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Space Charge Coupling Impedance for a Straight and Toroidal Vacuum Chamber

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SPACE CHARGE COUPLING IMPEDANCE FOR A STRAIGHT AND TOROIDAL VACUUM CHAMBER

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We like to calculate the fields generated by a beam of charged particles moving along the axis of two different cavity structures. These are shown in Figure 1 a and b.

The first structure is made of an infinitely long straight vacuum chamber of circular cross-section of radius b. The second structure is a toroidal vacuum chamber of rectangular cross-section of height h and width $w = R_o - R_i$, where R_o and R_i are respectively the outer and inner radius of the vacuum chamber.

Both of these structures can be represented with the same cylindrical coordinates (r, θ, z) but with different orientation. The beam has zero transverse dimension, moves with the velocity v, parallel to the z-axis in the case of the straight pipe at a distance $r = a$, and along the azimuth θ in the toroidal vacuum chamber at the radius R

between R_i and R_o and at the elevation $z = z_0$.

The following charge density is associated to the beam.

For the straight pipe:

$$\rho = Ne\, g(z - vt)\, \delta(\theta)\, \frac{\delta(r-a)}{a} \qquad (1)$$

where $g(z)$ is the longitudinal charge distribution which we assume here to have periodicity $2\pi R$ and

$$\int_0^{2\pi R} g(z)\, dz = 1$$

N is the total number of particles in a period.

For the toroidal vacuum chamber:

$$\rho = Ne\, g(\theta - \omega_0 t)\, \delta(z - z_0)\, \frac{\delta(r-R)}{R} \qquad (2)$$

where $g(\theta)$ is the azimuthal charge distribution with period 2π and

$$\int_0^{2\pi} g(\theta)\, d\theta = 1$$

N again is the total number of circulating particles and $\omega_0 = v/R$ the angular frequency.

If the charge is conserved, the equation of continuity holds

$$\operatorname{div} \vec{j} + \frac{1}{c} \frac{\partial \rho}{\partial t} = 0$$

where $\vec{j}$ is the current density. For the straight pipe

$$\vec{j} \equiv (0, 0, j)$$

is directed along the z-axis. For the toroidal vacuum chamber

$$\vec{j} \equiv (0, j, 0)$$

is directed along the azimuth. In any case

$$j = \beta c \rho \qquad (3)$$

with $v = \beta c$.

Because of the periodicity involved

$$R\, g(z - vt) = \sum_n g_n\, e^{in(z/R - \omega_0 t)} \qquad (4)$$

and

$$g(\theta - \omega_0 t) = \sum_n g_n\, e^{in(\theta - \omega_0 t)}\,. \qquad (5)$$

In the following we shall retain only one Fourier harmonic with amplitude g_n at the harmonic number n. We will be satisfied to calculate the corresponding fields generated.

We shall also assume that the vacuum chamber is made of perfectly conductive walls in both cases. This will require the following boundary conditions: the electric field tangential components and the magnetic field perpendicular components have to vanish identically at the vacuum chamber walls.

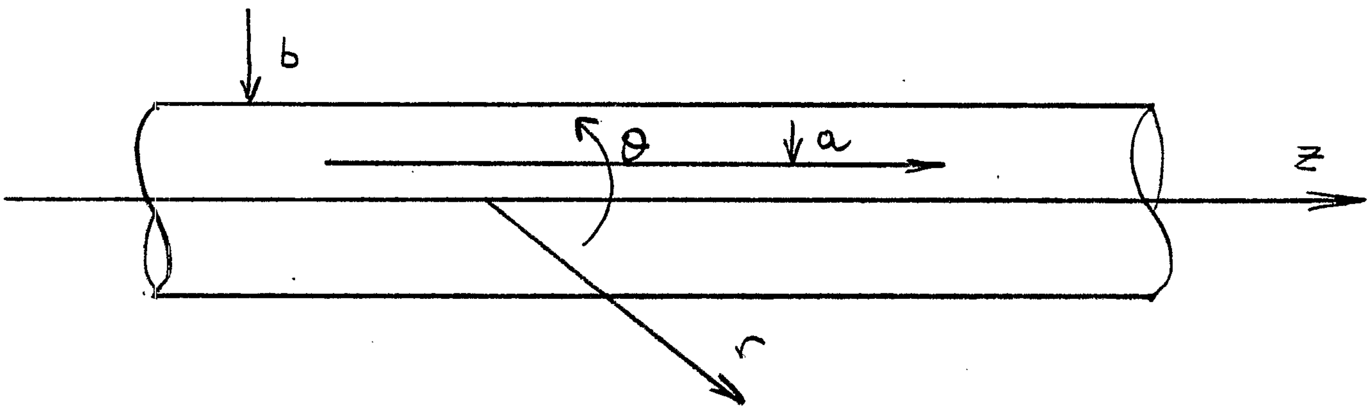


Fig. 1a Straight Vacuum Chamber

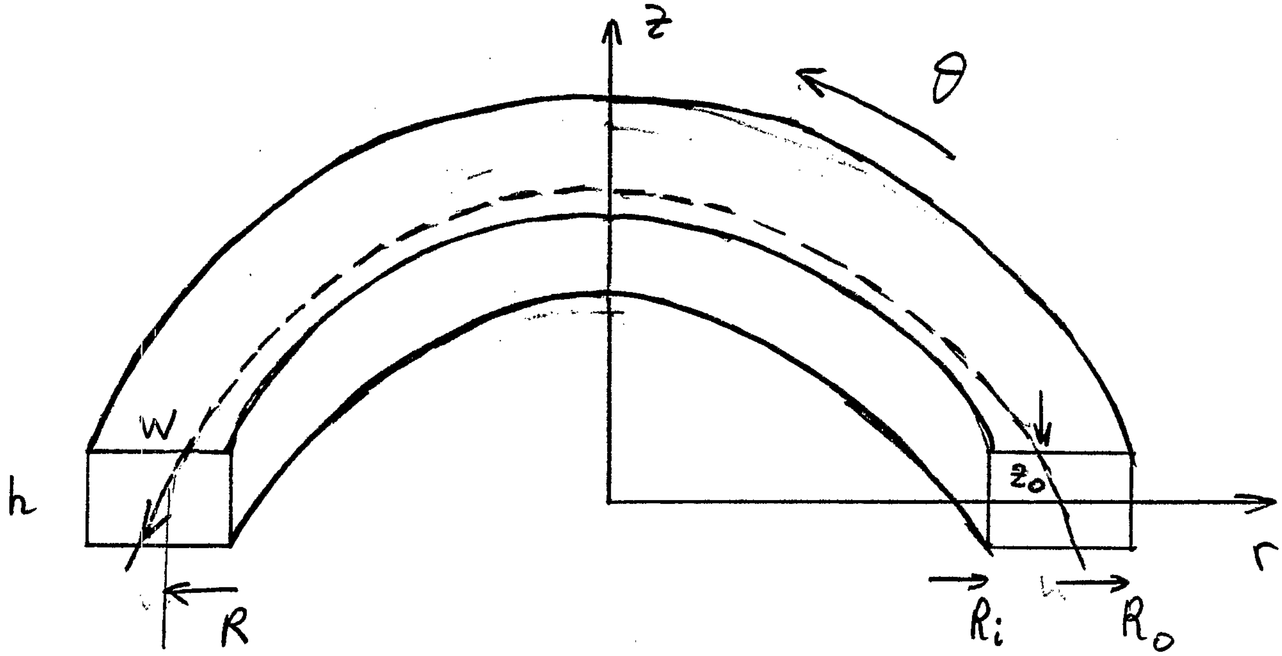


Fig. 1b Toroidal Vacuum Chamber

The fields generated by the beam are given by the scalar potential V and the vector potential $\vec{A}$ satisfying respectively the following wave equations

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -4\pi\rho \qquad (6)$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -4\pi\frac{\vec{J}}{c} \qquad (7)$$

and the Lorentz condition

$$\operatorname{div}\vec{A} + \frac{1}{c}\frac{\partial V}{\partial t} = 0 \qquad (8)$$

the fields are given by

$$\vec{E} = -\operatorname{grad} V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad (9)$$

$$\vec{B} = \operatorname{rot}\vec{A} \qquad (10)$$

Since the scalar potential V can be obtained from the Lorentz condition (8), it is enough to solve eq. (7) for $\vec{A}$ and to express the fields in terms of $\vec{A}$ alone -

We will adopt the following method - We first solve the omogeneous equation

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0$$

The general solution can be expressed as the sum of inifinite resonating modes - If λ denotes one of these modes, the corresponding resonating frequency is ω_λ and the vector potential is $\vec{A}_\lambda$, so that

$$\vec{A}_\lambda = \vec{a}_\lambda (r, \theta, z)\, e^{-i\omega_\lambda t}$$

and

$$\nabla^2 \vec{a}_\lambda + \frac{\omega_\lambda^2}{c^2} \vec{a}_\lambda = 0 \qquad (11)$$

where the amplitude $\vec{a}_\lambda$ of each mode satisfies the boundary conditions at the walls of the vacuum chamber -

Given the nature of the cylindrical coordinates used, it will be seen that each mode can be decomposed in two independent vectors, normal to each other

$$\vec{a}_\lambda \equiv (\vec{a}_{\perp\lambda}, \vec{a}_{z\lambda}) \qquad (12)$$

where $\vec{a}_{z\lambda}$ is directed along the z-axis and $\vec{a}_{\perp\lambda}$ is perpendicular to it. We will see that the beam can excite only one of these vectors but not the other. Which vector is excited depends on the shape of the vacuum chamber: straight or toroidal.

The field generated by the beam can then be obtained by superimposition of all the modes

$$\vec{A} = \sum_\lambda Q_\lambda(t)\, \vec{a}_\lambda(r, \theta, z) \qquad (13)$$

which we can invert in eq. (7). Taking into account also eq. (11), that gives

$$\sum_\lambda (\ddot{Q}_\lambda + \omega_\lambda^2 Q_\lambda)\, \vec{a}_\lambda = 4\pi c \vec{J} \qquad (14)$$

Let us multiply to the right both sides of eq. (14) by $\vec{a}_\mu^*$, which is the complex conjugate of $\vec{a}_\mu$, and integrate both sides over the volume surrounded by the vacuum chamber. In the case of the straight pipe, the volume can extend over one period $2\pi R$ length of the vacuum chamber. We have

$$\sum_\lambda \left(\ddot{Q}_\lambda + \omega_\lambda^2 Q_\lambda\right) \int \vec{a}_\lambda \vec{a}_\mu^* \, d\tau = \tag{15}$$

$$= 4\pi c \int \vec{J} \vec{a}_\mu^* \, d\tau$$

It can also be seen that the resonating modes described by the functions $\vec{a}_\lambda$ form a set of normal modes that, is

$$\int \vec{a}_\lambda \vec{a}_\mu^* \, d\tau = 0 \tag{16}$$

if $\lambda \neq \mu$, that is the two modes are different and

$$\int |\vec{a}_\lambda|^2 \, d\tau = I_\lambda \qquad (17)$$

with I_λ an arbitrary normalization constant which eventually will disappear from our final results. Thus eq. (15) reduces to

$$\ddot{Q}_\mu + \omega_\mu^2 Q_\mu = \frac{4\pi c}{I_\mu} \int \vec{j}\, \vec{a}_\mu^{\,*} \, d\tau \qquad (18)$$

Let us study the right-hand side of this equation for the two geometries shown in Figure 1a and b.

For the straight pipe we have seen that $\vec{j}$ is directed only along the z-axis; thus the beam can excite only the axial vectors $\vec{a}_{z\mu}$

$$\vec{j}\, \vec{a}_\mu^{\,*} = \vec{j}\, \vec{a}_{z\mu}^{\,*} = j\, a_{z\mu}^*$$

In this case we could also set $\vec{a}_{\perp\mu} = 0$ for any mode μ and replace (17) with

$$\int |a_{z\mu}|^2 \, d\tau = I_\mu \qquad (19)$$

From (1), (3) and (4) we have

$$\int \vec{j}\, \vec{a}_{\mu}^{\,*}\, d\tau =$$

$$= Ne\, \omega_0 g_n\, e^{-in\omega_0 t} \int e^{inz/R}\, a^{*}_{z_\mu}(a, 0, z)\, dz$$

The solution of (18) is

$$Q_\mu = \bar{Q}_\mu\, e^{-in\omega_0 t} \qquad (20)$$

with

$$\bar{Q}_\mu = \frac{4\pi c\, Ne\, \omega_0\, g_n\, F_\mu}{I_\mu\, (\omega_\mu^2 - n^2 \omega_0^2)} \qquad (21)$$

and

$$F_\mu = \int e^{inz/R}\, a^{*}_{z_\mu}(a, 0, z)\, dz \qquad (22)$$

For the case of the toroidal vacuum chamber, $\vec{j}$ is perpendicular to the z-axis, and the beam can excite only the transverse vectors $\vec{a}_{\perp}$

$$\vec{j}\,\vec{a}^{*}_{\mu} = \vec{j}\,\vec{a}^{*}_{\perp\mu} = j\,a^{*}_{\theta_\mu} .$$

We can also set $\vec{a}_{z_\mu} = 0$ for any mode μ and replace (17) with

$$\int |\vec{a}_{\perp\mu}|^2 \, d\tau = I_\mu \, . \qquad (23)$$

From (2), (3) and (5) we have

$$\int \vec{j}\,\vec{a}^{*}_{\mu}\, d\tau =$$

$$= N e \omega_0 g_\mu F_\mu e^{-in\omega_0 t}$$

where

$$F_\mu = R \int e^{in\theta} a^{*}_{\theta_\mu}(R, \theta, z_0)\, d\theta \qquad (24)$$

The solution of eq. (18) is then also given by eq-s (20 and 21) -

Straight Pipe

The solution to eq. (11) can be written as

$$\vec{a}_\lambda = \vec{f}_\lambda(r)\, e^{im\theta}\, e^{ikz}$$

where k is the propagation constant along the z-axis which is also the direction of motion of the beam. We have

$$\nabla^2 a_{r_\lambda} - \frac{a_{r_\lambda}}{r^2} - \frac{2}{r^2}\frac{\partial a_{\theta_\lambda}}{\partial \theta} + \frac{\omega_\lambda^2}{c^2} a_{r_\lambda} = 0 \qquad (25)$$

$$\nabla^2 a_{\theta_\lambda} - \frac{a_{\theta_\lambda}}{r^2} + \frac{2}{r^2}\frac{\partial a_{r_\lambda}}{\partial \theta} + \frac{\omega_\lambda^2}{c^2} a_{\theta_\lambda} = 0 \qquad (26)$$

and

$$\nabla^2 a_{z_\lambda} + \frac{\omega_\lambda^2}{c^2} a_{z_\lambda} = 0 \qquad (27)$$

We see that a_{r_λ} and a_{θ_λ} are coupled to each other through eq.s (25 and 26), whereas a_{z_λ} is independent from the other components. For this case we set $a_{r_\lambda} = a_{\theta_\lambda} = 0$ and concentrate on the solution of eq. (27), which we can also write as

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{df_{z\lambda}}{dr}\right) - \frac{m^2}{r^2} f_{z\lambda} - k^2 f_{z\lambda} + \frac{\omega_\lambda^2}{c^2} f_{z\lambda} = 0 \qquad (28)$$

If we define

$$q^2 = \frac{\omega_\lambda^2}{c^2} - k^2 \qquad (29)$$

the solution of this equation is a linear combination of the two Bessel functions $J_m(qr)$ and $N_m(qr)$. Since we demand the potentials and the fields to be finite in the centre $r=0$ of the pipe, the dependence on $N_m(qr)$ has to vanish, so that

$$f_{z\lambda} = D_0\, J_m(qr) \qquad (30)$$

where D_0 is an arbitrary constant.

Similarly we can write for the scalar potential

$$V_\lambda = U_\lambda\, e^{-i\omega_\lambda t} \qquad (31)$$

and

$$U_\lambda = u_\lambda(r)\, e^{im\theta}\, e^{ikz} \qquad (32)$$

and $u_\lambda(r)$ satisfies an equation like (28). From the Lorentz condition (8)

$$U_\lambda = -i \frac{c}{\omega_\lambda} \operatorname{div} \vec{a}_\lambda$$

$$= \frac{kc}{\omega_\lambda} a_{z_\lambda} \qquad (33)$$

so that

$$u_\lambda(r) = D_0 \frac{kc}{\omega_\lambda} J_m(qr) \qquad (34)$$

The boundary conditions to be satisfied for this case are

$$E_\theta = E_z = B_r = 0 \quad \text{at} \quad r = b$$

which are automatically satisfied if

$$J_m(qb) = 0 \qquad (35)$$

that is $qb = j_{ms}$ where j_{ms} is the s-th zero of the Bessel function of order m.

Insertion in (29) gives for the resonating frequency of the λ-th mode

$$\omega_\lambda = c\sqrt{k^2 + j_{ms}^2/b^2} \qquad (36)$$

Thus, each mode is described by a longitudinal propagation constant k and two mode numbers (m and s). The radial propagation constant is then given from (35) and the resonating frequency by (36). Each mode is then completely described apart from the constant D_0 which clearly will depend on k, m and s.

The normalization condition (19) now becomes

$$I_\mu = |D_0|^2 \int_0^b \left[J_m(qr)\right]^2 r\,dr \cdot 4\pi^2 R$$

$$= 2\pi^2 R b^2 |D_0|^2 \left[J_{m+1}(j_{ms})\right]^2 \qquad (37)$$

which relates D_0 to I_μ.

The form factor F_μ given by (22) becomes

$$F_\mu = D_0^* J_m(qa) \int e^{in\frac{z}{R}} e^{-ikz} dz$$

where D_0^* is the complex conjugate of D_0. If we require that the potential $a_{z\mu}$ has also periodicity $2\pi R$ in z, then the integral above is always zero unless

$$K = n/R \qquad (38)$$

in which case

$$F_\mu = 2\pi R\, D_0^* J_m(qa) \qquad (39)$$

Thus, given the excitation by the beam, the longitudinal propagation constant is determined and given by (38). Consequently the resonating frequency now becomes

$$\omega_\lambda = c\sqrt{n^2/R^2 + j_{ms}^2/b^2} \qquad (40)$$

The λ-th mode is described only by the indeces m and s. In the particular case the beam moves on the axis of the pipe, that is $a = 0$, then it is seen that only the modes with $m = 0$ can be excited and

$$\omega_\lambda = c \sqrt{n^2/R^2 + j_{os}^2/b^2} \tag{41}$$

$$F_\lambda = 2\pi R D_o^* \tag{42}$$

and j_{os} is the s-th zero of $J_o(z)$.

In the following we shall consider only this case.

Toroidal Vacuum Chamber

Also in this case eq.s (25, 26 and 27) hold, but here we shall set $a_{z_\lambda} = 0$ and concentrate on the solution of (25) and (26) -

It is convenient to write the solution in the following form

$$a_{r_\lambda} = f_{r_\lambda}(r)\, e^{i\bar{n}\theta} \left(D_1 \cos kz + D_2 \sin kz \right) \qquad (43)$$

$$a_{\theta_\lambda} = f_{\theta_\lambda}(r)\, e^{i\bar{n}\theta} \left(D_1 \cos kz + D_2 \sin kz \right) \qquad (44)$$

where D_1 and D_2 are two arbitrary constants -
Insertion of (43) and (44) in (25) and (26) gives

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_{r_\lambda}}{dr}\right) - \frac{\bar{n}^2+1}{r^2} f_{r_\lambda} + q^2 f_{r_\lambda} - 2i\frac{\bar{n}}{r^2} f_{\theta_\lambda} = 0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{f_{\theta_\lambda}}{dr}\right) - \frac{\bar{n}^2+1}{r^2} f_{\theta_\lambda} + q^2 f_{\theta_\lambda} + 2i\frac{\bar{n}}{r^2} f_{r_\lambda} = 0$$

and the solutions of these equations are:

$$f_{r_\lambda} = \frac{1}{q}\frac{df_\lambda}{dr} \qquad (45)$$

$$f_{\theta_\lambda} = i\frac{\bar{n}}{qr} f_\lambda \qquad (46)$$

with

$$f_\lambda(r) = N_{\bar{n}}(qR_i)\, J_{\bar{n}}(qr) - J_{\bar{n}}(qR_i)\, N_{\bar{n}}(qr) \qquad (47)$$

and q is still given by (29).

For the scalar potential we can make use of (31) and

$$U_\lambda = u_\lambda(r)\, e^{i\bar{n}\theta}\left(D_1 \cos Kz + D_2 \sin Kz\right). \qquad (48)$$

From the Lorentz condition (8)

$$U_\lambda = -i\frac{c}{\omega_\lambda} \operatorname{div} \vec{a}_\lambda$$

and

$$u_\lambda = i\frac{qc}{\omega_\lambda} f_\lambda \qquad (49)$$

There are two sets of boundary conditions to be satisfied:

(1) $E_r = E_\theta = B_z = 0$ at $z = \pm h/2$

(2) $E_\theta = E_z = B_r = 0$ at $R = R_i$ and $R = R_o$

The first set is satisfied if

$$K = \frac{\pi m}{h} \qquad (50)$$

and either $D_1 = 0$ and m is an even integer or $D_2 = 0$ and m is an odd integer. The second set of boundary conditions is satisfied if

$$f_\lambda (R_i) = f_\lambda (R_o) = 0$$

Since from (47) it is seen that indeed $f_\lambda (R_i) = 0$, we see that the radial propagation constant q is given by the zeros of the equation

$$N_{\bar{n}} (q R_i) J_{\bar{n}} (q R_o) - J_{\bar{n}} (q R_i) N_{\bar{n}} (q R_o) = 0 \qquad (51)$$

We will denote with q_s the s-th zero. Thus from (29) we see that the resonating frequency of the λ-th mode is

$$\omega_\lambda = c\sqrt{\frac{\pi^2 m^2}{h^2} + q_s^2} \tag{52}$$

Thus the λ-th mode is described by three mode numbers m, s and $\bar{n}$. The first two numbers in turn define the propagation constants K and q_s with (50) and (51). The resonating frequency is then derived with (52). Each mode is then completely described apart from a constant D_1 or D_2, since one of them has to be zero, which clearly depend on m, s and $\bar{n}$.

Let us denote with D_0 either D_1 or D_2, depending on which of the two is not zero, then the normalization integral (23) can be derived from (43), (44) and (45), (46)

$$I_\mu = \pi h |D_0|^2 \int_{R_i}^{R_0} \left[\left(\frac{df_\lambda}{q\,dr}\right)^2 + \left(\frac{\bar{n} f_\lambda}{q r}\right)^2 \right] r\,dr$$

If $f_n(\zeta)$ denotes a linear combination of Bessel functions of order n it is well known that the following relationships are satisfied

$$2 \frac{n}{\zeta} f_n = f_{n-1} + f_{n+1}$$

$$2 \frac{df_n}{d\zeta} = f_{n-1} - f_{n+1}$$

From (47) it is seen that $f_\lambda(r) = f_{\bar{n}}(qr)$. Thus

$$I_\mu = \frac{\pi h}{2} |D_0|^2 \int_{R_i}^{R_0} \left[f_{\bar{n}-1}^2(qr) + f_{\bar{n}+1}^2(qr) \right] r\,dr$$

To solve this integral see the Handbook of Mathematical Functions, edited by Abramowitz and Stegun, AMS 55, National Bureau of Standards, 9th printing, Nov. 1970, page 485. We obtain

$$I_\mu = \frac{\pi h}{4} |D_0|^2 \left| r^2 \left(f_{\bar{n}-1}^2 + f_{\bar{n}+1}^2 \right) \right|_{R_i}^{R_0}$$

At $r = R_i$ and $r = R_o$ the following relations hold

$$f_\lambda = f_{\bar n} = 0$$

$$f_{\bar n - 1} = - f_{\bar n + 1}$$

$$f_{\bar n + 1} = - \frac{1}{q} \frac{d f_{\bar n}}{dr}$$

so that

$$I_\mu = \frac{\pi h}{2} |D_o|^2 \left| r^2 f^2_{\bar n + 1} \right|_{R_i}^{R_o}$$

which with (47) and (51) becomes

$$I_\mu = \frac{2h}{\pi q^2} |D_o|^2 \left\{ \left[\frac{J_n(qR_i)}{J_n(qR_o)} \right]^2 - 1 \right\} \tag{53}$$

The form factor F_μ given by (24) with (44) and (46) is

$$F_\mu = - \frac{i \bar n}{q} f_\mu(qR) \left(D_1^* \cos kz_o + D_2^* \sin kz_o \right) \int e^{in\theta} e^{-i\bar n\theta} d\theta$$

which is always zero unless $\bar{n} = n$ in which case

$$F_\mu = -2\pi i \frac{n}{q} f_\mu (qR) \left(D_1^* \cos kz_0 + D_2^* \sin kz_0 \right) \qquad (54)$$

As a particular case $z_0 = 0$, that is the beam is located on the middle plane of the vacuum chamber, in which case

$$F_\mu = -2\pi i \frac{n}{q} f_\mu (qR) \, D_1^* \qquad (55)$$

and the beam can excite only those modes with odd integer m. In the following we shall consider only this case.

The total vector potential is given by (13).
Insertion of (20) and (21) gives

$$\vec{A} = 4\pi c N e \omega_0 g_n \, e^{-in\omega_0 t} \sum_\lambda \frac{F_\lambda \, \vec{a}_\lambda(r,\theta,z)}{I_\lambda(\omega_\lambda^2 - n^2\omega_0^2)} \qquad (56$$

For the straight pipe case, with the help of (37), (41) and (42)

$$A_z = \frac{4 N e \omega_0 g_n}{c b^2} \, e^{in\left(\frac{z}{R} - \omega_0 t\right)} \sum_s \frac{J_0(j_{0s}\, r/b)}{\left[J_1(j_{0s})\right]^2 \left(\frac{n^2}{\gamma^2 R^2} + \frac{j_{0s}^2}{b^2}\right)} \qquad (5$$

We have taken into account that $\omega_0 = \beta c/R$ and set $\gamma^2 = 1/(1-\beta^2)$.

For the toroidal vacuum chamber, we insert (52), (53) and (55) in eq. (56). We obtain for the azimuthal component of the vector potential

$$A_\theta = \frac{4\pi^3 n^2 N e \omega_0 g_n}{hc} \sum_{s,m} \frac{\left[f_n(q_s R)\, f_n(q_s r)/r\right] e^{in(\theta-\omega_0 t)}}{G_n(q_s R_i, q_s R_o)\left(\frac{\pi^2 m^2}{h^2} + q_s^2 - \frac{n^2}{R^2}\beta^2\right)} \tag{58}$$

where

$$G_n(z_i, z_o) = \left[\frac{J_n(z_i)}{J_n(z_o)}\right]^2 - 1 \tag{59}$$

The force acting on a particle of charge e in the beam, along the main direction of the beam, is

$-E_z$ in the case of the straight pipe and

$-E_\theta$ in the case of the toroidal vacuum chamber.

Again for the case of a toroidal vacuum chamber

$$A_r = -i\,\frac{4\pi^3 n N e \omega_0 g_n}{hc}\, e^{in(\theta-\omega_0 t)} \times$$

$$\times \sum_{s,m} \frac{f_n(q_s R)\; d f_n(q_s r)/dr}{G_n(q_s R_i, q_s R_o)\left(\frac{\pi^2 m^2}{h^2} + q_s^2 - \frac{n^2}{R^2}\beta^2\right)} \tag{60}$$

To satisfy the Lorentz condition (8) we add a scalar potential V. For the case of the straight pipe this is given simply by

$$A_z = \beta V \tag{61}$$

For the toroidal vacuum chamber we have

$$V = 4\pi^3 \, N e g_n \frac{e^{in(\theta - \omega_0 t)}}{h} \times$$

$$\times \sum_{s,m} \frac{f_n(q_s R)\, f_n(q_s r)\, q_s^2}{G_n(q_s R_i, q_s R_o)\left(\frac{\pi^2 m^2}{h^2} + q_s^2 - \frac{n^2}{R^2}\beta^2\right)} \tag{62}$$

In the following we shall estimate the fields on the beam axis, that is at $r = 0$ for the straight pipe and at $r = R$, $z = 0$ for the toroidal vacuum chamber.

For the straight pipe we have

$$E_z = -\frac{\partial V}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t}$$

From (57) and (61)

$$E_z = -i \frac{n A_z}{\beta \gamma^2 R} \tag{63}$$

and A_z is given by (57).

For the toroidal vacuum chamber

$$E_\theta = -\frac{1}{R}\frac{\partial V}{\partial \theta} - \frac{1}{c}\frac{\partial A_\theta}{\partial t}$$

From (58) and (62)

$$E_\theta = -in \frac{1}{R}\left(V - \beta A_\theta\right)$$

$$= -4\pi^3 i\, n\, Ne\, g_n \frac{e^{in(\theta - \omega_0 t)}}{hR} \times$$

$$\times \sum_{s,m} \frac{f_n^2(q_s R)\left(q_s^2 - \beta^2 n^2/R^2\right)}{G_n(q_s R_i, q_s R_o)\left(\frac{\pi^2 m^2}{h^2} + q_s^2 - \beta^2 \frac{n^2}{R^2}\right)} \tag{64}$$

We can finally derive the longitudinal Coupling Impedance. For the straight pipe this is

$$Z_n = - \frac{2\pi R\, E_z}{N e \omega_0 g_n \exp\left[in\left(\frac{z}{R} - \omega_0 t\right)\right]} \tag{65}$$

and for the toroidal vacuum chamber

$$Z_n = - \frac{2\pi R\, E_\theta}{N e \omega_0 g_n \exp\left[in\left(\theta - \omega_0 t\right)\right]} \tag{66}$$

Insertion of (63) in (65) with (57) gives for the straight pipe

$$Z_n = \frac{2\pi i n}{c \beta \gamma^2} S_n \tag{67}$$

with

$$S_n = 4 \sum_{s=1}^{\infty} \frac{1}{\left(\frac{n^2 b^2}{\gamma^2 R^2} + j_{0s}^2\right)\left[J_1(j_{0s})\right]^2} \tag{68}$$

Insertion of (64) in (66) gives for the toroidal vacuum chamber

$$Z_n = in \frac{4\pi}{\beta c} \cdot T_n \tag{69}$$

where

$$T_n = \sum_{s,m} \frac{f_n^2(q_s R)\left(q_s^2 R^2 - \beta^2 n^2\right) 2\pi^3 R/h}{G_n(q_s R_i, q_s R_o)\left(q_s^2 R^2 + \pi^2 m^2 \frac{R^2}{h^2} - \beta^2 n^2\right)} \tag{70}$$

and f_n is given by (47) and G_n by (59).

So far we have used the rationalized c.g.s. system of units. To convert to the M.K.S. system we multiply the r.h. side of (67) and (69) by $Z_0 c/4\pi$ where $Z_0 = 377$ ohm. Thus

$$Z_n = in \frac{Z_0 S_n}{2\beta\gamma^2} \quad \text{for the straight pipe} \tag{71}$$

$$Z_n = in \frac{Z_0}{\beta} \cdot T_n \quad \text{for the toroidal vacuum chamber} \tag{72}$$

The summation S_n, eq. (68), diverges since for s sufficiently large, each term goes like $1/s$. To remove the divergency we have to introduce a finite transverse size of the beam. We shall assume that the beam is still centered to the vacuum chamber but now has a uniform distribution over a cross section of radius d. We shall replace (1) with

$$\rho = N e\, g_n \frac{H(d-r)}{\pi d^2 R} e^{in\left(\frac{z}{R}-\omega_0 t\right)}$$

where $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$. Eq. (67) still holds but the summation S_n now is

$$S_n = 4 \sum_{s=1}^{\infty} \frac{2 J_1(j_{os} d/b) / (j_{os} d/b)}{\left(\frac{n^2 b^2}{\gamma^2 R^2} + j_{os}^2\right) \left[J_1(j_{os})\right]^2} \tag{73}$$

In the limit $d \to 0$ this expression reduces again to (68).

In the limit of small n $S_n \to S_0$

$$S_0 = 8 \sum_{s=1}^{\infty} \frac{J_1(j_{0s}\, d/b)/(j_{0s}\, d/b)}{j_{0s}^2 \left[J_1(j_{0s})\right]^2} \qquad (74)$$

which is valid for $nb/\gamma R < 1$. In Figure 2 we show with stars results of numerical evaluations of (74). They agree very well with the continuous curve which represents

$$S_0 = 1 + 2 \log \frac{b}{d} \qquad (75)$$

as we know it should be.

Fig. 3 plots the ratio S_n/S_0 according to (73) and (75) versus $nb/\gamma R$ for different ratios of beam to pipe radii d/b.

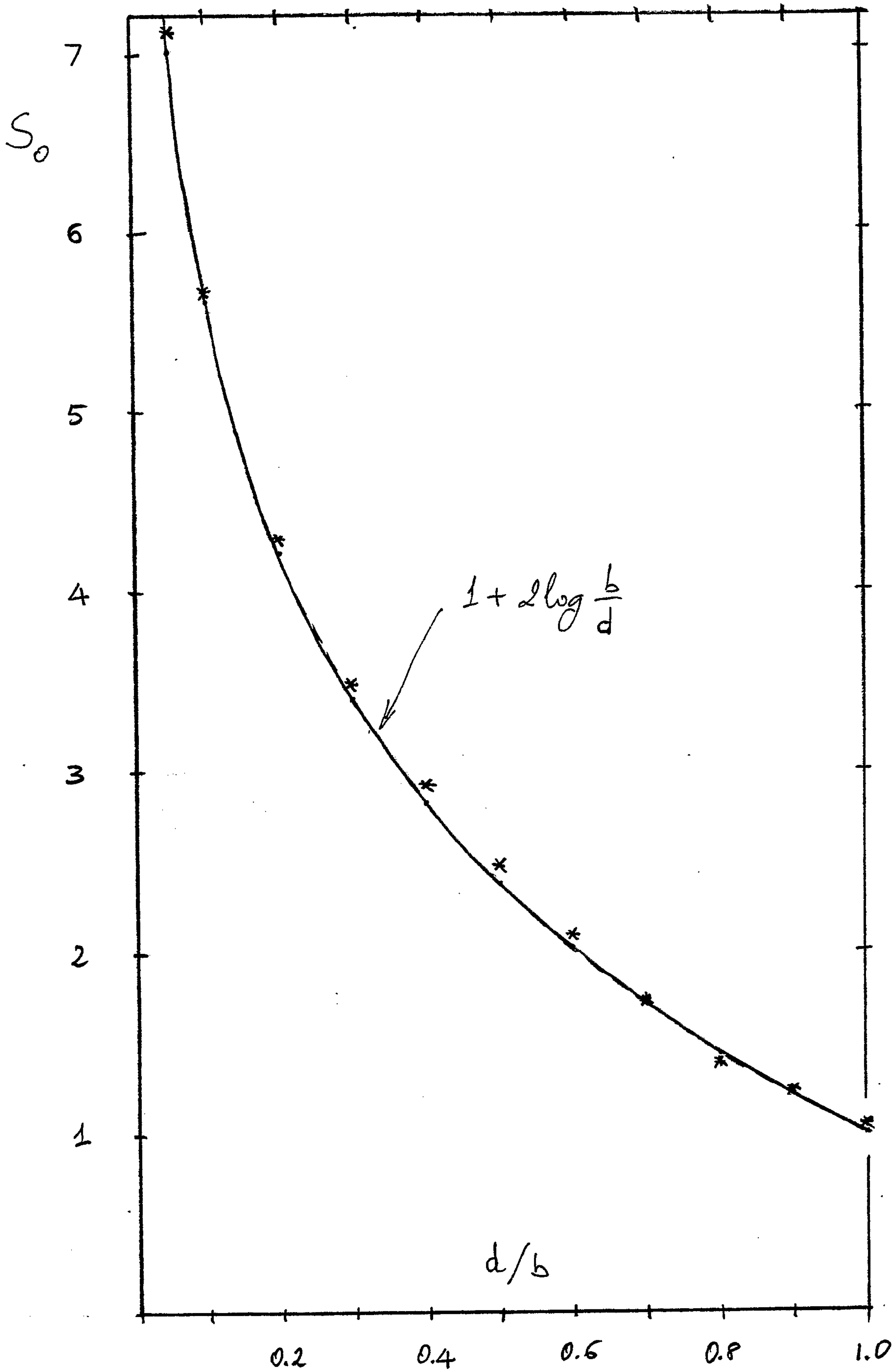
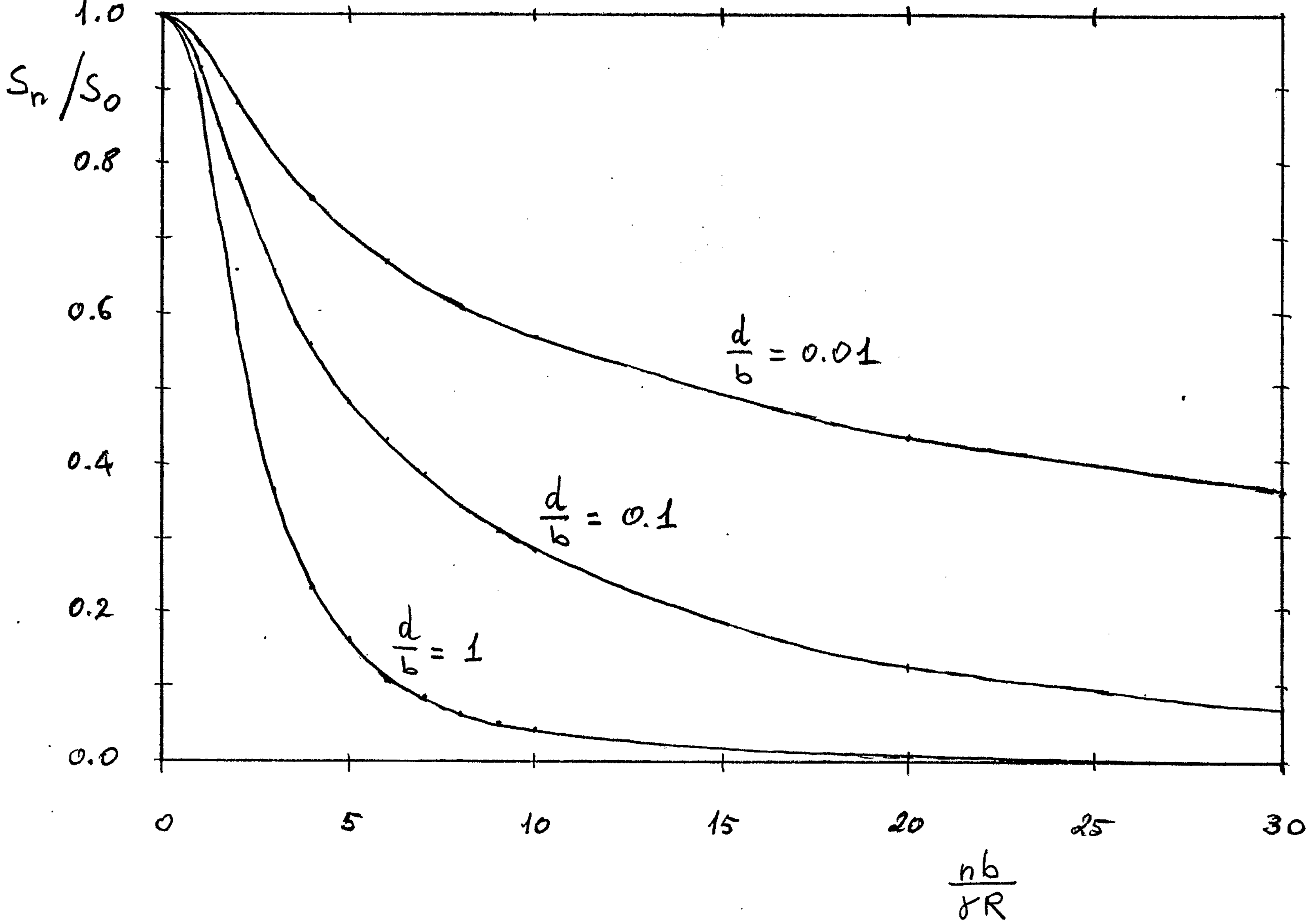


Fig. 2. S_o versus d/b

Fig. 3 Frequency Dependence of the Form Factor for the Straight Pipe for different d/b ratios.



If we return to the original model where the beam is located at $r = a$ and $\theta = 0$ with no transverse dimensions ($d=0$), then S_n from eq. (68) should be replaced with

$$S_n = \sum_{m,s} \frac{4\left[J_m(j_{ms}\, a/b)\right]^2}{\left(\frac{n^2 b^2}{\gamma^2 R^2} + j_{ms}^2\right)\left[J_{m+1}(j_{ms})\right]^2} \qquad (76)$$

It is seen that $S_n = 0$, and thus also the coupling impedance Z_n vanishes, at $a = b$. In between, for a given set of mode numbers m and s, S_n is an oscillatory function of the beam location, always positive.

In conclusion the case of the straight pipe is well understood. The method here employed to calculate the longitudinal coupling impedance for this geometry gives results which are in agreement with those obtained with other methods. Thus it is reasonable to expect that the same method will provide also the right results for the toroidal pipe case.

Discussion of the Toroidal Vacuum Chamber

The longitudinal coupling impedance for this case is given by (72) and (70), which should be compared respectively to (71) and (76) for the case of the straight pipe. There are several correspondances between the two sets of equations:

(i) The dependence on the beam position $r = R$ is mostly included in $f_n^2(q_s R)$ which is the equivalent of $[J_m(j_{ms} a/b)]^2$ in eq. (76). Also for the case of the toroidal vacuum chamber the coupling impedance vanishes at the walls $r = R_i$ and $r = R_o$ and has an oscillatory behaviour in between. There is nevertheless a dependence on the beam position $r = R$ also in other places at the numerator and denominator of eq. (70).

(ii) The difference at the numerator of eq. (70), $q_s^2 R^2 - \beta^2 n^2$, has an equivalent in the difference $1 - \beta^2$ appearing in eq. (71). Both of them are the results of the algebraic contribution from the scalar V and vector $\vec{A}$ potentials to estimate the electric field

$$\vec{E} = -\mathrm{grad}\, V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} .$$

Whereas A_z and V are always in the same ratio for any pair of mode numbers m and s for the straight pipe, this is not true for the toroidal vacuum chamber. This is one of the major differences between the two cases. We will see that for a low harmonic n the coupling impedance for a toroidal vacuum chamber depends on the beam energy (γ) very weakly. Moreover for large values of n, depending on the beam energy and on the dimension of the vacuum chamber, the coupling impedance can change sign. In the case of the straight pipe, the coupling impedance is always positive, no matter how large is n.

(iii) The form factor $G_n(q_s R_i, q_s R_o)$ at the denominator of eq. (70) is equivalent to the form factor $[J_1(j_{ms})]^2$ at the denominator of eq. (76). They are the result of the normalization operated on the expressions for the free modes of oscillations. They are always positive.

(iv) Finally the difference $\omega_x^2 - n^2\omega_0^2$ enters at the denominator of both eq.s (70) and (76). But in the straight pipe case this difference is ultimately $(n^2b^2/\gamma^2R^2 + j_{ms}^2)$ as always positive, whereas for the toroidal vacuum chamber the difference is

$$q_s^2 R^2 + \pi^2 m^2 \frac{R^2}{h^2} - \beta^2 n^2 \qquad (77)$$

In this case, as we have seen before, the two terms $q_s^2 R^2$ and $\beta^2 n^2$ do not cancel each other as in the straight pipe case. We shall see that for sufficiently large harmonic, depending on the beam energy and vacuum chamber dimensions, the quantity given by (77) can identically vanishes, giving rise to beam induced resonances. At these resonance the coupling impedance can be infinitely large and switch sign from one side to the other of the resonance.

Long Wavelength Approximation

If $z \gg n$, we can use the following asymptotic expressions for the Bessel functions

$$J_n(z) = \sqrt{2/\pi z}\ \cos\left(z - \frac{n}{2}\pi - \frac{\pi}{4}\right)$$

$$N_n(z) = \sqrt{2/\pi z}\ \sin\left(z - \frac{n}{2}\pi - \frac{\pi}{4}\right)$$

Then eq. (51) becomes (with $\bar{n} = n$)

$$\frac{2}{\pi\sqrt{R_i R_o}} \sin q_s (R_i - R_o) = 0$$

which is satisfied if

$$q_s = \frac{\pi}{w} s$$

with $s = 1, 2, 3, \ldots$

It is easily seen that all the above is valid as long as

$$n \lesssim R_i/w \qquad (78)$$

which in the following will define the long wave-length range.

With the same approximation, from (47)

$$f_n^2(q_s R) = \frac{4}{q_s^2 \pi^2 R R_i} \sin^2\left(\frac{R-R_i}{w} \pi s\right)$$

which still shows the property to vanish at $r = R_i$ and $R = R_o$ and to oscillate in between. In the following let us assume that the beam is located in the middle of the vacuum chamber, at $r = R_i + w/2$, in which case

$$f_n^2(q_s R) = \frac{4}{q_s^2 \pi^2 R R_i}$$

for s an odd integer and $f_n^2 = 0$ for s an even integer. In the following s, like m, in the summations of eq. (70) is an odd integer.

With the same approximation:

$$G_n(q_s R_i, q_s R_o) = w/R_i$$

Finally $\beta^2 n^2$ can be neglected when compared to $q_s^2 R^2$ at both the numerator and denominator of (70). Thus, in conclusion, in the long wavelength range given by (78), for $R = R_i + w/2$, the summation T_n, eq. (70), takes the value

$$T_o = \frac{8\,w}{\pi\,h} \sum_{s,m} \frac{1}{s^2 + m^2 \frac{w^2}{h^2}} \qquad (79)$$

where s and m are odd integer.

The summation in eq. (79) diverges. To avoid the divergence we shall introduce a finite beam width but we shall still assume that the beam height is infinitesimally small. We assume that the beam has a uniform distribution over a width 2d around the centre at r = R. To accomplish this we replace the charge distribution eq. (2) with

$$\rho = Ne\, g(\theta - \omega_0 t)\, \delta(z - z_0)\, \frac{H(r-R, d)}{2dr}$$

where $H(x,d) = 1$ for $|x| < d$ and $H(x,d) = 0$ for $|x| > d$. The result is to operate the replacement

$$f_\mu(qR) \rightarrow \frac{R}{2d} \int_{R-d}^{R+d} f_\mu(qr)\, dr/r \qquad (80)$$

in eq. (54). The same replacement then will apply to one of the $f_n(q_s R)$'s which appear at the r.h. side of eq. (70). In the long wavelength approximation (80) becomes

$$f_\mu(qR) \rightarrow \frac{R}{2d} \frac{2}{\pi q} \int_{R-d}^{R+d} \frac{\sin q(R_i - r)}{r^2}\, dr$$

If now $R \gg d$ and $R = R_i + \frac{w}{2}$, we can move the r^{-2} factor from under the signal of integration to the outside. The result is to replace eq. (79) with

$$T_0 = \frac{8w}{\pi h} \sum_{s,m} \frac{\sin(\pi s d/w)}{\frac{\pi s d}{w}\left(s^2 + m^2 \frac{w^2}{h^2}\right)} \qquad (81)$$

where again s and m are odd integers.

The form factor T_0 has been calculated numerically according to eq. (81). The results are shown in Fig. 4. The dashed lines summarize the results for $w/h = 2$ and $w/h = 3$. The results for $w/h = 1$ are given by the stars and compared with

$$T_0 = 1 - \log(2d/w) \qquad (82)$$

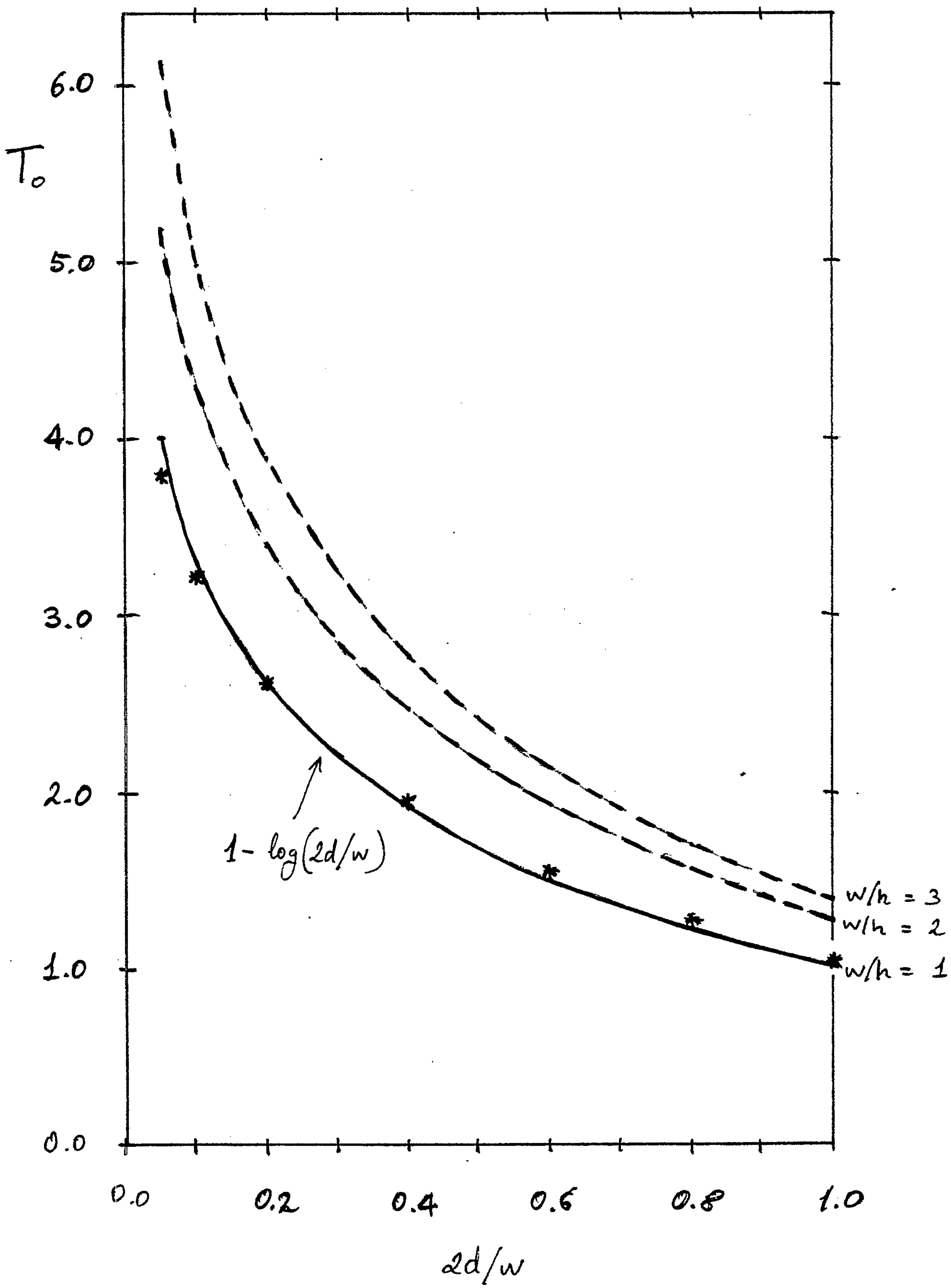
shown by the continuous curve, demonstrating very good agreement for this case.

In the long wavelength range where $n < R_i/w$, the coupling impedance is simply given by

$$Z_n = i n \frac{Z_0}{\beta} T_0 \qquad (83)$$

For the special case of a square vacuum chamber, $w = h$, T_0 is given by (82). The impedance is always positive and increases linearly with n like in the case for the straight pipe. But for the toroidal vacuum chamber there is only a weak dependence with the beam energy through β.

Fig. 4 The form factor T_0 versus $2d/w$ for different vacuum chamber aspect ratios w/h



Short Wavelength Approximation

The short-wavelength range can be defined by

$$n \gtrsim R_0/w \qquad (84)$$

The propagation mode numbers q_s, which are the zeroes of eq. (51) are discussed and evaluated in Appendix A for the special case $\eta = W/2R \ll 1$. There are two types of zeros. The first type is

$$q_s^2 = \frac{n^2}{R^2} + \frac{\pi^2 s^2}{w^2} \qquad (85)$$

with $s = s_0, s_0+1, s_0+2, \ldots$ and $s_0^2 \sim \eta^3 n^2$. The second type is

$$q_s^2 = n^2 \frac{1-2\eta}{R^2} + \left(3\pi \frac{4s-1}{4}\right)^{2/3} \frac{n^{4/3}}{R^2} \qquad (86)$$

where $s = 1, 2, \ldots, s_0 - 1$. Thus if $\eta^3 n^2 \ll 1$ only zeroes of the first type, eq. (85), exist. The second type of zeros do exist only if $\eta^3 n^2 \gg 1$.

In the long wavelength approximation, eq. (78), the first term at the right-hand side of (85) could have been ignored. When (84) applies both terms are to be taken into account.

With a zero of the first type one should insert (85) in the summation of (70). The quantities f_n^2 and G_n have of course to be calculated accordingly as it is shown in Appendix B. The insertion in the other factors gives, at the numerator,

$$q_s^2 R^2 - \beta^2 n = \pi^2 s^2 \frac{R^2}{w^2} + \frac{n^2}{\gamma^2} \approx \pi^2 s^2 \frac{R^2}{w^2}$$

and the second term at the r.h. side can be neglected again as long as $n < \gamma R/w$. The same is true with the insertion at the denominator

$$q_s^2 R^2 + \pi^2 m^2 \frac{R^2}{h^2} - \beta^2 n^2 =$$

$$= \pi^2 s^2 \frac{R^2}{w^2} + \pi^2 m^2 \frac{R^2}{h^2} + \frac{n^2}{\gamma^2}$$

$$\approx \pi^2 s^2 \frac{R^2}{w^2} + \pi^2 m^2 \frac{R^2}{h^2}$$

Observe that with the zeroes of the first type each term in the summation of (70) is positive. The denominator (77) never vanishes. Considering the results of Appendix B thus it appears that eq. (83) with (81) is a good approximation for the coupling impedance for harmonics

$$n < \gamma R/w \quad . \tag{87}$$

A different behaviour appears when one or more zeroes of the second type appear. The largest difference is the behaviour of the denominator (77) which now can be written as follows with (86)

$$q_s^2 R^2 + \pi^2 m^2 \frac{R^2}{h^2} - \beta^2 n^2 =$$

$$= n^2 \left(\frac{1}{\gamma^2} - 2\eta \right) + \pi^2 m^2 \frac{R^2}{h^2} + \left(3\pi \, \frac{4s-1}{4} \right)^{2/3} n^{4/3} \tag{88}$$

This quantity can vanish if

$$2\eta > 1/\gamma^2 \tag{89}$$

This is the condition for the beam to induce a resonance, since at least one of the terms in the summation (70) diverges. In this case, of course, only that term will predominate over all the others, and the coupling impedance is infinitely large.

The case the resonance condition (89) is not satisfied, which occurs for sufficiently low beam energy, is discussed in Appendix B. Here we assume that (89) is satisfied. The procedure is the following:

1. For a given harmonic number n one checks that the quantity $n^2\eta^3 > 1$, otherwise there are no zeroes of the second type.
2. One estimates
$$s_0 \approx n^2\eta^3 \tag{90}$$
3. For given n and for given s, between 1 and s_0, one estimates that odd integer value of m which makes the r.h. side of (88) to vanish. That is

$$m \simeq \frac{h}{\pi R}\left[n^2\left(2\eta - \frac{1}{\gamma^2}\right) - (3\pi s)^{2/3} n^{4/3} \right]^{1/2} \qquad (91)$$

In order for m to be real and positive, we require

$$\frac{s^2}{s_0^2} < \frac{8}{9\pi^2}\left(1 - \frac{1}{2\eta\gamma^2}\right)^3 \qquad (92)$$

It is easily seen that the term $s = s_0$ cannot satisfy this relation and therefore does not add a contribution to the beam induced resonance. In order for the lowest order term $s = 1$ to satisfy the relation (92), the harmonic number n has to be sufficiently large, namely

$$n^2\eta^3 > \frac{9\pi^2}{8\left(1 - \frac{1}{2\eta\gamma^2}\right)^3} \qquad (93)$$

Thus, for given η and γ, provided the resonance condition (89) is satisfied, the relation (93) provides the lowest harmonic n for which one resonating term is real. For larger values of n, the number of resonating modes s is given by (92) and the corresponding mode number m by (91). This ends our discussion of the behaviour of the toroidal vacuum chamber in the short-wavelength range. Appendix C shows the calculation of the remaining terms entering the summation (70).

Appendix A

Evaluation of the Zeros of Cross-Product Bessel Functions -

The problem is to find the zeros of eq. (51) which we like to write here again

$$N_n(qR_i)\,J_n(qR_o) - J_n(qR_i)\,N_n(qR_o) = 0 \qquad (A1)$$

Consider the Bessel equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_n}{dr}\right) - \frac{n^2}{r^2}f_n + q^2 f_n = 0 \qquad (A2)$$

The most general solution to this equation is a linear combination of the two Bessel functions $J_n(qr)$ and $N_n(qr)$. We are interested to that particular solution that vanishes identically for $r = R_i$ and $r = R_o$. Thus the solution of (A2) can be written as

$$f_n = N_n(qR_i)\,J_n(qr) - J_n(qR_i)\,N_n(qr) \qquad (A3)$$

where q has characteristic values that can be obtained as the zeros of eq. (A1).

Let us introduce the variables

$$
\begin{aligned}
w &= R_o - R_i \\
\bar{R} &= (R_o + R_i)/2 \\
\eta &= w/2\bar{R} \\
\delta^2 &= \eta^2 \left(q^2 \bar{R}^2 - n^2 \right)
\end{aligned}
\qquad (A4)
$$

and replace r with the new variables x so that

$$r = \bar{R}\,(1 + \eta x) \qquad (A5)$$

As r varies between R_i and R_o x ranges between -1 and $+1$. It is also convenient to make the transformation

$$\varphi_n = (1 + \eta x)^{1/2} f_n \qquad (A6)$$

in terms of which and the new variables defined above the Bessel equation (A2) becomes

$$\frac{d\varphi_n^2}{dx^2} + \left[\delta^2 + \frac{(\eta^2/4) + \eta^3 n^2 (2 + \eta x) x}{(1 + \eta x)^2} \right] \varphi_n = 0 \qquad (A7)$$

The new boundary conditions are

$$\varphi_n = 0 \quad \text{at} \quad x = +1 \quad \text{and} \quad x = -1 \qquad \text{(A8)}$$

We are interested only in those cases where $\eta \ll 1$, that is $w \ll 2\bar{R}$. We shall consider two ranges of harmonic numbers : (i) $\eta^3 n^2 \ll 1$, (ii) $\eta^3 n^2 \gg 1$.

(i) $\eta^3 n^2 \ll 1$. In this range eq. (A7) reduces to

$$\frac{d^2 \varphi_n}{dx^2} + \delta^2 \varphi_n = 0$$

the general solution of which being a linear combination of $\cos(\delta x)$ and $\sin(\delta x)$. In order to satisfy the boundary conditions (A8) it is easily seen that the only possible eigenvalues for δ are

$$\delta = \frac{\pi}{2} s \qquad s = 1, 2, 3, \ldots$$

When this is combined with (A4) the characteristic values of q_s are determined

$$q_s^2 = \frac{n^2}{\bar{R}^2} + \frac{\pi^2 s^2}{w^2} \qquad \text{(A9)}$$

(iii) $\eta^3 n^2 >> 1$ - In this range eq. (A 7) reduces to

$$\frac{d^2\varphi_n}{dx^2} + \left(\delta^2 + 2\eta^3 n^2 x\right)\varphi_n = 0 \qquad (A10)$$

With the transformation

$$z = \frac{\delta^2 + 2\eta^3 n^2 x}{(2\eta^3 n^2)^{2/3}}$$

we see that the general solution of (A10) is a linear combination of the two Airy functions $Ai(-z)$ and $Bi(-z)$. Both of these are plotted in Fig. 5. Thus

$$\varphi_n = c_1\, Ai(-z) + c_2\, Bi(-z) \qquad (A11)$$

as boundary conditions we require now

$$\begin{aligned} c_1\, Ai(-z_-) + c_2\, Bi(-z_-) &= 0 \\ c_1\, Ai(-z_+) + c_2\, Bi(-z_+) &= 0 \end{aligned} \qquad (A12)$$

where

$$z_\pm = \frac{\delta^2 \pm 2\eta^3 n^2}{(2\eta^3 n^2)^{2/3}} \qquad (A13)$$

c_1 and c_2 are two chosen constants satisfying (A12).

An inspection of Fig. 5 shows clearly that the boundary conditions (A12) are satisfied if $c_2 = 0$, $z_+ > 0$ and z_- negative with large absolute value. Thus we only require that

$$Ai(-z_+) = 0 \qquad (A14)$$

The zeros of the Airy function are known

$$z_+ \approx \left(3\pi \frac{4s-1}{8}\right)^{2/3} \qquad s = 1, 2, 3, \ldots$$

and from (A13) the eigenvalues of δ^2 are

$$\delta^2 = -2\eta^3 n^2 + \left(3\pi \frac{4s-1}{4}\right)^{2/3} \eta^2 n^{4/3}$$

Finally from (A4)

$$q_s^2 = n^2 \frac{1-2\eta}{\bar{R}^2} + \left(3\pi \frac{4s-1}{4}\right)^{2/3} \frac{n^{4/3}}{\bar{R}^2} \qquad (A15)$$

This derivation is valid as long as

$$s^2 < \eta^3 n^2 \qquad (A16)$$

Usually only for the first zero, $s = 1$.

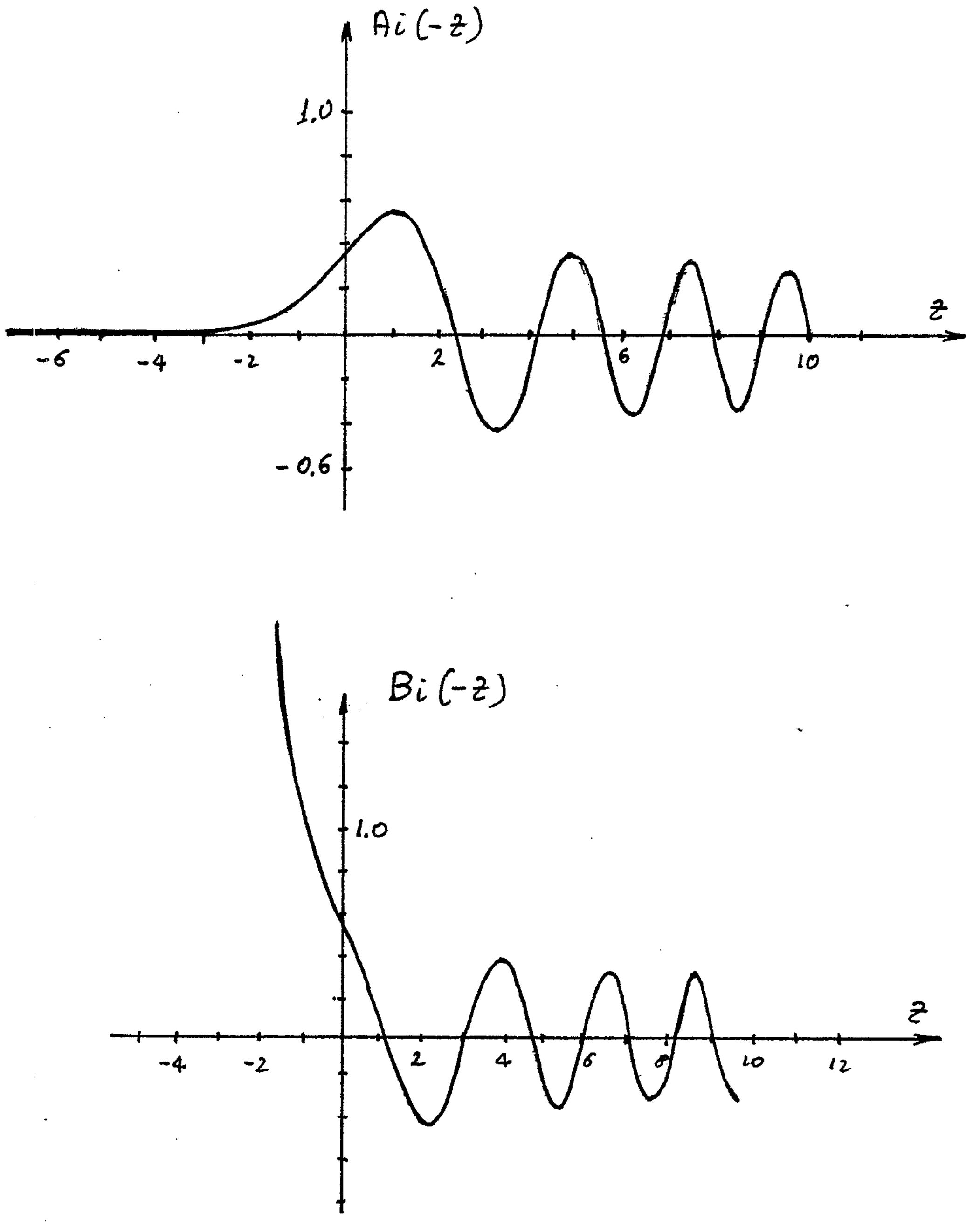


Fig. 5 The Airy's Functions

For larger zeros, that is when $s^2 > \eta^3 n^2$, both z_+ and z_- are positive, in which case, to satisfy both boundary conditions (A12), we require

$$Ai(-z_-)\,Bi(-z_+) - Ai(-z_+)\,Bi(-z_-) = 0 \qquad (A17)$$

With very good approximation, as long as both z_+ and z_- are large, we have

$$Ai(-z) \simeq \frac{\sin(\zeta + \frac{\pi}{4})}{\pi^{1/2} z^{1/4}} \qquad (A18a)$$

$$Bi(-z) \simeq \frac{\cos(\zeta + \frac{\pi}{4})}{\pi^{1/2} z^{1/4}} \qquad (A18b)$$

where $\zeta = \frac{2}{3} z^{3/2}$. Thus (A17) is satisfied if

$$\sin(\zeta_+ - \zeta_-) = 0$$

or

$$z_+^{3/2} - z_-^{3/2} = \frac{3}{2}\pi s \qquad (A19)$$

where again $s = 1, 2, 3, \ldots$

• With (A13), (A19) becomes

$$\left(\delta^2 + 2\eta^3 n^2\right)^{3/2} - \left(\delta^2 - 2\eta^3 n^2\right)^{3/2} = \frac{3\pi s}{2\left(2\eta^3 n^2\right)^{1/2}} \qquad \text{(A20)}$$

which is again satisfied by $\delta = \pi s/2$.

Conclusions

Let us order the zeros of the eq. (A1) according to q_s with $s = 1, 2, 3, \ldots$.

If $s^2 < \eta^3 n^2$, then q_s is given by (A15).

If $s^2 > \eta^3 n^2$, then q_s is given by (A9).

This result is independent of the magnitude of $\eta^3 n^2$. Clearly if $\eta^3 n^2 \ll 1$ then no zeroes with the form (A15) are possible.