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The Stationary RF Bucket

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Introduction

As a result of intrabeam scattering, the bunches in RHIC will fill the entire bucket during the normal operation period. It follows that in analyzing the related rf problems the full nonlinear theory is required. In the pertinent literature, the nonlinear solutions were obtained by numerical integration.[1-3] For stationery buckets the integration can be carried out leading to analytical expressions.[4] In the present note, these expressions are rederived and used to produce numerical tables.

Synchrotron oscillation equations

The equations of synchrotron oscillations can be derived from the Hamiltonian

$$H(w,\phi) = |\eta| \frac{h^2\ \omega_\infty^2}{\gamma\ E_o A} \frac{w^2}{2} - \frac{Q\ eV}{2\pi h} \left\{ \cos\phi - \cos\phi_s + (\phi - \phi_s) \sin\phi_s \right\}$$

in the canonically conjugate energy variable w and position variable ϕ, which describe the motion around the synchronous particle (subscript s).

Following definitions apply:

$$w = \frac{\Delta E\ A}{h\ \beta\ \omega_\infty}$$

$$\omega_\infty = c/R_o$$

$$\omega_{rf} = h\ \beta\ \omega_\infty$$

$$\eta = \frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2}$$

and ΔE = energy deviation/amu

A = atomic mass unit

Q = charge state

E_o = rest mass/amu.

The stationary bucket is defined by $\sin\phi_s = 0$ leading to

$$H(w,\phi) = |\eta| \frac{h^2\ \omega_\infty^2}{\gamma\ E_o\ A} \frac{w^2}{2} + \frac{QeV}{\pi h} \sin^2\phi/2$$

The phase space trajectories are given by H = constant,

$$\left(\frac{w}{w_B}\right)^2 + \sin^2\phi/2 = N \qquad \text{for } 0 < N < 1$$

with the constant N known as the Cole-Morton parameter. The bucket half height (N = 1) is given by

$$w_B = \left(\frac{2}{\pi} \frac{\gamma E_o\ eV}{|\eta|\ \omega_\infty^2\ h^3} QA\right)^{1/2}$$

or, in terms of energy, by

$$\Delta_B = \left(\frac{\Delta E}{E}\right)_B = \left(\frac{2}{\pi} \frac{\beta^2\ eV}{h\ |\eta|\ \gamma\ E_o} \frac{Q}{A}\right)^{1/2}$$

The dimensions of a partial bucket $(0 < N < 1)$ are

Half-height $\quad w_N = \sqrt{N}\, w_B$

Half-width $\quad \phi_N = 2 \arcsin \sqrt{N}$

$$\approx 2\sqrt{N} \qquad \text{for } N \ll 1$$

Bunch half height and bucket half height are, therefore, related by

$$\Delta_E = \Delta_B \sin \phi_N/2$$

The stationary bucket factor

The phase space area/amu of a partial bucket is given by

$$A_N = \frac{4}{A} \int_0^{\phi_N} w \, d\phi$$

$$= \frac{4}{A} w_B \int_0^{\phi_N} (N - \sin^2\phi/2)^{1/2} \, d\phi$$

By means of the coordinate transformation

$$\sin\phi/2 = \sqrt{N}\, x$$

and the usual definition fo the stationary bucket factor

$$\alpha_N = \frac{A_N A}{8 w_B}$$

one finds

$$\alpha_N = N \int_0^1 \frac{(1-x^2)^{1/2}}{(1-Nx^2)^{1/2}} \, dx$$

$$= (N-1)\, K(N) + E(N)$$

with K(N) and E(N) the complete elliptical integrals of the first and second kind.

In the small-amplitude limit

$$\alpha_N \backsimeq \frac{\pi}{2} N \qquad\qquad N << 1$$

$$A_N \backsimeq 2 \pi N w_B = \pi w_N \phi_N$$

Furthermore $\lim_{N \to 1} \alpha_N = 1$

<u>Sychrotron oscillation frequency</u>

The period of one synchrotron oscillation

$$T = \frac{2\pi}{\Omega}$$

is obtained by integration of

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial w} = 2 \Omega_o \sqrt{N - \sin^2\phi/2}$$

where the small-amplitude synchrotron oscillation frequency is anticipated, with

$$\Omega_o = \omega_\infty \sqrt{\frac{|\eta| \; h \; eV \; Q}{2 \; \pi \; \gamma E_o A}}$$

From

$$\int_0^{T/4} dt = \frac{1}{2\Omega_o} \int_0^{\phi_N} \frac{d\phi}{\sqrt{N - \sin^2\phi/2}}$$

follows

$$\Omega_o/\Omega = \frac{2}{\pi} \int_0^1 (1-x^2)^{-1/2} \; (1-Nx^2)^{-1/2} \; dx$$

and finally

$$\Omega = \frac{\pi \, \Omega_o}{2 \, K(N)}$$

In the small amplitude approximation

$$\Omega \simeq \Omega_o \qquad \text{for } N << 1$$

as expected.

References

1. F.T. Cole & P.L. Morton, Report UCID 10130 (LRL 1964)

2. H. Koziol, Report CERN 67-29 (CERN 1967)

3. M.Q. Barton, CBA Technical Note 393 (BNL 1982)

4. J.C. Herrera, private communication.

N	HALF-HEIGHT (/Δ_B)	HALF-WIDTH (/π)	ALPHA	OMEGA (/Ω_o)
.0001	.01	.6366E-02	.7854E-04	1.000
.0004	.02	.1273E-01	.3142E-03	.9999
.0009	.03	.1910E-01	.7069E-03	.9998
.0016	.04	.2547E-01	.1257E-02	.9996
.0025	.05	.3184E-01	.1964E-02	.9994
.0036	.06	.3822E-01	.2829E-02	.9991
.0049	.07	.4460E-01	.3851E-02	.9988
.0064	.08	.5098E-01	.5031E-02	.9984
.0081	.09	.5737E-01	.6368E-02	.9980
.0100	.10	.6377E-01	.7864E-02	.9975
.0121	.11	.7017E-01	.9518E-02	.9970
.0144	.12	.7658E-01	.1133E-01	.9964
.0169	.13	.8300E-01	.1330E-01	.9958
.0196	.14	.8942E-01	.1543E-01	.9951
.0225	.15	.9585E-01	.1772E-01	.9943
.0256	.16	.1023	.2017E-01	.9935
.0289	.17	.1088	.2278E-01	.9927
.0324	.18	.1152	.2555E-01	.9918
.0361	.19	.1217	.2848E-01	.9909
.0400	.20	.1282	.3158E-01	.9899
.0441	.21	.1347	.3483E-01	.9888
.0484	.22	.1412	.3825E-01	.9877
.0529	.23	.1477	.4183E-01	.9865
.0576	.24	.1543	.4557E-01	.9853
.0625	.25	.1609	.4948E-01	.9841
.0676	.26	.1674	.5355E-01	.9827
.0729	.27	.1740	.5779E-01	.9813
.0784	.28	.1807	.6220E-01	.9799
.0841	.29	.1873	.6677E-01	.9784
.0900	.30	.1940	.7151E-01	.9768
.0961	.31	.2007	.7642E-01	.9752
.1024	.32	.2074	.8150E-01	.9735
.1089	.33	.2141	.8674E-01	.9718
.1156	.34	.2209	.9216E-01	.9700
.1225	.35	.2276	.9776E-01	.9681
.1296	.36	.2344	.1035	.9662
.1369	.37	.2413	.1095	.9642
.1444	.38	.2482	.1156	.9621
.1521	.39	.2550	.1219	.9600
.1600	.40	.2620	.1283	.9578
.1681	.41	.2689	.1350	.9555
.1764	.42	.2759	.1418	.9532
.1849	.43	.2830	.1488	.9508
.1936	.44	.2900	.1560	.9483
.2025	.45	.2972	.1634	.9458
.2116	.46	.3043	.1710	.9431
.2209	.47	.3115	.1787	.9404
.2304	.48	.3187	.1867	.9376
.2401	.49	.3260	.1948	.9348
.2500	.50	.3333	.2031	.9318

N	HALF-HEIGHT ($/\Delta_B$)	HALF-WIDTH ($/\pi$)	ALPHA	OMEGA ($/\Omega_o$)
.2601	.51	.3407	.2117	.9288
.2704	.52	.3481	.2204	.9256
.2809	.53	.3556	.2293	.9224
.2916	.54	.3632	.2385	.9191
.3025	.55	.3707	.2478	.9157
.3136	.56	.3784	.2573	.9122
.3249	.57	.3861	.2671	.9086
.3364	.58	.3939	.2770	.9049
.3481	.59	.4017	.2872	.9011
.3600	.60	.4097	.2976	.8972
.3721	.61	.4177	.3082	.8932
.3844	.62	.4257	.3191	.8890
.3969	.63	.4339	.3301	.8848
.4096	.64	.4421	.3414	.8804
.4225	.65	.4505	.3530	.8758
.4356	.66	.4589	.3647	.8712
.4489	.67	.4674	.3768	.8664
.4624	.68	.4760	.3890	.8614
.4761	.69	.4848	.4016	.8563
.4900	.70	.4936	.4144	.8511
.5041	.71	.5026	.4274	.8456
.5184	.72	.5117	.4407	.8400
.5329	.73	.5210	.4543	.8342
.5476	.74	.5303	.4682	.8282
.5625	.75	.5399	.4824	.8220
.5776	.76	.5496	.4969	.8155
.5929	.77	.5595	.5117	.8089
.6084	.78	.5696	.5268	.8019
.6241	.79	.5798	.5423	.7947
.6400	.80	.5903	.5580	.7872
.6561	.81	.6011	.5742	.7794
.6724	.82	.6121	.5907	.7713
.6889	.83	.6233	.6076	.7628
.7056	.84	.6349	.6249	.7538
.7225	.85	.6468	.6426	.7445
.7396	.86	.6591	.6608	.7346
.7569	.87	.6718	.6794	.7242
.7744	.88	.6849	.6985	.7131
.7921	.89	.6986	.7182	.7014
.8100	.90	.7129	.7384	.6888
.8281	.91	.7278	.7592	.6752
.8464	.92	.7436	.7808	.6605
.8649	.93	.7604	.8030	.6444
.8836	.94	.7784	.8261	.6266
.9025	.95	.7978	.8502	.6065
.9216	.96	.8193	.8754	.5833
.9409	.97	.8437	.9020	.5554
.9604	.98	.8725	.9305	.5200
.9801	.99	.9099	.9617	.4680
1.0000	1.00	1.000	1.000	0.