

Particle Losses Due To Diffusion Processes In Presence of an Aperture Limitation

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PARTICLE LOSSES DUE TO
DIFFUSION PROCESSES
IN PRESENCE OF AN APERTURE LIMITATION.

A. G. Ruggiero

(BNL, April 12, 1984)

Particle Distribution Function

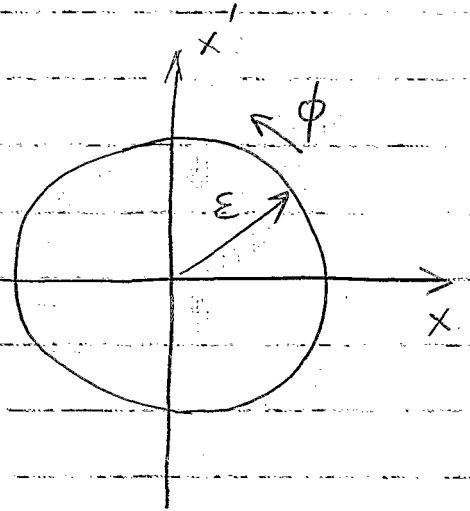
$$f = f(x, x', t)$$

$N(t)$, number of particles

$$N(t) = \iint f(x, x', t) dx dx'$$

Transform the variables
from x, x' to ϵ, ϕ

$$f = f(\epsilon, \phi, t)$$



ϵ is the amplitude-emittance

What is the transport equation that f satisfies?

Assume there is no dissipation. The motion preserves energy. Lagrange's theorem applies.

$$\text{total time derivative} = \frac{df}{ds} = 0$$

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + \dot{x} \frac{\partial f}{\partial x} + \ddot{x} \frac{\partial f}{\partial \dot{x}} = 0$$

or

$$\frac{\partial f}{\partial s} + \dot{\phi} \frac{\partial f}{\partial \phi} + \dot{\epsilon} \frac{\partial f}{\partial \epsilon} = 0$$

In a storage ring the time t is replaced by the longitudinal coordinate s .

ϵ is an invariant of motion, $\dot{\epsilon} = 0$
then

$$\frac{\partial f}{\partial s} + \dot{\phi} \frac{\partial f}{\partial \phi} = 0$$

where $\dot{\phi} = \omega$ is about the Larmor frequency

We can assume that

$$\frac{\partial f}{\partial \phi} = 0$$

and

$$f = f(\epsilon, t)$$

For a stationary distribution $\frac{\partial f}{\partial t} = 0$ and

$$f = f(\epsilon) \quad \text{of } \epsilon \text{ alone}$$

In conclusion

$$\frac{\partial f}{\partial t} = 0$$

If one introduces diffusion D the equation changes to

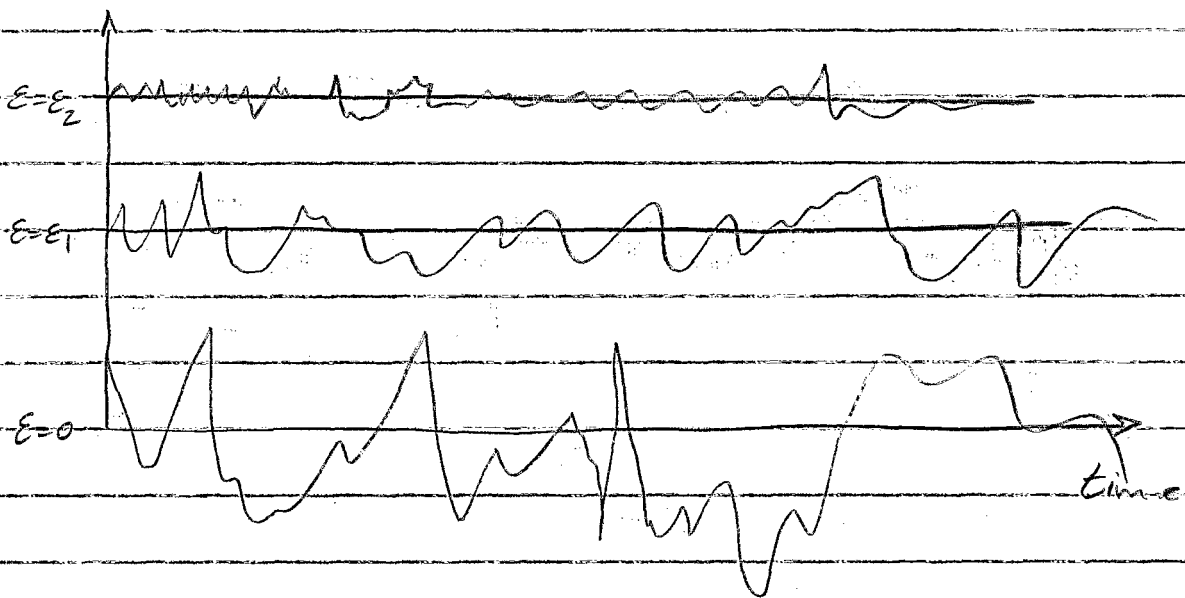
$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \epsilon} \left(\epsilon D \frac{\partial f}{\partial \epsilon} \right)$$

Fokker-Planck equation

D is the average rms increase of the emittance per unit of time

$$D = \frac{d\langle \epsilon \rangle}{dt}$$

It is a strong function of the amplitude-emittance ϵ



We will assume here

$$D = D(\epsilon)$$

let us change variables

$$\tau = \frac{1}{\epsilon_a} \int_0^t D(t') dt'$$

$$y = \frac{\epsilon}{\epsilon_a}$$

where ϵ_a is an aperture limit

The He equation is

$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial y} \left(y \frac{\partial f}{\partial y} \right)$$

This is a partial differential equation. To solve it we require knowledge of two boundary conditions

1) $f(y, \tau=0) = f_0(y)$ initial distribution

in particular $f_0(y=1) = 0$

2.) $f(y=1, \tau) = 0$ that is there are no particles at the wall (sink) at any time τ

Let us solve the equation

$$f = \sum_n c_n f_n(y) e^{-\beta_n z}$$

By insertion one can find

$$f_n(y) = J_0(\lambda_n \sqrt{y})$$

and

$$\beta_n = \frac{\lambda_n^2}{4}$$

λ_n are the zeros of the Bessel function $J_0(z)$.

Boundary condition 2) is automatically satisfied since

$$J_0(\lambda_n) = 0$$

The coefficient c_n are to be calculated from the boundary condition 1).

$$\sum_n c_n J_0(\lambda_n \sqrt{y}) = f_0(y)$$

or by inversion

$$c_n = \frac{1}{J_1^2(\lambda_n)} \int_0^1 f_0(y) J_0(\lambda_n \sqrt{y}) dy$$

We are interested in the number of particles $N(\tau)$ surviving after time τ

$$N(\tau) = \int_0^1 f(y, \tau) dy$$

$$= 2 \sum_n \frac{e^{-\lambda_n^2 \tau / 4}}{\lambda_n J_1(\lambda_n)} \int_0^1 f_0(y) J_0(\lambda_n \sqrt{y}) dy$$

Observe that at no time a gaussian distribution is possible

The results for some interesting cases:

(1) $f_0(y) = \delta(y)$

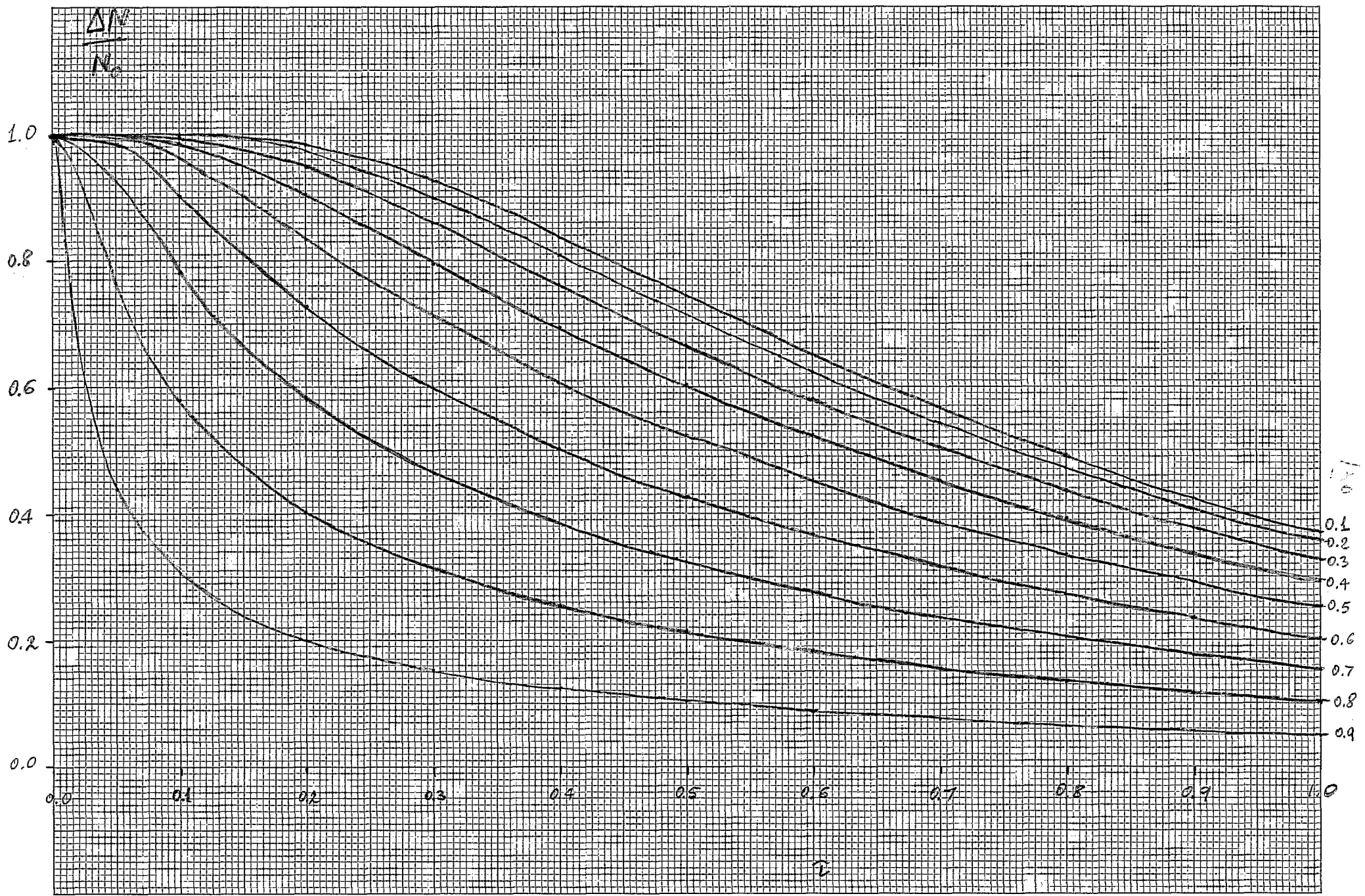
$$N(\tau) = 2 \sum_n e^{-\lambda_n^2 \tau / 4} / \lambda_n J_1(\lambda_n)$$

(2) $f_0(y) = 1 \quad 0 \leq y < 1$

$$N(\tau) = 4 \sum_n e^{-\lambda_n^2 \tau / 4} / \lambda_n^2$$

(3) $f_0(y) = \lambda_1 J_0(\lambda_1 \sqrt{y}) / 2 J_1(\lambda_1)$

$$N(\tau) = e^{-\lambda_1^2 \tau / 4}$$



1.0

0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

1.0

9

The good field aperture is

$$\pm 25.2 \text{ mm}$$

We take $\beta_{\max} = 51.6 \text{ m}$

$$\eta_{\max} = 1.385 \text{ m}$$

For $\Delta p/p = 0$ $\epsilon_a = 12.3 \pi \text{ mm-mrad}$

For $\Delta p/p = 0.5\%$ $\epsilon_a = 6.5 \pi \text{ mm-mrad}$

The initial normalised emittance is

$$10 \pi \text{ mm-mrad} \text{ for } (95\% \text{ of beam})$$

Even at $y = 10$

$$\epsilon_{\text{init}} = 1 \pi \text{ mm-mrad} \text{ (")}$$

So

$$\epsilon_{\text{init}} \ll \epsilon_a$$

We can take for the initial distribution a delta-function

Observe that

$$\tau = \frac{1}{\epsilon_a} \int_0^t D(t') dt' = \frac{1}{\epsilon_a} \int_0^t \frac{\langle \Delta \epsilon \rangle}{dt} dt$$

$$= \frac{\Delta \epsilon(t)}{\epsilon_a}$$

where $\Delta \epsilon(t)$ is the rms emittance increase after time t

The worst case is at $y=12$ where (G.P.)

$$\Delta \epsilon(t=2 \text{ hours}) = \frac{34.5 - 10}{6 \times 12} = 0.34 \pi \text{ mm mrad}$$

So τ is very small anyway. There are very negligible losses (see also next Table) certainly less than 0.1%.

$N(\tau)$

$$\tau = \frac{\Delta E(t)}{\epsilon_a}$$

$$y_0 = 0$$

$$y_0 = 0.2$$

$$y_0 = 0.4$$

0.02

0.99987

0.99958

0.99879

0.04

0.99972

0.99945

0.99866

0.06

0.99959

0.99932

0.99776

0.08

0.99945

0.99906

0.99420

0.10

0.99924

0.99829

0.98675

Diffusion in the Momentum Plane

Let $f = f(p, t)$ be the distribution function

The F.P. equation now is

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial p} \left(D_p \frac{\partial f}{\partial p} \right)$$

where now $D_p = \frac{d\langle p^2 \rangle}{dt}$

we can assume again $D_p = D_p(t)$

Introduce p_a as aperture limit and

$$z = \frac{p}{p_0} \quad \text{and} \quad x = \frac{1}{p_a^2} \int_0^t D_p(t') dt'$$

$$= \left[\frac{\Delta p(t)}{p_a} \right]^2$$

where $\Delta p(t)$ is the rms increase in momentum deviation after time t .

Then we have

$$f = f(z, \tau)$$

and

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial z^2}$$

which has solution

$$f = \sum_{n=0}^{\infty} c_n (\cos \beta_n z) e^{-\beta_n^2 \tau}$$

having assumed that $f(-z) = f(z)$

There are again two boundary conditions

$$1.) \quad f(1, \tau) = 0 \quad (\text{"sink"})$$

this is satisfied by letting

$$\beta_n = \frac{\pi}{2} (2n+1)$$

$$2.) \quad f(z, 0) = f_0(z) \quad \text{initial distribution}$$

This condition requires

$$\sum_{n=0}^{\infty} c_n \cos(\beta_n z) = f_0(z)$$

that is by inversion

$$c_n = 2 \int_0^1 f_0(z) \cos \beta_n z \, dz$$

For $f_0(z) = \delta(z) \Rightarrow \underline{c_n = 2}$

For the number of particles survived at the time τ

$$N(\tau) = \sum_{n=0}^{\infty} c_n e^{-\beta_n^2 \tau} \int_0^1 \cos \beta_n z \, dz$$

$$= \sum_{n=0}^{\infty} c_n (-1)^n \frac{e^{-\beta_n^2 \tau}}{\beta_n}$$

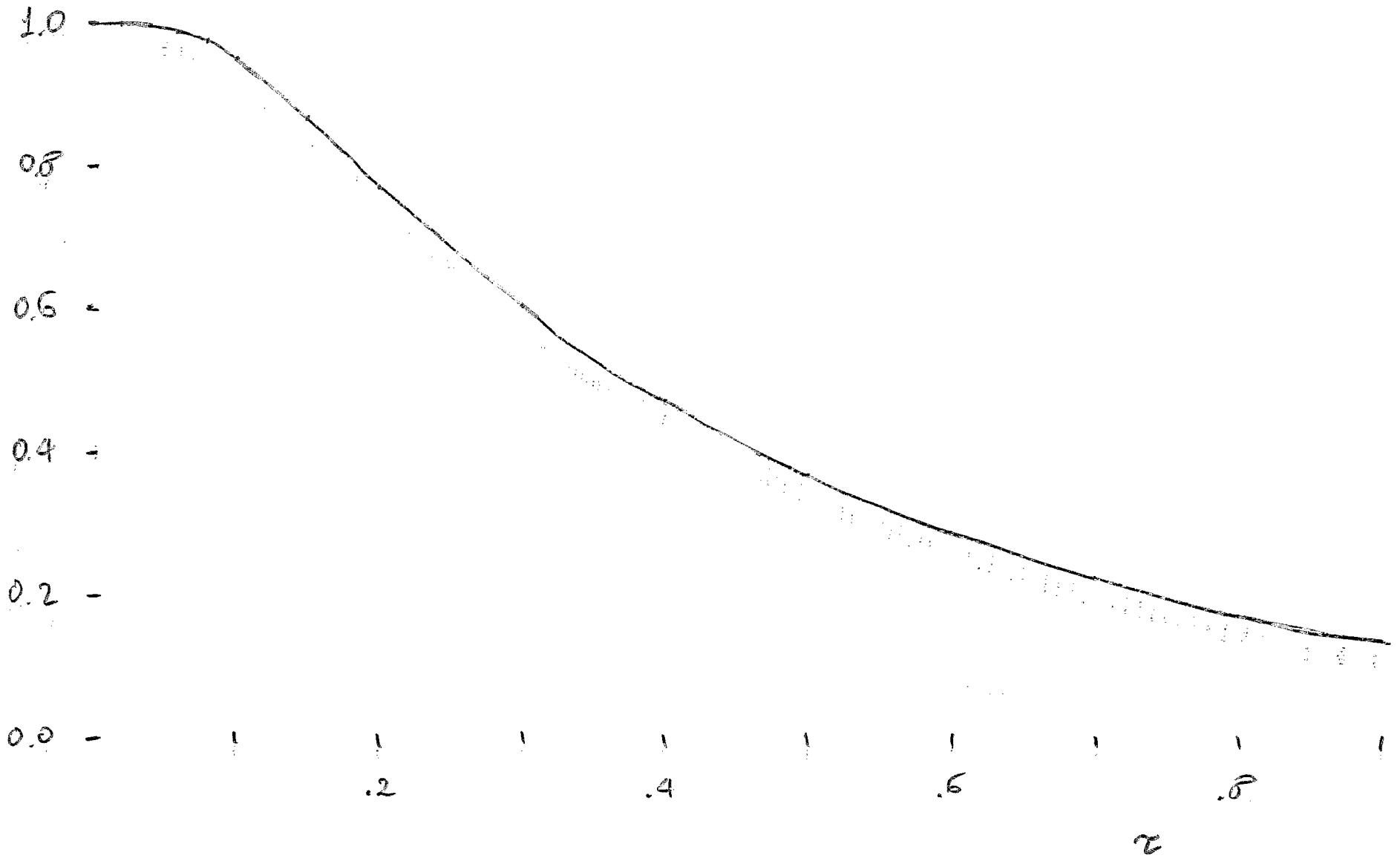
For $f_0(z) = \delta(z)$

$$N(\tau) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta_n} e^{-\beta_n^2 \tau}$$

Results for $f_0(z) = \delta(z)$

<u>z</u>	<u>N(z)</u>
0.02	0.999893
0.04	.999079
0.06	.992102
0.08	.975039
0.10	.949173
0.12	.917406
0.14	.882289
0.16	.845648
0.18	.808683
0.20	.772153

$N(\tau) / N_0$



From G. Parson Calculations (V=1MV)
t=2 hours

δ	12	20	30	50	75	100
P_a	3.49	6.13	9.08	3.93	2.91	2.45
Δp	1.22	1.37	1.57	1.14	.921	.759
τ	0.122	0.050	0.030	0.084	0.100	0.104
$N(\tau)$	0.91	0.99676	0.9998		0.949	0.94

$$F_{0.5} \quad \tau = \left(\frac{\Delta p}{P_a} \right)^2 = \left(\frac{1}{2.5} \right)^2 = 0.16$$

$$N(\tau) = 0.84565$$