

# Particle Losses Due To Diffusion Processes In Presence of an Aperture Limitation

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PARTICLE LOSSES DUE TO  
DIFFUSION PROCESSES  
IN PRESENCE OF AN APERTURE LIMITATION

A. G. Ruggiero

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## Particle Distribution Function

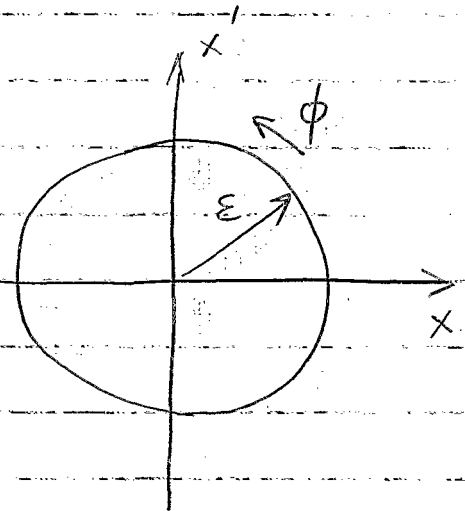
$$f = f(x, x', t)$$

$N(t)$ , number of particles

$$N(t) = \iint f(x, x', t) dx dx'$$

Transform the variables  
from  $x, x'$  to  $\epsilon, \phi$

$$f = f(\epsilon, \phi, t)$$



$\epsilon$  is the amplitude-emittance

What is the transport equation that  $f$  satisfies?

Assume there is no dissipation. The motion preserves energy. Liouville's theorem applies.

$$\text{total time derivative} = \frac{df}{ds} = 0$$

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + x'' \frac{\partial f}{\partial x'} = 0$$

or

$$\frac{\partial f}{\partial s} + \phi' \frac{\partial f}{\partial \phi} + \varepsilon' \frac{\partial f}{\partial \varepsilon} = 0$$

In a storage ring the time  $t$  is replaced by the longitudinal coordinate  $s$ .

$\varepsilon$  is an invariant of motion,  $\varepsilon' = 0$ .  
then

$$\frac{\partial f}{\partial s} + \nu \frac{\partial f}{\partial \phi} = 0$$

where  $\phi' = \nu$  is about the betatron tune

We can assume that

$$\frac{\partial f}{\partial \phi} = 0$$

and

$$f = f(\epsilon, t)$$

For a stationary distribution  $\partial f / \partial t = 0$  and

$$f = f(\epsilon) \quad \text{of } \epsilon \text{ alone}$$

In conclusion

$$\frac{\partial f}{\partial t} = 0$$

If one introduces diffusion  $D$  the equation changes to

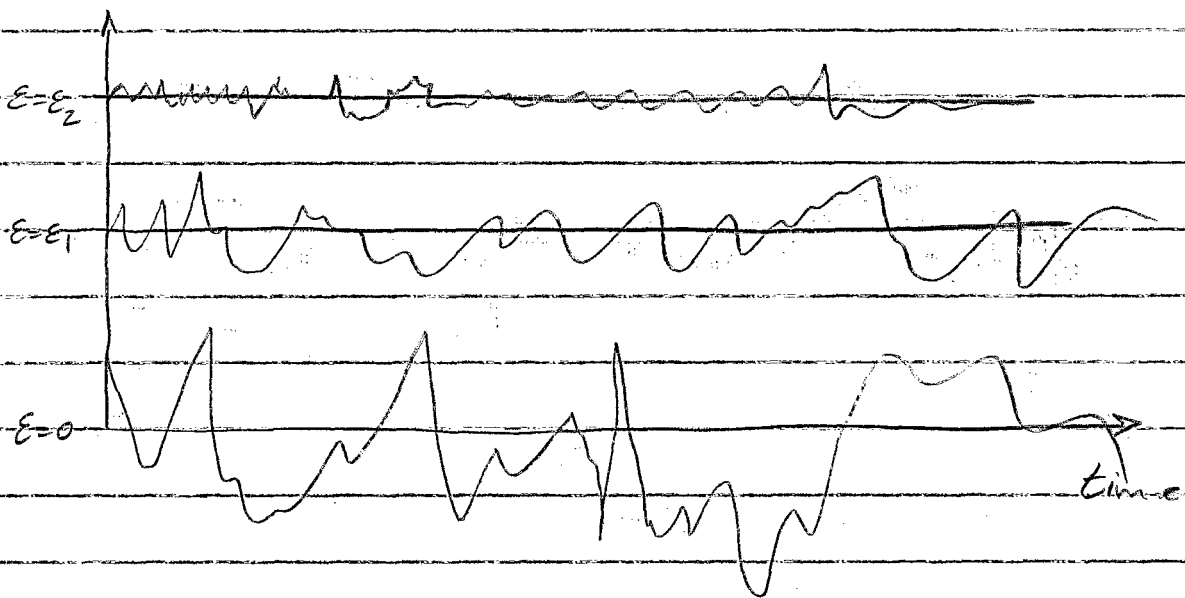
$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \epsilon} \left( \epsilon D \frac{\partial f}{\partial \epsilon} \right)$$

Fokker-Planck equation

$D$  is the average rms increase of the emittance per unit of time

$$D = \frac{d\langle \epsilon \rangle}{dt}$$

It is a strong function of the amplitude-emittance  $\epsilon$



We will assume here

$$D = D(t)$$

let us change variables

$$\tau = \frac{1}{\varepsilon_a} \int_0^t D(t') dt'$$

$$y = \frac{\varepsilon}{\varepsilon_a}$$

where  $\varepsilon_a$  is an aperture limit

Then the equation is

$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial y} \left( y \frac{\partial f}{\partial y} \right)$$

This is a partial differential equation. To solve it one requires knowledge of two boundary conditions.

1)  $f(y, \tau=0) = f_0(y)$  initial distribution

in particular  $f_0(y=1) = 0$

2)  $f(y=1, \tau) = 0$  that is there are no particles at the wall (sink) at any time  $\tau$



Let us solve the equation

$$f = \sum_n c_n f_n(y) e^{-\beta_n z}$$

By insertion one can find

$$f_n(y) = J_0(\lambda_n \sqrt{y})$$

and

$$\beta_n = \frac{\lambda_n^2}{4}$$

$\lambda_n$  are the zeros of the Bessel function  $J_0(z)$ .

Boundary condition 2) is automatically satisfied since

$$J_0(\lambda_n) = 0$$

The coefficient  $c_n$  are to be calculated from the boundary condition 1).

$$\sum_n c_n J_0(\lambda_n \sqrt{y}) = f_0(y)$$

or by inversion

$$c_n = \frac{1}{J_1^2(\lambda_n)} \int_0^1 f_0(y) J_0(\lambda_n \sqrt{y}) dy$$

We are interested in the number of particles  $N(\tau)$  surviving after time  $\tau$

$$N(\tau) = \int_0^1 f(y, \tau) dy$$

$$= 2 \sum_n \frac{e^{-\lambda_n^2 \tau / 4}}{\lambda_n J_1(\lambda_n)} \int_0^1 f_0(y) J_0(\lambda_n \sqrt{y}) dy$$

Observe that at no time a gaussian distribution is possible

The results for some interesting cases:

$$(1) \quad f_0(y) = \delta(y)$$

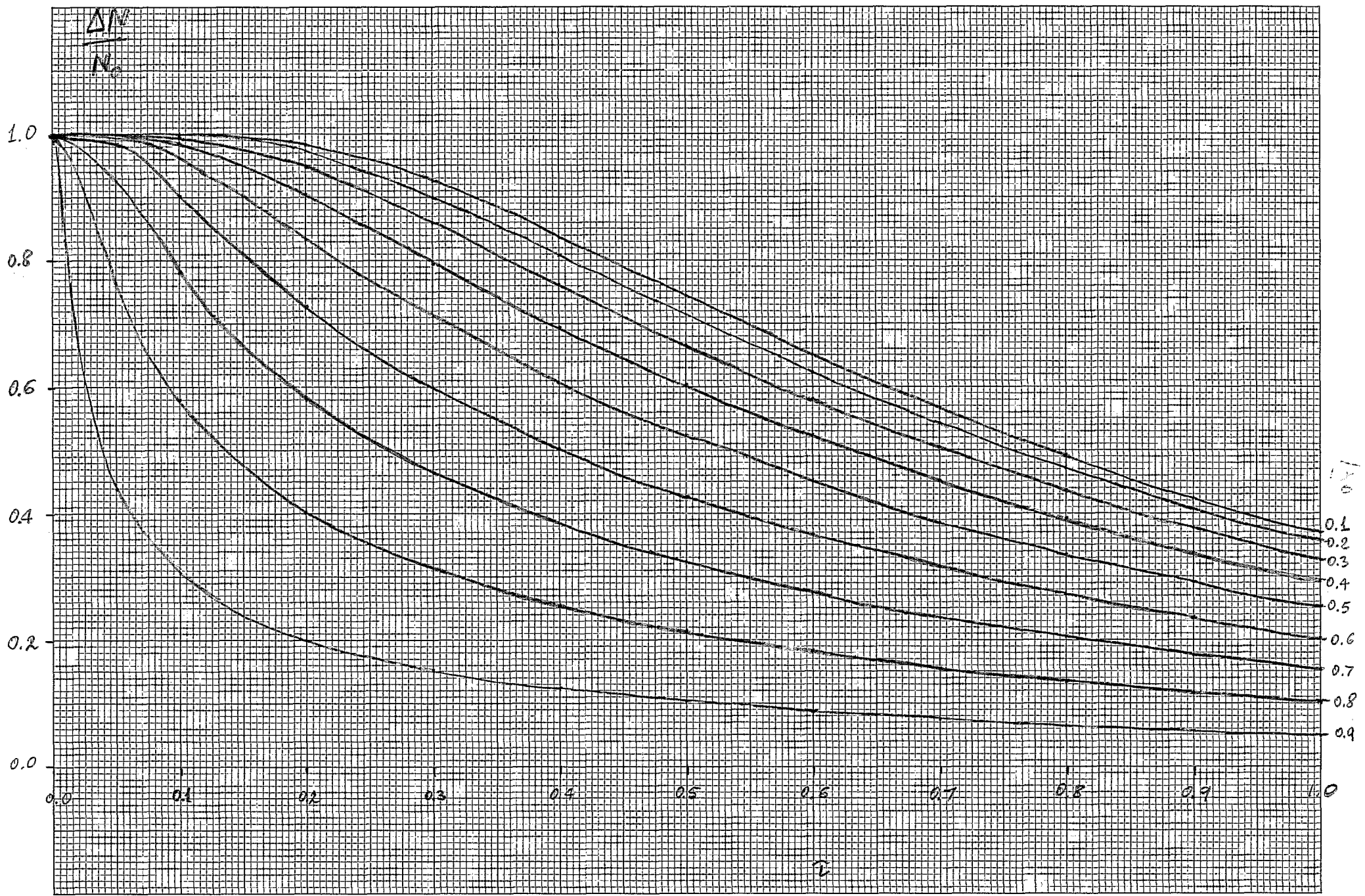
$$N(\tau) = 2 \sum_n e^{-\lambda_n^2 \tau / 4} / \lambda_n J_1(\lambda_n)$$

$$(2) \quad f_0(y) = 1 \quad 0 \leq y < 1$$

$$N(\tau) = 4 \sum_n e^{-\lambda_n^2 \tau / 4} / \lambda_n^2$$

$$(3) \quad f_0(y) = \lambda_1 J_0(\lambda_1 \sqrt{y}) / 2 J_1(\lambda_1)$$

$$N(\tau) = e^{-\lambda_1^2 \tau / 4}$$



The good field aperture is

$$\pm 25.2 \text{ mm}$$

$$\begin{aligned} \text{We take } \beta_{\max} &= 51.6 \text{ m} \\ \eta_{\max} &= 1.385 \text{ m} \end{aligned}$$

$$\text{For } \Delta p/p = 0 \quad \epsilon_a = 12.3 \pi \text{ mm-mrad}$$

$$\text{For } \Delta p/p = 0.5\% \quad \epsilon_a = 6.5 \pi \text{ mm-mrad}$$

The initial normalised emittance is

$$10 \pi \text{ mm-mrad} \quad \text{for (95\% of beam)}$$

Even at  $y = 10$

$$\epsilon_{\text{init}} = 1 \pi \text{ mm-mrad} \quad ( \text{ " } )$$

So

$$\epsilon_{\text{init}} \ll \epsilon_a$$

We can take for the initial distribution a delta-function

Observe that

$$\begin{aligned} \tau &= \frac{1}{\varepsilon_a} \int_0^t D(t') dt' = \frac{1}{\varepsilon_a} \int_0^t \frac{\langle \Delta \varepsilon \rangle}{dt} dt \\ &= \frac{\Delta \varepsilon(t)}{\varepsilon_a} \end{aligned}$$

where  $\Delta \varepsilon(t)$  is the rms emittance increase after time  $t$ .

The worst case is at  $y=12$  where (G.P.)

$$\Delta \varepsilon(t=2 \text{ hours}) = \frac{34.5 - 10}{6 \times 12} = 0.34 \pi \text{ mm mrad}$$

So  $\tau$  is very small anyway. There are very negligible losses (see also next Table) certainly less than 0.1%.

$N(\tau)$

$$\tau = \frac{\Delta \varepsilon(t)}{\varepsilon_a}$$

$$y_0 = 0$$

$$y_0 = 0.2$$

$$y_0 = 0.4$$

0.02

0.99987

0.99958

0.99879

0.04

0.99972

0.99945

0.99866

0.06

0.99959

0.99932

0.99776

0.08

0.99945

0.99906

0.99420

0.10

0.99924

0.99829

0.98675

## Diffusion in the Momentum Plane

Let  $f = f(p, t)$  be the distribution function

The F.P. equation now is

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial p} \left( D_p \frac{\partial f}{\partial p} \right)$$

where now  $D_p = \frac{d\langle p^2 \rangle}{dt}$

we can assume again  $D_p = D_p(t)$

Introduce  $p_a$  as aperture limit and

$$z = \frac{p}{p_0} \quad \text{and} \quad x = \frac{1}{p_a^2} \int_0^t D_p(t') dt'$$

$$= \left[ \frac{\Delta p(t)}{p_a} \right]^2$$

where  $\Delta p(t)$  is the rms increase in momentum deviation after time  $t$ .



Then we have

$$f = f(z, \tau)$$

and

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial z^2}$$

which has solution

$$f = \sum_{n=0}^{\infty} c_n (\cos \beta_n z) e^{-\beta_n^2 \tau}$$

having assumed that  $f(-z) = f(z)$

There are again two boundary conditions

1.)  $f(1, \tau) = 0$  ("sink")

this is satisfied by letting

$$\beta_n = \frac{\pi}{2} (2n+1)$$

2.)  $f(z, 0) = f_0(z)$  initial distribution

This condition requires

$$\sum_{n=0}^{\infty} c_n \cos(\beta_n z) = f_0(z)$$

that is by inversion

$$c_n = 2 \int_0^1 f_0(z) \cos \beta_n z \, dz$$

For  $f_0(z) = \delta(z) \Rightarrow \underline{\underline{c_n = 2}}$

For the number of particles survived at the time  $\tau$

$$N(\tau) = \sum_{n=0}^{\infty} c_n e^{-\beta_n^2 \tau} \int_0^1 \cos \beta_n z \, dz$$

$$= \sum_{n=0}^{\infty} c_n (-1)^n \frac{e^{-\beta_n^2 \tau}}{\beta_n}$$

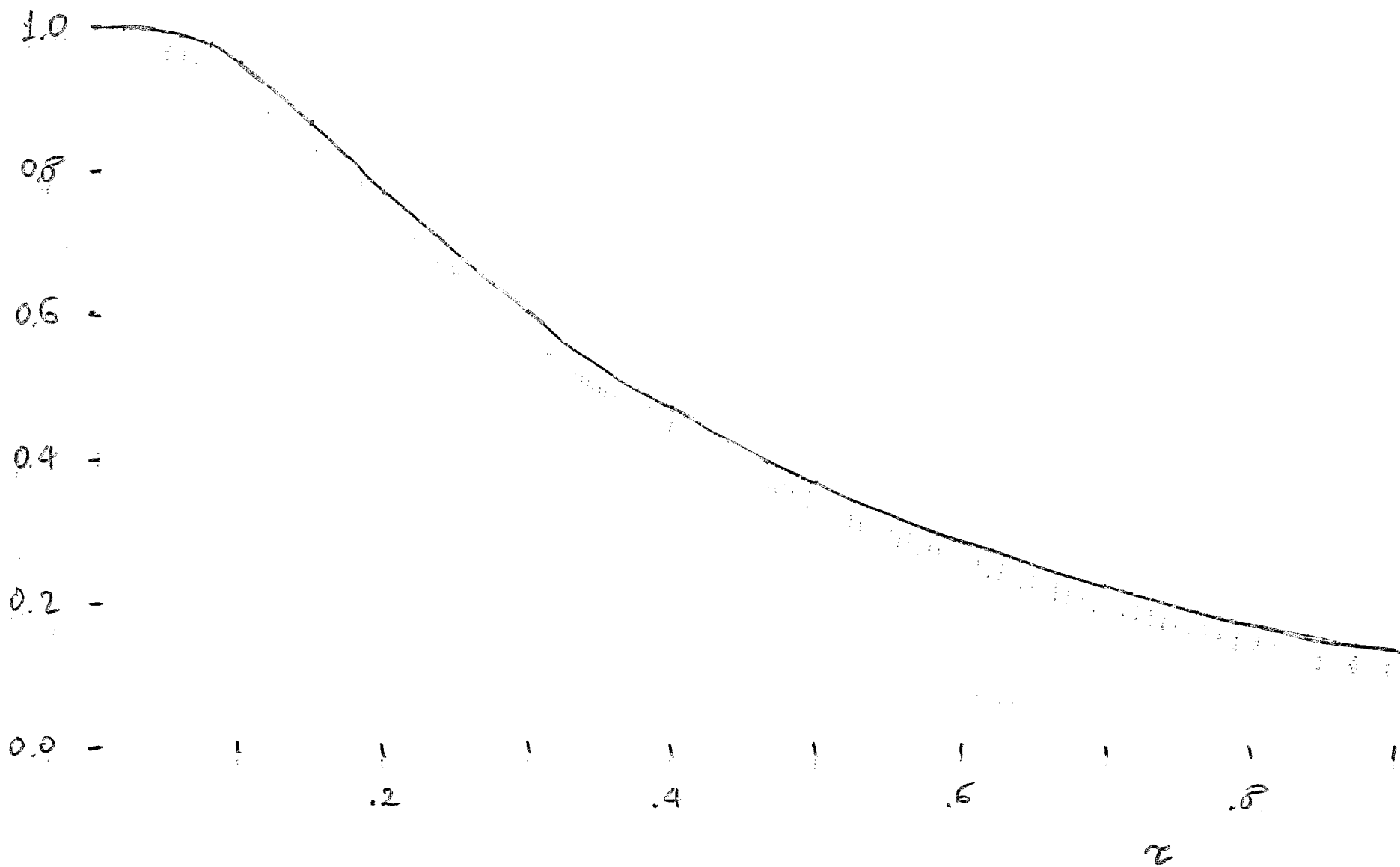
For  $f_0(z) = \delta(z)$

$$N(\tau) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta_n} e^{-\beta_n^2 \tau}$$

Results for  $f_0(z) = \delta(z)$

<u><math>z</math></u>	$N(z)$
0.02	0.999893
0.04	.999079
0.06	.992102
0.08	.975039
0.10	.949173
0.12	.917406
0.14	.882289
0.16	.845648
0.18	.808683
0.20	.772151

$N(\tau)/N_0$



From G. Parson Calculations  $(V=1MV)$   
 $t=2\text{ hours}$

$\delta$	12	20	30	50	75	100
$P_a$	3.49	6.13	9.08	3.93	2.91	2.45
$\Delta p$	1.22	1.37	1.57	1.14	.921	.789
$\tau$	0.122	0.050	0.030	0.084	0.100	0.104
$N(\tau)$	0.91	0.99676	0.9998		0.949	0.94

$$F_{0.5} \quad \tau = \left( \frac{\Delta p}{P_a} \right)^2 = \left( \frac{1}{2.5} \right)^2 = 0.16$$

$$N(\tau) = 0.84565$$