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A General Scheme of Solenoid Compensation Using Skew-Quadrupoles. An Example: Decoupling of the STAR Detector in RHIC

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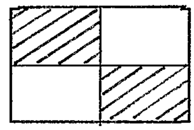
1. Introduction

Detectors placed at interaction points of various colliders contain solenoids producing longitudinal magnetic fields which linearly couple the horizontal and the vertical betatron motions, (x-y coupling). Compensation the linear coupling is usually required[1–3] in order to avoid a loss of luminosity. In the previous note[4] the compensating schemes using antisolenoids were described, and subsequently applied to the decoupling of the STAR detector which will be placed in RHIC. In this note I shall describe a complementary scheme of compensating the linear coupling by employing skew-quadrupoles. As before, I will illustrate a general method on the case of STAR detector. Significantly new features are introduced by the antisymmetric insertions of the 1991 RHIC lattice, for which the calculations are performed.

2. Description of the Decoupling Method

We start with a ring which is globally decoupled, and has 4×4 transfer matrix $T_0\,(s'', s')$ which is of the block-diagonal form

$$T_0 = \left[\begin{array}{c|c} T_{0x} & \mathbf{0} \\ \hline \mathbf{0} & T_{0y} \end{array}\right] =$$



. (2.1)

This form ensures that the horizontal (x, p_x), and the vertical (y, p_y) variables transform independently among themselves (are decoupled) when T_0 acts on the state- vector

$$z = \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} . \tag{2.2}$$

An installment of a detector, containing the solenoid magnetic field, produces the linear coupling which destroys the block-diagonal structure of the total transfer matrix. To compensate for, the additional couplers (solenoids, skew-quadrupoles) are needed. Together they form an insertion, which needs to be properly balanced so the resulting full-turn transfer matrix will be of the block-diagonal form again. To be specific, let us consider an insertion (AB) consisting of even number 2N of skew-quadrupoles placed symmetrically around a solenoid, as shown in Fig. 1.

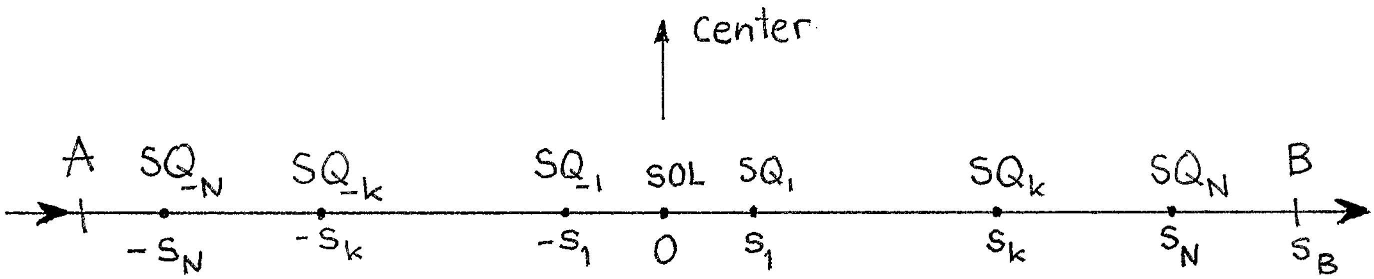


Fig. 1. Schematic layout of a (AB) insertion.

The coordinates $s_k, k = \pm 1, \ldots, \pm N$ are attached to the centers of the k-th skew-quadrupoles of length ℓ_k, as shown in Fig. 2. The solenoid has the $s_0 = 0$ coordinate, placed at its center

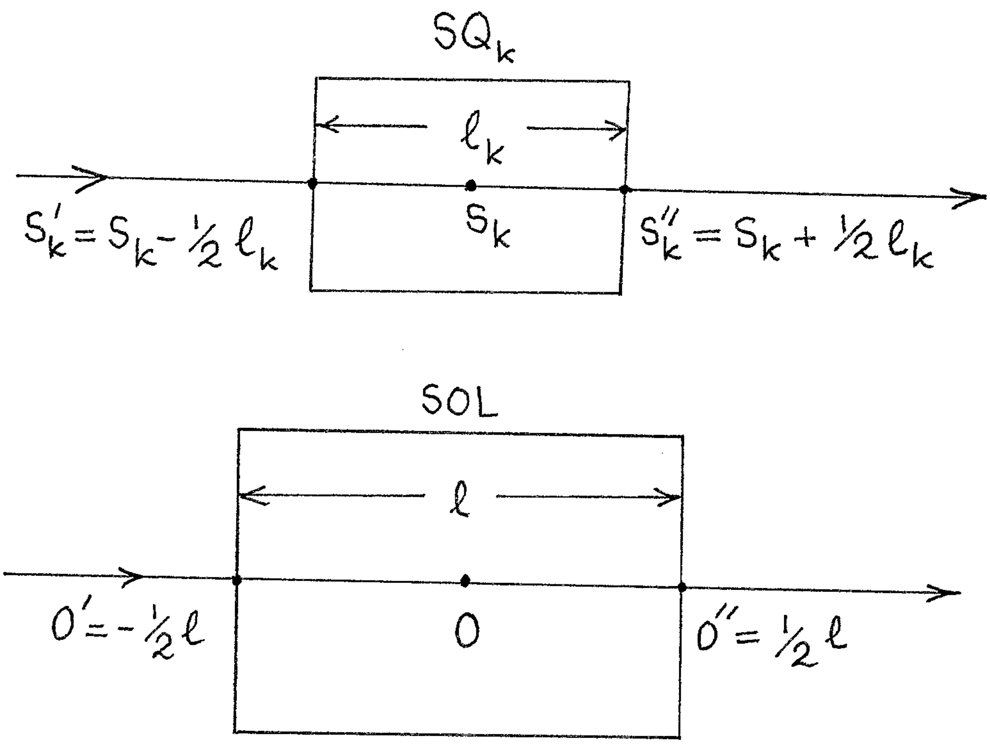


Fig. 2.: Coordinates associated with the k-th skew-quadrupole, (a), and with the solenoid at $s_0 = 0$, (b).

The transfer matrix T_{AB} through the insertion is given by the expression

$$T_{AB} = T_0\left(s_B, s''_N\right) T_N\left(s''_N, s'_N\right) T_0\left(s'_N, s''_{N-1}\right) T_{N-1}\left(s''_{N-1}, s'_{N-1}\right) \ldots T_{SOL}\left(0'', 0'\right) \ldots$$
$$\ldots T_0\left(s'_2, s''_1\right) T_{-N}\left(s''_1, s'_1\right) T_0\left(s'_1, s_A\right). \tag{2.3}$$

Using so called the "projection on the coupler" concept[5–6]

$$P_k\left(s_k, 0\right) \equiv T_0\left(0, s''_k\right) T_k\left(s''_k, s'_k\right) T_0\left(s'_k, 0\right), k = \pm 1, \ldots, \pm N, \tag{2.4}$$

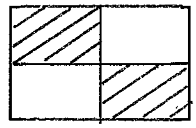
for the skew-quadrupoles, and

$$P_{SOL}\left(0, 0\right) \equiv T_0\left(0, 0''\right) T_{SOL}\left(0'', 0'\right) T_0\left(0', 0\right), \tag{2.5}$$

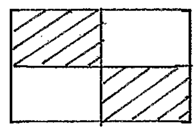
for the solenoid, it is possible to rewrite the transfer matrix as follows

$$T_{AB} = T_0\left(s_B, 0\right) P_N\left(s_N, 0\right) \cdots P_1\left(s_1, 0\right) P_{SOL}\left(0, 0\right) P_{-1}\left(-s_1, 0\right) \cdots P_{-N}\left(-s_N, 0\right) T_0\left(0, s_A\right). \tag{2.6}$$

As it was stated earlier, the (AB) insertion is exactly decoupled when its transfer matrix T_{AB} has the block-diagonal form

$$T_{AB} = $$  . (2.7)

It is clear that the full-turn transfer matrix will also be of this form since the structure is preserved upon a matrix multiplication. The basic formula (2.6) shows that the necessary and sufficient condition for the insertion to be decoupled is that the product of the projections itself will also be of the block-diagonal form

$$P_N \cdots P_1 \; P_{SOL} \; P_{-1} \cdots P_{-N} = $$  . (2.8)

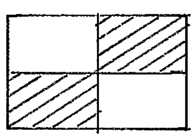
It will now be demonstrated that for small skew-quads strengths q_k, similarly as for a small solenoid's strength θ, (comp (3.15) in ref. [4]) the following asymptotic expansions hold

$$P_k\,(s,0) = \mathbf{1}_4 + q_k R_k\,(s,0) + 0\left(q_k^2\right), k = \pm 1, \ldots, \pm N,$$

and

$$P_{SOL}\,(s,0) = \mathbf{1}_4 + \theta S\,(s,0) + 0\left(\theta^2\right), \tag{2.9}$$

$\left(\theta = \frac{B_s \ell}{2(B\rho)}, \ell - \text{length}, (B\rho) - \text{magnetic rigidity}\right)$, where, what is remarkable, both R_k and S matrices are block-anti-diagonal

$$R_k = $$  $$, \quad S = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{1}_2 \\ \hline -\mathbf{1}_2 & \mathbf{0} \end{array}\right], k = \pm 1, \ldots, \pm N. \tag{2.10}$$

Therefore, to the first-order in the strength parameters,

$$\begin{aligned} P_N \cdots P_1 \; P_{SOL} \; P_{-1} \cdots P_{-N} &= (\mathbf{1}_4 + q_N R_N + \cdots) \cdots (\mathbf{1}_4 + q_1 R_1 + \cdots)(\mathbf{1}_4 + \theta \; S + \cdots) \\ &(\mathbf{1}_4 + q_{-1} R_{-1} + \cdots) \cdots (\mathbf{1}_4 + q_{-N} R_{-N} + \cdots) = \\ &= \mathbf{1}_4 + \theta \; S + \sum_{k=1}^{N} (q_k R_k + q_{-k} R_{-k}) + \cdots . \end{aligned} \tag{2.11}$$

This expression is of the block-diagonal form only if contributions to block-anti-diagonal part cancel between themselves

$$\theta \; S + \sum_{k=1}^{N} (q_k R_k + q_{-k} R_{-k}) = \mathbf{0}. \tag{2.12}$$

These are the decoupling conditions, to the first-order, of the (AB) insertion. Notice, that when the above decoupling conditions hold, one has, to the second-order (according to the formula (2.11)),

$$P_N \cdots P_1 \, P_{SOL} \, P_{-1} \cdots P_N = \mathbf{1}_4 + 0\left(\theta^2, q_k^2\right), \tag{2.13}$$

and the basic formula (2.6) yields

$$T_{AB} = T_0\left(S_B, 0\right) T_0\left(0, S_A\right) = T_0\left(S_B, S_A\right) + 0\left(\theta^2, q_k^2\right). \tag{2.14}$$

This means that the (AB) insertion becomes transparent for a beam, to the second-order in the strength parameters, when it is decoupled.

3. The R-Matrice for a Skew-Quadrupole

In order to be able to solve the decoupling conditions (2.12) for the q_k's one needs to know the R_k matrices. The S-matrix is known and its derivation was presented in the previous note.[4]

The transfer matrix of a thin skew-quadrupole of length ℓ is obtained from the transfer matrix of (vertically focusing) quadrupole by a rotation on $\pi/4$

$$T_{SQ}(\ell) = R^{-1}(\pi/4)\ T_Q(\ell)\ R(\pi/4), \tag{3.1}$$

where for the vertically focusing quadrupole we have

$$T_Q(\ell) = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline f^{-1} & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & f^{-1} & 1 \end{array}\right], \tag{3.2}$$

where

$$f^{-1} \equiv \frac{\ell}{(B\rho)} \frac{\partial B_x}{\partial x}\Big|_{y=0} \equiv (\beta_x\beta_y)^{-1/2}\, q\ . \tag{3.3}$$

Taking into account the rotation matrices

$$\mathrm{R}(\pi/4) = \frac{1}{\sqrt{2}}\left[\begin{array}{c|c|c|c} 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline -1 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 1 \end{array}\right], \mathrm{R}^{-1}(\pi/4) = \frac{1}{\sqrt{2}}\left[\begin{array}{c|c|c|c} 1 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \end{array}\right], \tag{3.4}$$

we get for the transfer matrix of a thin skew-quadrupole

$$T_{SQ}(\ell) = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & f^{-1} & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline f^{-1} & 0 & 0 & 1 \end{array}\right] \equiv \left[\begin{array}{c|c} \mathbf{1}_2 & t \\ \hline t & \mathbf{1}_2 \end{array}\right], \tag{3.5}$$

where the 2×2 matrix t is

$$t \equiv f^{-1}\left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] = (\beta_x\beta_y)^{-1/2}\, q \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right]. \tag{3.6}$$

According to the formula (2.4) we have now for the projection on the k-th skew-quadrupole

$$P_k(s_k,0) = T_0\left(0,s_k''\right)\ T_{SQ}(\ell_k)\ T_0\left(s_k',0\right), k=1,\ldots,N\ , \tag{3.7}$$

where, in the thin lens approximation

$$\begin{aligned} T_0\left(0,s_k''\right) &= T_0\left(0,s_k+\frac{1}{2}\ell_k\right) = T_0(0,s_k)+\cdots, \\ T_0\left(s_k',0\right) &= T_0\left(s_k-\frac{1}{2}\ell_k,0\right) = T_0(s_k,0)+\cdots. \end{aligned} \tag{3.8}$$

As the result, the projection has the expansion

$$\begin{aligned} &P_k(s_k,0) = T_0^{-1}(s_k,0)\ T_{SQ}(\ell_k)\ T_0(s_k,0)+\cdots = \\ &= \mathbf{1}_4 + \left[\begin{array}{c|c} \mathbf{0} & T_{0x}^{-1}(s_k,0)\,t(s_k)\,T_{0y}(s_k,0) \\ \hline T_{0y}^{-1}(s_k,0)\,t(s_k)\,T_{0x}(s_k,0) & \mathbf{0} \end{array}\right] + \cdots \end{aligned} \tag{3.9}$$

Comparing with the asymptotic expansion (2.9) we obtain for the R_k-matrice the expression

$$R_k(s_k,0) = \left[\begin{array}{c|c} \mathbf{0} & Q(s_k,0) \\ \hline -\overline{Q}(s_k,0) & \mathbf{0} \end{array}\right], \tag{3.10}$$

were we have denoted

$$\begin{aligned} Q(s_k,0) &= q^{-1}(s_k)\ T_{0x}^{-1}(s_k,0)\ t(s_k)\ T_{0y}(s_k,0) \equiv Q_k, \\ &= \left[\beta_x(s_k)\,\beta_y(s_k)\right]^{-1/2} T_{0x}^{-1}(s_k,0) \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] T_{0y}(s_k,0), \end{aligned} \tag{3.11}$$

and $\overline{Q}$ is a symplectic conjugate* of Q,

$$\overline{Q}(s_k,0) = -\left[\beta_x(s_k)\,\beta_y(s_k)\right]^{-1/2} T_{0y}^{-1} \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right]\ T_{0x}(s_k,0) \equiv \overline{Q}_k. \tag{3.12}$$

Denoting, for the sake of brevity

$$\begin{aligned} &\beta_x(s) \equiv \beta_x,\ \beta_x(0) \equiv \beta_x^*,\ \alpha_x(s) \equiv \alpha_x,\ \alpha_x(0) \equiv \alpha_x^*, \\ &\beta_y(s) \equiv \beta_y,\ \beta_y(0) \equiv \beta_y^*,\ \alpha_y(s) \equiv \alpha_y,\ \alpha_y(0) = \alpha_y^*, \end{aligned} \tag{3.13a}$$

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* A symplectic conjugate of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\overline{A} \equiv \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. For A symplectic one has that $\overline{A} = A^{-1}$.

and

$$T_{0x}(s,0) \equiv T_{0x},\ \psi_x(s,0) = \int_0^s \frac{ds}{\beta_x} \equiv \psi_x,\ c_x \equiv \cos\psi_x,\ s_k \equiv \sin\psi_x,$$
$$T_{0y}(s,0) \equiv T_{0y},\ \psi_y(s,0) = \int_0^s \frac{ds}{\beta_y} \equiv \psi_y,\ c_y \equiv \cos\psi_y,\ s_y \equiv \sin\psi_y, \tag{3.13b}$$

and taking into account that, for RHIC,

$$\alpha_x^* = \alpha_y^* = 0,$$

and

$$\beta_x^* = \beta_y^* = \beta^*, \tag{3.14}$$

we get for the transfer matrices of the globally decoupled lattice

$$T_{0x,y} = \left[\begin{array}{c|c} \left(\frac{\beta}{\beta^*}\right)^{1/2} c & (\beta\beta^*)^{1/2} s \\ \hline -(\beta\beta^*)^{-1/2}(\alpha c + s) & \left(\frac{\beta^*}{\beta}\right)^{1/2}(c - \alpha s) \end{array}\right]_{x,y}, \tag{3.15}$$

and for the inverse

$$T_{0x,y}^{-1} = \left[\begin{array}{c|c} \left(\frac{\beta^*}{\beta}\right)^{1/2}(c - \alpha s) & -(\beta\beta^*)^{1/2} s \\ \hline (\beta\beta^*)^{-1/2}(\alpha c + s) & \left(\frac{\beta}{\beta^*}\right)^{1/2} c \end{array}\right]_{x,y}. \tag{3.16}$$

From this using the formula (3.11) we get the Q_k-matrix

$$Q(s_k,0) = \left[\begin{array}{c|c} -s_x(k)\,c_y(k) & -\beta^* s_x(k)\,s_y(k) \\ \hline \frac{1}{\beta^*} c_x(k)\,c_y(k) & c_x(k)\,s_y(k) \end{array}\right] \equiv Q_k, \tag{3.17}$$

and its symplectic conjugate

$$\overline{Q}(s_k,0) = \left[\begin{array}{c|c} c_x(k)\,s_y(k) & \beta^* s_x(k)\,s_y(k) \\ \hline -\frac{1}{\beta^*} c_x(k)\,c_y(k) & -s_x(k)\,c_y(k) \end{array}\right] \equiv \overline{Q}_k, k = 1, \cdots, N. \tag{3.18}$$

For the symmetrically placed skew-quadrupoles the Q_k-matrices will be different since the 1991 RHIC lattice is taken as to be anti-symmetric one. Taking into account that

$$Q_{-k} \equiv Q(-s_k, 0), \tag{3.19}$$

and

$$\psi_x(-k) = \psi_x(-s_k, 0) = \int_0^{-s_k} \frac{ds}{\beta_x(s)} = -\int_0^{s_k} \frac{ds}{\beta_x(-s)} = -\int_0^{s_k} \frac{ds}{\beta_y(s)} = -\psi_y(k), \tag{3.20}$$

and similarly

$$\psi_y(-k) = -\psi_x(k), \tag{3.21}$$

as the result of the anti-symmetric condition which is adopted in the 1991 RHIC lattice

$$\beta_x(-s) = \beta_y(s). \tag{3.22}$$

As the consequence, we have the relations

$$\begin{aligned} c_x(-k) &= c_y(k), \quad c_y(-k) = c_x(k), \\ s_x(-k) &= -s_y(k), \quad s_y(-k) = -s_x(k), \end{aligned} \tag{3.23}$$

which lead to the following formula for the Q_{-k}

$$Q_{-k} = \left[\begin{array}{c|c} s_y(k)\, c_x(k) & -\beta^* s_x(k)\, s_y(k) \\ \hline \frac{1}{\beta^*} c_x(k)\, c_y(k) & -c_y(k)\, s_x(k) \end{array}\right], k = 1, \cdots, N. \tag{3.24}$$

Knowing this we can analyze the decoupling conditions in details.

4. The Decoupling Schemes Using Skew-Quadrupoles

Using (2.10) and (3.10) and (3.17), (3.18) and (3.24) the decoupling conditions (2.12) become

$$\theta\left[\begin{array}{c|c} \mathbf{0} & \mathbf{1}_2 \\ \hline -\mathbf{1}_2 & \mathbf{0} \end{array}\right]+\sum_{k=1}^{N} q_k\left[\begin{array}{c|c} \mathbf{0} & Q_k \\ \hline -\overline{Q}_k & \mathbf{0} \end{array}\right]+\sum_{k=1}^{N} q_{-k}\left[\begin{array}{c|c} \mathbf{0} & Q_{-k} \\ \hline -\overline{Q}_{-k} & \mathbf{0} \end{array}\right]=0. \quad (4.1)$$

They are equivalent to the following conditions in terms of the 2×2 submatrices

$$\theta\mathbf{1}_2+\sum_{k=1}^{N}\left(q_k Q_k+q_{-k}Q_{-k}\right)=0. \quad (4.2)$$

The lower-left submatrices lead to the condition which is just symplectic conjugate of the last one, and thus may be ignored as not independent.

Using again the formulae (3.17) and (3.24) we may write the decoupling condition (4.2) in components

$$\theta-\sum_{k=1}^{N} q_k s_x(k)\, c_y(k)+\sum_{k=1}^{N} q_{-k} s_y(k)\, c_x(k)=0, \quad (4.3)$$

$$\theta+\sum_{k=1}^{N} q_k c_x(k)\, s_y(k)-\sum_{k=1}^{N} q_{-k} c_y(k)\, s_x(k)=0, \quad (4.4)$$

$$\sum_{k=1}^{N}\left(q_k+q_{-k}\right) s_x(k)\, s_y(k)=0, \quad (4.5)$$

$$\sum_{k=1}^{N}\left(q_k+q_{-k}\right) c_x(k)\, c_y(k)=0. \quad (4.6)$$

Let us notice, at first, that the standard scheme with oppositely powered skew-quadrupole pairs is not possible here since the assumptions

$$q_k=-q_{-k},\; k=1,\ldots,N, \quad (4.7)$$

contradict the first two equations, when $\theta\neq 0$

$$\begin{aligned}\theta-\sum_{k=1}^{N} q_k \sin\left[\psi_x(k)+\psi_y(k)\right]&=0,\\ \theta+\sum_{k=1}^{N} q_k \sin\left[\psi_x(k)+\psi_y(k)\right]&=0.\end{aligned} \quad (4.8)$$

This is a consequence of the antisymmetry of the 1991 lattice which we use here.

It is possible, however, to find a symmetric solution for which symmetrically placed skew-quadrupoles are powered in the same way,

$$q_k = q_{-k}, \qquad k = 1, \ldots, N. \tag{4.9}$$

In this case the first two equations (4.3) and (4.4) coincide and we get the conditions

$$\sum_{k=1}^{N} q_k \sin\left[\psi_x(k) - \psi_y(k)\right] = \theta, \tag{4.10}$$

$$\sum_{k=1}^{N} q_k \cos\left[\psi_x(k) - \psi_y(k)\right] = 0, \tag{4.11}$$

$$\sum_{k=1}^{N} q_k \cos\left[\psi_x(k) + \psi_y(k)\right] = 0. \tag{4.12}$$

The last two equations follow by adding and by subtracting, respectively, the conditions (4.5) and (4.6).

Various decoupling schemes correspond to various choices of the number N, of the skew quadrupole pairs. We shall examine closely the lowest possibilities: $N = 1, 2$ and 3.

5. Symmetrically Placed & Equally Powered Pairs of Skew-Quads

A. For $N = 1$ one gets the condition on the skew-quadrupole strength

$$q_1 \sin\left[\psi_x(1) - \psi_y(1)\right] = \theta, \tag{5.1}$$

and restrictions on the locations and the phase advances between 0 and s

$$\sin\left[\psi_x(s_1, 0)\right] \sin\left[\psi_y(s_1, 0)\right] = 0, \tag{5.2}$$

and

$$\cos\left[\psi_x(s_1, 0)\right] \cos\left[\psi_y(s_1, 0)\right] = 0, \text{(magic phases!)}. \tag{5.3}$$

They appear to be restrictive and rather rigid. A good deal of luck is needed to find these magic phases, if they exist at all.

B. For $N = 2$ we get the conditions

$$q_1 \sin\left[\psi_x(1) - \psi_y(1)\right] + q_2 \sin\left[\psi_x(2) - \psi_y(2)\right] = \theta, \tag{5.4}$$

$$q_1 s_x(1)\, s_y(1) + q_2 s_x(2)\, s_y(2) = 0, \tag{5.5}$$

$$q_1 c_x(1)\, c_y(1) + q_2 c_x(2)\, c_y(2) = 0. \tag{5.6}$$

The last two equations require that their determinant vanishes otherwise they yield the trivial solution: $q_1 = 0, q_2 = 0$. Hence, again we obtain the same restriction on phase advances and the locations s_1, s_2 as previously found in (4.19)

$$s_x(1)\, s_y(1)\, c_x(2)\, c_y(2) - s_x(2)\, s_y(2)\, c_x(1)\, c_y(1) = 0, \text{(magic phases!)}. \tag{5.7}$$

Once the rather special locations s_1, s_2 in a ring are known, the decoupling strengths q_1, q_2 can be found from the first two equations (5.4) and (5.5).

A minimal scheme in which the skew-quadrupole locations are not restricted requires at least three pairs of them, since then the number of unknowns q_1, q_2, q_3 agrees with the number of equations.

C. For $N = 3$ the decoupling condition yields

$$q_1 \sin\left[\psi_x(1) - \psi_y(1)\right] + q_2 \sin\left[\psi_x(2) - \psi_y(2)\right] + q_3 \sin\left[\psi_x(3) - \psi_y(3)\right] = \theta, \tag{5.8}$$

$$q_1 \cos\left[\psi_x(1) - \psi_y(1)\right] + q_2 \cos\left[\psi_x(2) - \psi_y(2)\right] + q_3 \cos\left[\psi_x(3) - \psi_y(3)\right] = 0, \tag{5.9}$$

$$q_1 \cos\left[\psi_x(1) + \psi_y(1)\right] + q_2 \cos\left[\psi_x(2) + \psi_y(2)\right] + q_3 \cos\left[\psi_x(3) + \psi_y(3)\right] = 0. \tag{5.10}$$

Denoting, for the sake of brevity

$$\delta_k \equiv \psi_x(k) - \psi_y(k),$$

and

$$\sigma_k \equiv \psi_x(k) + \psi_y(k), \quad k = 1, 2, 3 \tag{5.11}$$

we may write the solution as follows

$$q_1 = \Delta^{-1} \begin{vmatrix} c(\delta_2) & c(\sigma_2) \\ c(\delta_3) & c(\sigma_3) \end{vmatrix} \theta, \tag{5.12}$$

$$q_2 = -\Delta^{-1} \begin{vmatrix} c(\delta_1) & c(\sigma_1) \\ c(\delta_3) & c(\sigma_3) \end{vmatrix} \theta, \tag{5.13}$$

$$q_3 = \Delta^{-1} \begin{vmatrix} c(\delta_1) & c(\sigma_1) \\ c(\delta_2) & c(\sigma_2) \end{vmatrix} \theta, \tag{5.14}$$

where the determinant Δ is

$$\Delta = \begin{vmatrix} s(\delta_1) & c(\delta_1) & c(\sigma_1) \\ s(\delta_2) & c(\delta_2) & c(\sigma_2) \\ s(\delta_3) & c(\delta_3) & c(\sigma_3) \end{vmatrix} \neq 0, (\text{assumed!}). \tag{5.15}$$

The position s_1, s_2, s_3 are the skew-quadrupoles and, correspondingly, phase advances are not restricted here, cf. Fig. 3., unless the determinant Δ vanishes for some unlucky choice.

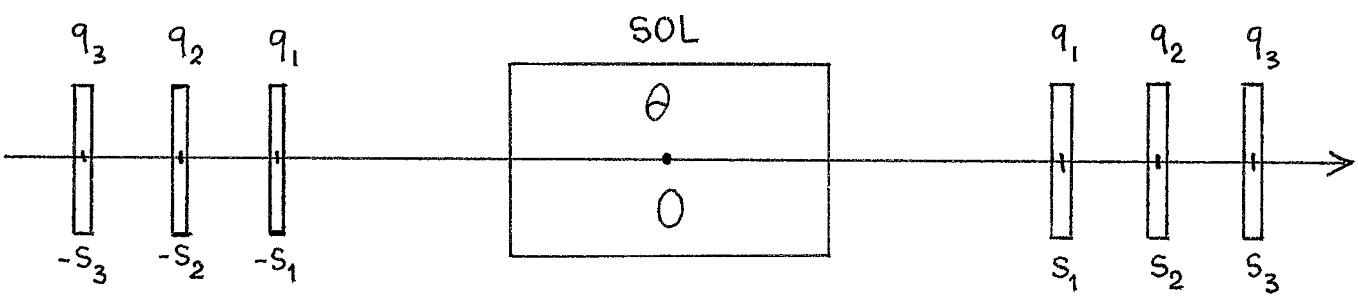

Fig. 3.: Decoupling scheme with three pairs of symmetrically placed, and equally powered skew-quadrupoles.

We shall apply this scheme for the decoupling of the STAR detector solenoid in RHIC.

6. Two Triplets of Skew-Quads (YELLOW RING at 6 o'clock)

Let us assume that the STAR detector is placed at 6 o'clock in RHIC rings. We will consider first the YELLOW RING, and later we will decouple the BLUE one, as well.

The phase advances and the β-functions are shown in Figures 4 and 5. It is tempting to choose the following locations for the right triplet of skew-quadrupoles

$$\begin{aligned} s_1 &= 34.66 \text{ m}, \quad \text{at } Q_3, \\ s_2 &= 81.942 \text{ m}, \quad \text{at } Q_4, \\ s_3 &= 120.514 \text{ m}, \quad \text{at } Q_8, \end{aligned} \tag{6.1}$$

The corresponding left triplet is symmetrically located

$$\begin{aligned} &-s_1, \text{ at } Q_3, \\ &-s_2, \text{ at } Q_4, \\ &-s_3, \text{ at } Q_8. \end{aligned} \tag{6.2}$$

The phases advances are

$$\begin{aligned} \Psi_x(1) &= -0.242 \times 2\pi \ , \ \Psi_y(1) = -0.241 \times 2\pi, \\ \Psi_x(2) &= -0.321 \times 2\pi \ , \ \Psi_y(2) = -0.665 \times 2\pi, \\ \Psi_x(3) &= -0.699 \times 2\pi \ , \ \Psi_y(3) = -0.857 \times 2\pi, \end{aligned} \tag{6.3}$$

Using the formula (5.15) we get for the determinant

$$\Delta = 0.698, \tag{6.4}$$

and for the decoupling strengths

$$\begin{aligned} q_1 &= -0.036\ \theta_*, \\ q_2 &= 2.204\ \theta_*, \\ q_3 &= 0.633\ \theta_*, \end{aligned} \tag{6.5}$$

where the STAR solenoid parameter θ_* is given by

$$\theta_* = \frac{B_* \ell_*}{2\,(B\rho)} \tag{6.6}$$

Taking into account the formula (3.3), and the facts

$$\begin{aligned} B_* &= 0.5 \text{ T}, \\ \ell_* &= 4 \text{ m}, \end{aligned} \tag{6.7}$$

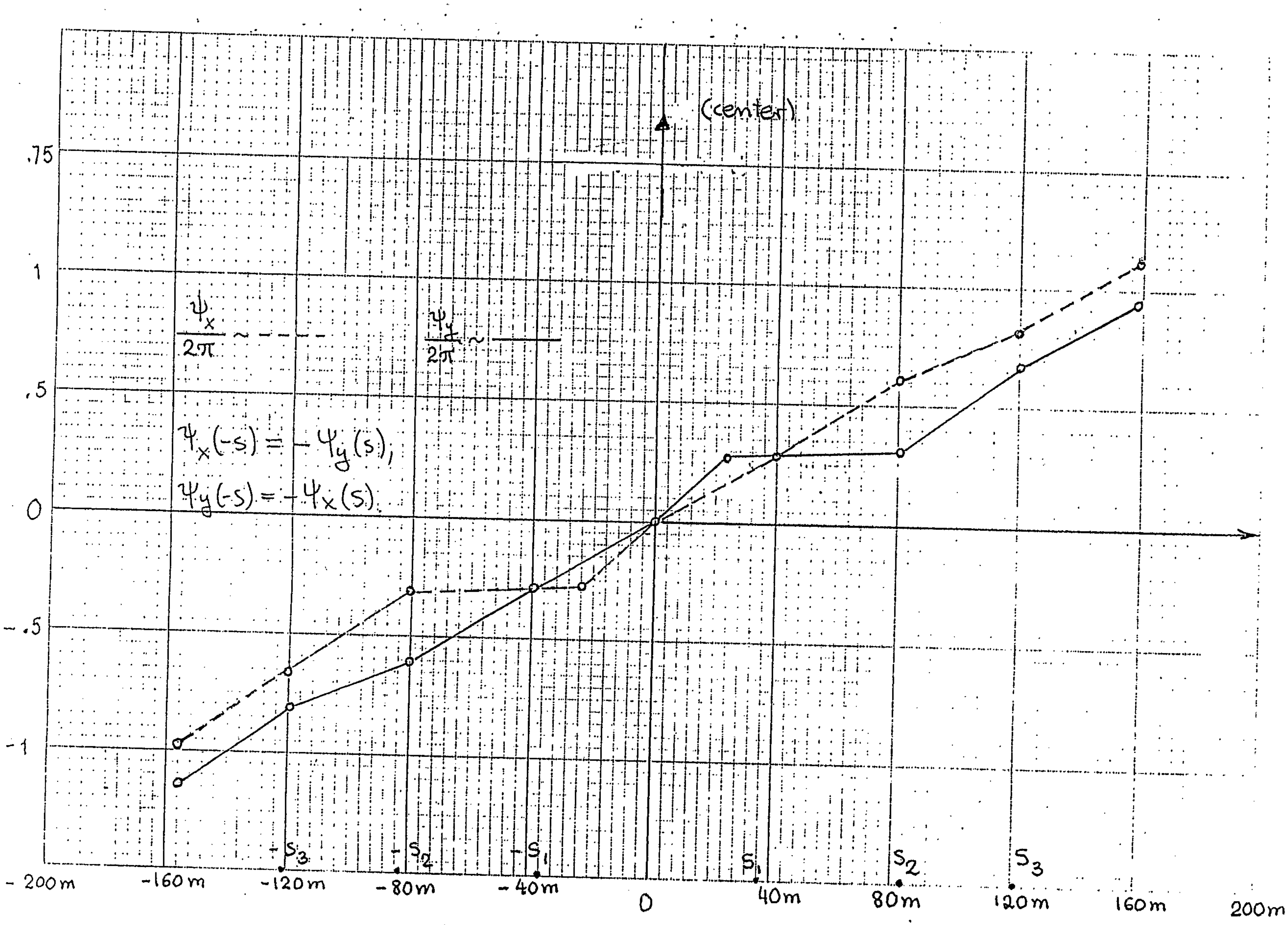

Fig. 4: Phase advances in YELLOW RING around 6 o'clock (not quite in scale!). RHIC Anti-Symmetric Lattice 1991, $\beta^* = 2$ m, YELLOW RING.

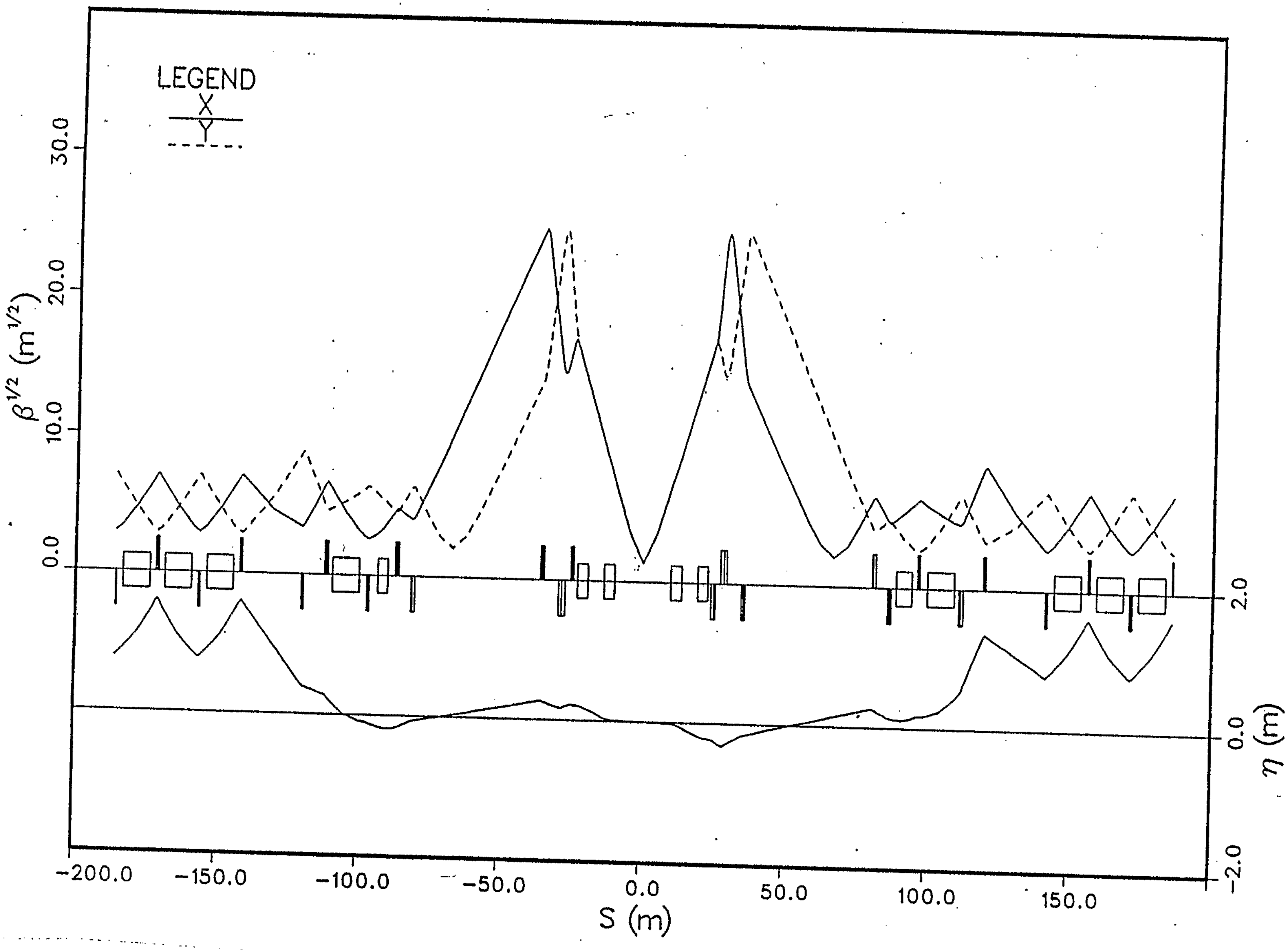

Fig. 5: β-Functions around 6 o'clock IP in 1991 RHIC Anti-Symmetric Lattice, $\beta^* = 2$ m.

and that, for $\beta^* = 2$ m, 1991 RHIC lattice, one has

$$\begin{aligned} \beta_x(s_1) &= 586.617 \text{ m}, & \beta_y(s_1) &= 247.214 \text{ m}, \\ \beta_x(s_2) &= 18.027 \text{ m}, & \beta_y(s_2) &= 40.584 \text{ m}, \\ \beta_x(s_3) &= 12.589 \text{ m}, & \beta_y(s_3) &= 77.764 \text{ m}, \end{aligned} \tag{6.8}$$

we find the skew-quadrupole gradients multiplied by their lengths

$$\begin{aligned} \ell_1 \frac{\partial B_x^{(1)}}{\partial x}\bigg|_{y=0,s_1} &= -0.945 \text{ Gauss}, \\ \ell_2 \frac{\partial B_x^{(2)}}{\partial x}\bigg|_{y=0,s_2} &= 814.84 \text{ Gauss}, \\ \ell_3 \frac{\partial B_x^{(3)}}{\partial x}\bigg|_{y=0,s_3} &= 202.31 \text{ Gauss}. \end{aligned} \tag{6.9}$$

The symmetrically placed skew-quadrupoles have equal strengths.

7. Decoupling of the STAR Detector (BLUE RING at 6 o'clock)

The phase advances in the BLUE RING, around 6 o'clock, are the same as shown in Fig. 4. Only the direction of the s variable is reversed as shown in Fig. 6. This means that the decoupling proceeds exactly in the same way as for the YELLOW RING, and the decoupling strengths are the same as previously found.

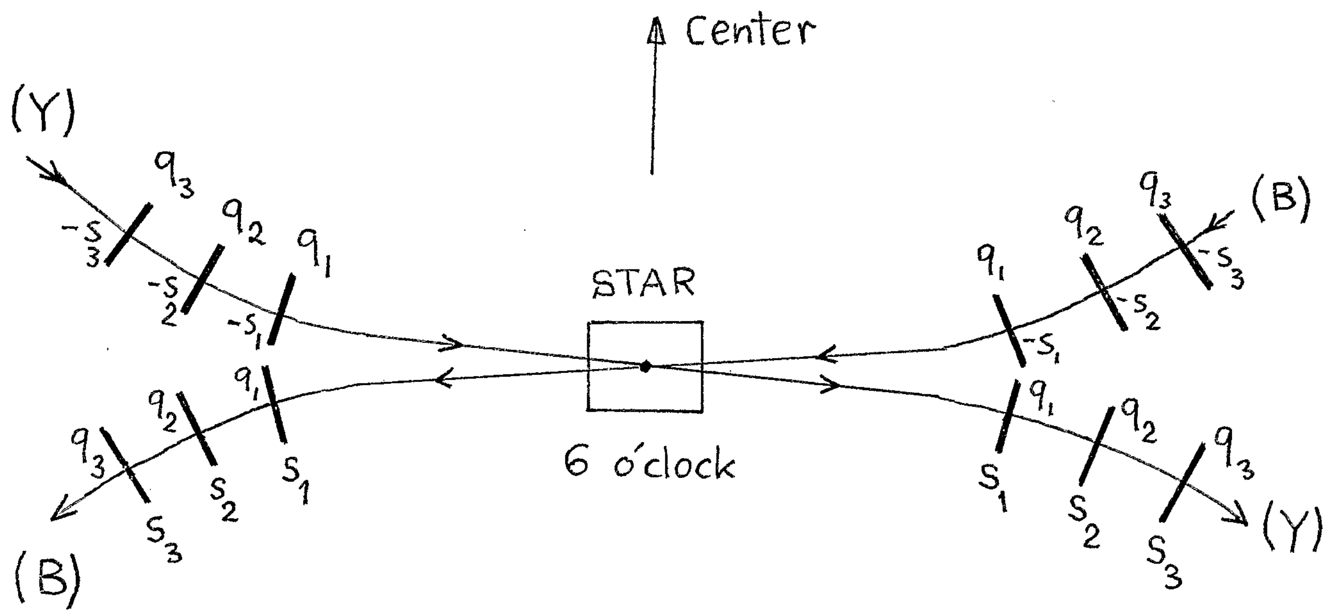

Fig. 6.: Schematic layout of the decoupling triplets, their positions and strengths, in both rings around 6 o'clock IP.

8. Decoupling of the STAR Detector Placed at 8 o'clock, (YELLOW RING)

If the STAR detector would be placed at 8 o'clock in RHIC rings instead of 6 o'clock, the decoupling scheme would need to be changed because of the anti-symmetric lattice. Namely, one has the relations between the β-functions at the neighboring locations

$$\beta_x^{(8)}(s_0+s) = \beta_y^{(6)}(s), \tag{8.1}$$

$$\beta_y^{(8)}(s_0+s) = \beta_x^{(6)}(s), \tag{8.2}$$

and

$$\beta_x^{(6)}(-s) = \beta_y^{(6)}(s), \tag{8.3}$$

$$\beta_y^{(6)}(-s) = \beta_x^{(6)}(s), \tag{8.4}$$

where s_0 is a distance between the 8 o'clock and 6 o'clock Interaction Points. One derives from it the relations (for s-small),

$$\Psi_x^{(a)}(-s) = -\Psi_y^{(a)}(s), \tag{8.5}$$

$$\Psi_y^{(a)}(-s) = -\Psi_x^{(a)}(s), \; a = 6, 8, \tag{8.6}$$

$$\Psi_x^{(8)}(s_0+s) = \Delta\Psi_x + \Psi_y^{(6)}(s), \tag{8.7}$$

$$\Psi_x^{(8)}(s_0+s) = \Delta\Psi_y + \Psi_x^{(6)}(s), \tag{8.8}$$

$$\Psi_x^{(8)}(s_0-s) = \Delta\Psi_x + \Psi_y^{(6)}(s), \tag{8.9}$$

$$\Psi_y^{(8)}(s_0-s) = \Delta\Psi_y + \Psi_x^{(6)}(s), \tag{8.10}$$

where, following the YELLOW RING, we have

$$\Delta\Psi_{x,y} = \int_{s_8}^{s_6} \frac{ds}{\beta_{x,y}^{(8)}(s)} = \int_{0}^{s_0} \frac{ds}{\beta_{x,y}^{(8)}(s)}. \tag{8.11}$$

This means that the plot of the phase-advances at 6 o'clock is as that at 8 o'clock with x and y labels interchanged. As the result the determinant Δ changes its sign

$$\Delta^{(8)} = -\Delta^{(6)} = -0.698, \tag{8.12}$$

and so the decoupling strengths as it is seen from the formula (5.12) - (5.14)

$$\begin{aligned} q_1^{(8)} &= -q_1^{(6)}, \\ q_2^{(8)} &= -q_2^{(6)}, \\ q_3^{(8)} &= -q_3^{(6)}. \end{aligned} \tag{8.13}$$

So they should be powered in opposite to those at 6 o'clock. This situation repeats when moving to the next IPs. For instance, for a hypothetical solenoid placed in every six Interaction Points of RHIC, the decoupling strengths would be

$$q_k^{(10)} = -q_k^{(8)} = q_k^{(6)} = -q_k^{(4)} = q_k^{(2)} = -q_k^{(0)}, \quad k = 1,2,3, \quad \left(q^{(0)} \equiv q^{(12)}\right). \tag{8.14}$$

Below, we list examples of the decoupling schemes used in various electron accelerators.[9]

Name	Number of skew-quad pairs	Comments
BERC	1	1/4 way round ring, with magic phases
CESR	4	2 partially rotated quad pairs, & 2 purely skew pairs
DORIS	0	antisolenoids inside experiment
LEP	4	complete correction
PETRA	0	~ global compensation between 3 experimental solenoids
TRISTAN	3	some difficulties at injection

High energy proton accelerators are not listed here since the X-Y linear coupling due to experimental solenoids is perhaps less significant than the non-linear coupling due to other sources like dipoles, for example. A global decoupling scheme is designed to reduce the X-Y coupling present in a machine, experimental solenoids including.

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