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Beta-Function Distortions Due to Linear Coupling

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1. Introduction

Since a coupling between transverse x and y degrees of freedom is expected in RHIC it is important to examine its influence on beta-functions.¹ We shall calculate the shifts of the beta-functions produced by point skew-quadrupoles distributed around the ring. The X - Y coupling is linear in this case² and its effect can be calculated exactly assuming that k -th skew-quadrupole of length ℓ_k is located at s_k in the ring and has strength q_k .

$$q_k = (\beta_x \beta_y)^{1/2} \frac{\ell_k}{\rho} a_{1|s=s_k}, \quad k = 1, \dots, N \quad (1.1)$$

where β_x, β_y are beta-functions of a perfect machine.³

It is known that in the presence of linear coupling there exists a matrix R such that in passing to new variables u, u', v, v' the transverse motions are decoupled i.e.,

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix} = R \begin{bmatrix} u \\ u' \\ v \\ v' \end{bmatrix}, \quad (1.2)$$

$$T = \begin{bmatrix} M & m \\ m & N \end{bmatrix} = R \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} R^{-1}. \quad (1.3)$$

Here T is a single turn 4×4 symplectic transfer matrix for a coupled machine and A, B are symplectic 2×2 submatrices describing uncoupled transverse motions

$$A = \begin{bmatrix} \cos \mu_1 + \alpha_1 \sin \mu_1 & \beta_1 \sin \mu_1 \\ -\gamma_1 \sin \mu_1 & \cos \mu_1 - \alpha_1 \sin \mu_1 \end{bmatrix}, \quad (1.4)$$

$$B = \begin{bmatrix} \cos \mu_2 + \alpha_2 \sin \mu_2 & \beta_2 \sin \mu_2 \\ -\gamma_2 \sin \mu_2 & \cos \mu_2 - \alpha_2 \sin \mu_2 \end{bmatrix}, \quad (1.5)$$

$\alpha_k, \beta_k, \gamma_k, \mu_k, k = 1, 2$ are the usual Courant-Snyder parameters of decoupled motions.

We would like to calculate the beta-function distortions

$$\Delta\beta_x = \beta_1 - \beta_x, \quad (1.6)$$

$$\Delta B_y = \beta_2 - \beta_y, \quad (1.7)$$

assuming that the linear coupling is small. This can be done using general formulae that express the A, B submatrices in terms of the T matrix

$$A = M + (t + \delta)^{-1} (\bar{m} + m) m, \quad (1.8)$$

$$B = N - (t + \delta)^{-1} (m + \bar{n}) n, \quad (1.9)$$

where

$$t = \frac{1}{2} \text{Tr} (M - N), \quad (1.10)$$

and

$$\delta = (t^2 + |\bar{m} + n|)^{1/2}. \quad (1.11)$$

Here \bar{m} stands for a symplectic conjugate of m

$$m = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \rightarrow \bar{m} = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}. \quad (1.12)$$

The single turn transfer matrix T can be written as a polynomial

$$T = \sum_{k=0}^N T^{(k)}, \quad (1.13)$$

where $T^{(k)}$ are given via, k -th order in the q 's, driving terms. In this note we shall calculate the beta-function distortions up to the second order in the q 's.

2. Calculations of the Beta-Function Distortions

According to the formula (1.8), (1.9) one has the relations

$$A_{12} = \beta_1 \sin \mu_1 = M_{12} + (t + \delta)^{-1} [(\bar{m} + n) m]_{12}, \quad (2.1)$$

$$B_{12} = \beta_2 \sin \mu_2 = N_{12} - (t + \delta)^{-1} [(m + \bar{n}) m]_{12}. \quad (2.2)$$

Taking into account the tune splitting

$$\mu_1 = \mu_x + \Delta\mu_1, \quad (2.3)$$

$$\mu_2 = \mu_y + \Delta\mu_2, \quad (2.4)$$

where $\Delta\mu_1, \Delta\mu_2$ are expressed through second-order driving terms, one gets from the expansion (1.13), assuming $\alpha_x(0) = \alpha_y(0) = 0$, the results

$$\begin{aligned} \frac{\Delta\beta_x}{\beta_x} = & \frac{1}{2} \left(d_{cs}^{(2)} - d_{sc}^{(2)} \right) \cot^2 \mu_x + \left[\frac{1}{2} \left(\check{d}_{cc}^{(2)} + \check{d}_{ss}^{(2)} \right) - d_{ss}^{(2)} \right] \cot \mu_x + \\ & + d_{cs}^{(2)} + (t + \delta)^{-1} \left\{ - \left[\left(d_{sc}^{(1)} \right)^2 + \left(d_{ss}^{(1)} \right)^2 \right] \sin \mu_y \cot \mu_x - \left(d_{sc}^{(2)} - d_{cs}^{(2)} \right) \cos \mu_y + \right. \\ & \left. + \left(d_{cc}^{(1)} d_{sc}^{(1)} + d_{ss}^{(1)} d_{cs}^{(1)} \right) \sin \mu_y \right\} + O(q^4), \end{aligned} \quad (2.5)$$

$$\frac{\Delta\beta_y}{\beta_y} = \frac{\Delta\beta_x}{\beta_x} \Big|_{x \leftrightarrow y} \quad (2.6)$$

Here the driving terms of the first order $d^{(1)}$ and of the second order $d^{(2)}$ are defined as follows:

$$\begin{bmatrix} d_{ss}^{(1)} \\ d_{sc}^{(1)} \\ d_{cs}^{(1)} \\ d_{cc}^{(1)} \end{bmatrix} = \sum_{r=1}^N q_r \begin{bmatrix} \sin \mu_x^r \sin \mu_y^r \\ \sin \mu_x^r \cos \mu_y^r \\ \cos \mu_x^r \sin \mu_y^r \\ \cos \mu_x^r \cos \mu_y^r \end{bmatrix}, \quad (2.7)$$

and

$$\begin{bmatrix} d_{ss}^{(2)} \\ d_{sc}^{(2)} \\ d_{cs}^{(2)} \\ d_{cc}^{(2)} \end{bmatrix} = \sum_{1 \leq r < s \leq N} q_r q_s \sin(\mu_y^s - \mu_y^r) \begin{bmatrix} \sin \mu_x^s \sin \mu_x^r \\ \sin \mu_x^s \cos \mu_x^r \\ \cos \mu_x^s \sin \mu_x^r \\ \cos \mu_x^s \cos \mu_x^r \end{bmatrix}, \quad (2.8)$$

and μ_x^r, μ_y^r are phase advances

$$\mu_x^r = \int_0^{S_r} \frac{ds}{\beta_x}, \quad \mu_y^r = \int_0^{S_r} \frac{ds}{\beta_y}. \quad (2.9)$$

Additional sets of driving terms, denoted as $\check{d}_{ss}^{(1)}$, $\check{d}_{ss}^{(2)}$ etc, are obtained from the above equations by simply exchanging x and y ,

$$\check{d}^{(k)}(x, y) = d^{(k)}(y, x), k = 1, 2 \quad (2.10)$$

It is easy to notice the relations

$$\begin{aligned} \check{d}_{ss}^{(1)} &= d_{ss}^{(1)}, \\ \check{d}_{sc}^{(1)} &= d_{cs}^{(1)}, \\ \check{d}_{cs}^{(1)} &= d_{sc}^{(1)}, \\ \check{d}_{cc}^{(1)} &= d_{cc}^{(1)}. \end{aligned} \quad (2.11)$$

It is interesting to check if the beta-function distortions disappear after correction of the tune-splitting which requires, among others, that the following equalities hold

$$\begin{aligned} d_{cc}^{(2)} - d_{ss}^{(2)} &= 0, \\ \check{d}_{cc}^{(2)} - \check{d}_{ss}^{(2)} &= 0, \\ d_{cs}^{(2)} - d_{sc}^{(2)} &= 0, \\ \check{d}_{cs}^{(2)} - \check{d}_{sc}^{(2)} &= 0. \end{aligned} \quad (2.12)$$

Applying them to the formula (2.5), (2.6) one finds that residual beta-function distortions are present

$$\begin{aligned} \frac{\Delta\beta_x}{\beta_x} &= -d_{ss}^{(2)} \cot \mu_x + d_{cs}^{(2)} + (t + \delta)_{|\Delta\nu=0}^{-1} \left\{ - \left[\left(d_{sc}^{(1)} \right)^2 + \left(d_{ss}^{(1)} \right)^2 \right] \sin \mu_y \cot \mu_x + \right. \\ &\quad \left. + \left(d_{cc}^{(1)} d_{sc}^{(1)} + d_{ss}^{(1)} d_{cs}^{(1)} \right) \sin \mu_y \right\} + O(q^4), \end{aligned} \quad (2.13)$$

$$\frac{\Delta\beta_y}{\beta_y} \Big|_{\Delta\nu=0} = \frac{\Delta\beta_x}{\beta_x} \Big|_{\Delta\nu=0, x \leftrightarrow y}, \quad (2.14)$$

and, according to the formula (1.10), (1.11)

$$t + \delta \Big|_{\Delta\nu=0} = 2 (\cos \mu_x - \cos \mu_y) + O(q^4). \quad (2.15)$$

One sees that passing to the limit $\nu_x - \nu_y \rightarrow 0$ is delicate here, and higher order terms in the last expansion should be included. This is essential since RHIC is designed to operate at almost equal tunes: $\nu_x = 28.826$, $\nu_y = 28.821$.

Our results should be compared with perturbative calculations of the beta-function distortions.^{4,5} Clearly, the residual beta-function distortions can be removed if we correct driving terms which appear in the formula (2.13), (2.14).

Assuming that skew-quadrupole errors are randomly distributed around the ring and taking into account that

$$N \langle q^2 \rangle = G_0^2 \quad (2.16)$$

where, for RHIC we take

$$G_0 \simeq 0.25, \quad (2.17)$$

we get the estimate for the average distortion

$$\langle \frac{\Delta\beta_x}{\beta_x} \rangle \simeq -0.25, \quad (2.18)$$

and similar for $\langle \frac{\Delta\beta_y}{\beta_y} \rangle$.

Even larger beta-function distortions are expected in SSC in which case $G_0 \simeq 0.5$ which yields for the average beta-function distortion ≈ -1.2 .

3. References

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