mm-mrad

For $\Delta p/p = 0.5\%$ $\varepsilon_a = 6.5\ \pi$ mm mrad

The initial normalized emittance is

$10\ \pi$ mm mrad for (95% of beam)

Even at $\gamma = 10$

$\varepsilon_{init} = 1\pi$ mm. mrad (")

So

$\varepsilon_{init} \ll \varepsilon_a$

We can take for the initial distribution a delta-function.

Observe that

$$\tau = \frac{1}{\varepsilon_a}\int_0^t D(t')\,dt' = \frac{1}{\varepsilon_a}\int_0^t \frac{\langle\Delta\varepsilon\rangle}{dt}\,dt$$

$$= \frac{\Delta\varepsilon(t)}{\varepsilon_a}$$

where $\Delta\varepsilon(t)$ is the rms emittance increase after time t.

The worst case is at $\gamma = 12$ where (G.P.)

$$\Delta\varepsilon(t = 2\,\text{hours}) = \frac{34.5 - 10}{6 \times 12} = 0.34\ \pi\ \text{mm mrad}$$

So τ is very small anyway. There are very negligible losses (see also next Table) certainly less than 0.1%.

$\tau = \dfrac{\Delta\varepsilon(t)}{\varepsilon_a}$	$N(\tau)$		
	$y_0 = 0$	$y_0 = 0.2$	$y_0 = 0.4$
0.02	0.99987	0.99958	0.99879
0.04	0.99972	0.99945	0.99866
0.06	0.99959	0.99932	0.99776
0.08	0.99945	0.99906	0.99420
0.10	0.99924	0.99829	0.98675

Diffusion in the Momentum Plane

Let $f = f(p, t)$ be the distribution function

The F.P. equation now is

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial p}\left(D_p \frac{\partial f}{\partial p}\right)$$

where now $D_p = \dfrac{d\langle p^2\rangle}{dt}$

we can assume again $D_p = D_p(t)$

Introduce p_a as aperture limit and

$$z = \frac{p}{p_a} \qquad \text{and} \qquad \tau = \frac{1}{p_a^2}\int_0^t D_p(t')\,dt'$$

$$= \left[\frac{\Delta p(t)}{p_a}\right]^2$$

where $\Delta p(t)$ is the rms increase in momentum deviation after time t.

Then we have

$$f = f(z, \tau)$$

and

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial z^2}$$

which has solution

$$f = \sum_{n=0}^{\infty} c_n (\cos \beta_n z)\, e^{-\beta_n^2 \tau}$$

having assumed that $f(-z) = f(z)$

There are again two boundary conditions

1.) $f(1, \tau) = 0$ ("sink")

this is satisfied by letting

$$\beta_n = \frac{\pi}{2}(2n+1)$$

2.) $f(z, 0) = f_0(z)$ initial distribution

This condition requires

$$\sum_{n=0}^{\infty} c_n \cos(\beta_n z) = f_0(z)$$

that is by inversion

$$c_n = 2 \int_0^1 f_0(z) \cos \beta_n z \, dz$$

For $f_0(z) = \delta(z) \implies \underline{\underline{c_n = 2}}$

For the number of particles survived at the time τ

$$N(\tau) = \sum_{n=0}^{\infty} c_n e^{-\beta_n^2 \tau} \int_0^1 \cos \beta_n z \, dz$$

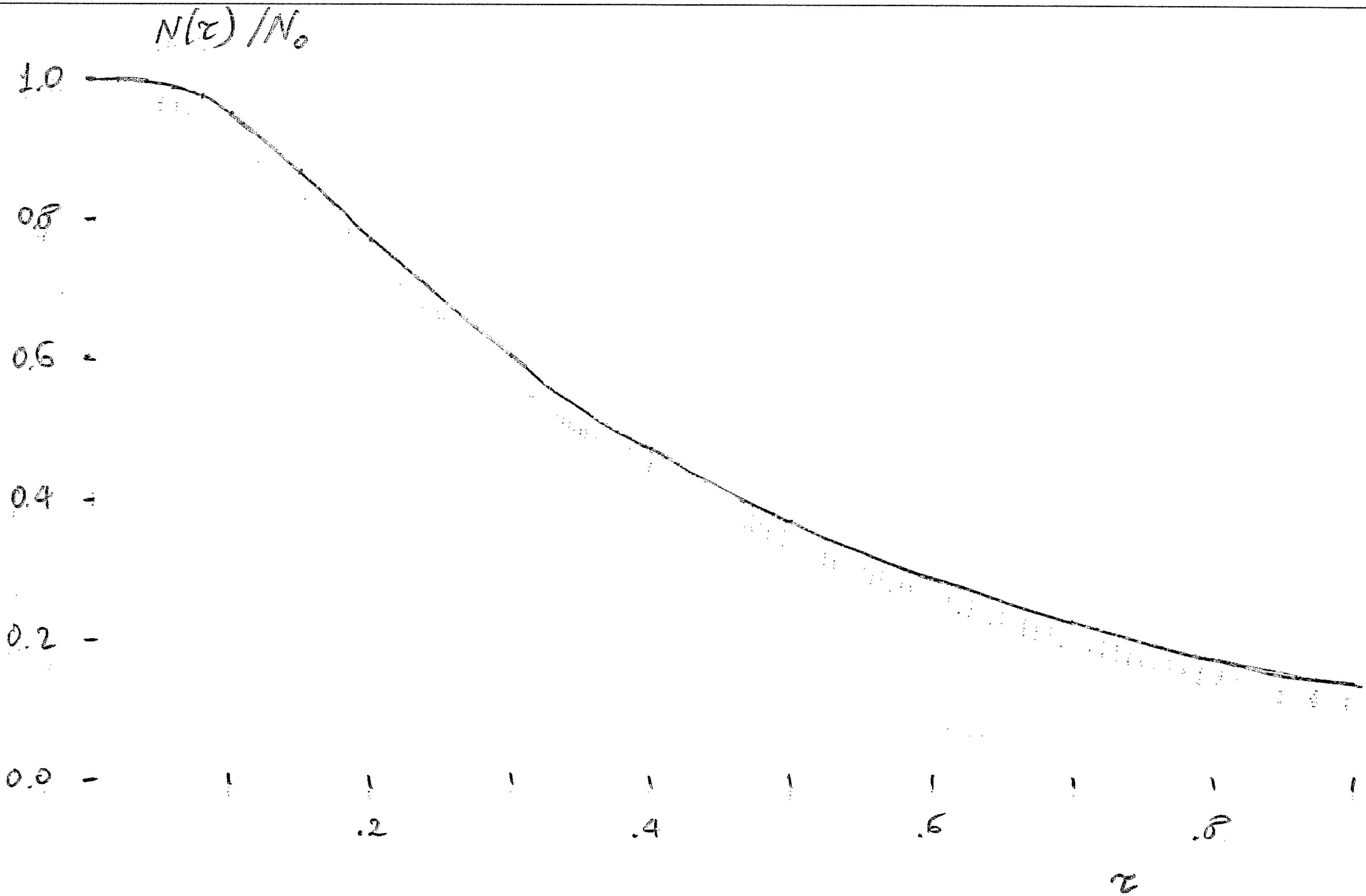
$$= \sum_{n=0}^{\infty} c_n (-1)^n \frac{e^{-\beta_n^2 \tau}}{\beta_n}$$

For $f_0(z) = \delta(z)$

$$N(\tau) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta_n} e^{-\beta_n^2 \tau}$$

Results for $f_0(z) = \delta(z)$

τ	$N(\tau)$
0.02	0.999893
0.04	.999079
0.06	.992102
0.08	.975039
0.10	.949173
0.12	.917406
0.14	.882289
0.16	.845648
0.18	.808683
0.20	.772152


N(τ) /N₀
1.0
0.8
0.6
0.4
0.2
0.0
.2
.4
.6
.8
τ

From G. Parzen Calculations ($V = 1 MV$)
$t = 2$ hours

γ	12	20	30	50	75	100
P_a	3.49	6.13	9.08	3.93	2.91	2.45
Δ_p	1.22	1.37	1.57	1.14	.921	.789
τ	0.122	0.050	0.030	0.084	0.100	0.104
$N(\tau)$	0.91	0.99676	0.9998		0.949	0.94

For $\tau = \left(\frac{\Delta p}{P_a}\right)^2 = \left(\frac{1}{2.5}\right)^2 = 0.16$

$N(\tau) = 0.84565$