

Definition of Emittance in Tracking Studies

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Definition of Emittance in Tracking Studies

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DEFINITION OF EMITTANCE in TRACKING STUDIES

1. BEAM PARAMETERS GIVEN BY

ions / bunch N_B

normalized emittance (full coupling)

$$\epsilon_H = \epsilon_V = \epsilon_N$$

Actual emittance

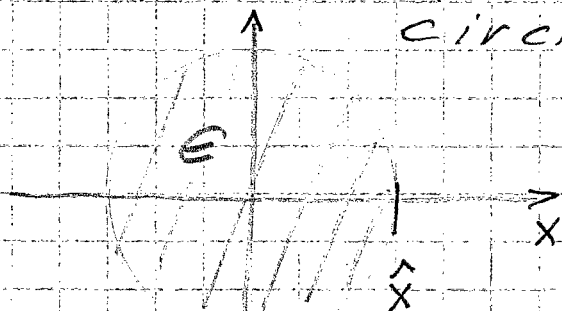
$$\epsilon = \epsilon_N / (N_B)$$

2. MOTION OF PARTICLES IN PHASE SPACE

ellipse in (x, x')

at QF/QD regular ellipse

circle in $(x, \beta x')$



Emittance defines maximum ellipse

Beam size in (x, y) space:

$$\hat{x} = \sqrt{\epsilon \beta_H}$$

$$\hat{y} = \sqrt{\epsilon \beta_V}$$

3. TRUNCATED GAUSSIAN BEAMS in 2-dimensional phase space

$$N_B = \int_E \frac{N_2}{2\pi \epsilon_{rms}} \exp\left(-\frac{1}{2} \frac{x^2/\beta + \beta x'^2}{\epsilon_{rms}}\right) dx dx'$$

N_B = # ions within emittance

N_2 = # ions in infinite phase plane

Integration over

$$x^2/\beta + \beta x'^2 \leq \epsilon$$

$$\frac{N_B}{N_2} = 1 - \exp\left(-\frac{1}{2} \frac{\epsilon}{\epsilon_{rms}}\right)$$

BNL - CONVENTION

$$\frac{N_B}{N_2} = 95\% \Rightarrow \epsilon_{rms} = \frac{\epsilon}{6}$$

at CERN

$$\epsilon = 4 \epsilon_{rms} \Rightarrow \frac{N_B}{N_{2(CERN)}} = 86\%$$

HOWEVER : 100% of beam within beam size by definition

4. One-dimensional Gaussian beam

The non-truncated beam has line density

$$\frac{N_1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2} \frac{x^2}{\sigma^2}\right)$$

with $\sigma = \sqrt{\epsilon_{rms} \beta}$

Limiting the beam size to $x \leq \hat{x}$ (scraper) reduces beam to

$$\int_{-\hat{x}}^{\hat{x}} \frac{N_1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2} \frac{x^2}{\sigma^2}\right) dx$$

$$= 2P\left(\frac{\hat{x}}{\sigma}\right) - 1$$

$$= 99\% \quad \text{if} \quad \hat{x} = \sqrt{6}\sigma$$

($\neq 95\%$)

Conclusion:

- Line density of truncated Gaussian beam is not Gaussian
- In practice, measurement of linear density distribution yields $\tilde{\sigma}$, which is basis for aperture requirement (6 $\tilde{\sigma}$ -rule)

5. 4 Dimensional Gaussian Beams

H/V uncoupled

$$N_B = N_4 \int_{E_H} \frac{1}{2\pi E_{Hrms}} \exp(\quad) dx dx' \\ \times \int_{E_V} \frac{1}{2\pi E_{Vrms}} \exp\left(-\frac{1}{2} \frac{y^2/\beta_V + \beta_V y'^2}{E_{Vrms}}\right) dy dy'$$

Truncation (i.e. scraping) imposes limits:

$$x \leq \hat{x} \rightarrow x^2/\beta_H + \beta_H x'^2 \leq E_H$$

$$y \leq \hat{y} \rightarrow y^2/\beta_V + \beta_V y'^2 \leq E_V$$

separate constraints ("intersecting cylinders")

$$N_B = N_4 \left[1 - \exp\left(-\frac{1}{2} \frac{E_H}{E_{Hrms}}\right) \right] \left[1 - \exp\left(-\frac{1}{2} \frac{E_V}{E_{Vrms}}\right) \right]$$

Conclusion:

2-dimensional treatment is based

on truncation in one plane onlywith uncoupled motion(then $N_4 = N_2$)

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6. Gaussian Beams in 4-dim phase space
fully coupled ($\epsilon_H = \epsilon_V = \epsilon$)

$$N_B = \int_{E_T} \frac{N_4}{(2\pi \epsilon_{rms})^2} \exp\left(-\frac{1}{2} \frac{x^2/\beta_H + \beta_H x'^2 + y^2/\beta_V + \beta_V y'^2}{\epsilon_{rms}}\right) dx dx' dy dy'$$

Integration over 4 dim ellipsoid
defined by beam size:

$$E_T = x^2/\beta_H + \beta_H x'^2 + y^2/\beta_V + \beta_V y'^2 \leq \hat{E}_T$$

Determine \hat{E}_T :

$$x = \hat{x} \quad \text{if } x'^2 = y = y' = 0$$

$$\hat{E}_T = \frac{\hat{x}^2}{\beta_H} \equiv \epsilon$$

$$N_B = N_4 \left[1 - \left(1 + \frac{\epsilon}{2\epsilon_{rms}} \right) \exp\left(-\frac{\epsilon}{2\epsilon_{rms}}\right) \right]$$

= 80% of N_4

Conclusion :

Scrapers in one-direction at $x = \hat{x} = \sqrt{6} \sigma$
reduces beam by 20% due to coupling!

HOWEVER: 100% of BEAM is within
beam size by definition

7. Dynamic APERTURE REQUIREMENTS

- The beam size grows due to intra beam scattering to

$$\epsilon_N = 30 \pi \text{ mm.mrad at } p = 30$$

$$\Rightarrow \epsilon = 1 \pi \text{ mm.mrad}$$

- Requirements for dynamic aperture are based on "6 σ -rule"
- Dynamic stability in tracking studies for initial conditions in 4D space

$$\epsilon_T = \frac{x^2}{\beta_H} + \beta_H x'^2 + \frac{y^2}{\beta_V} + \beta_V y'^2 \leq 6\epsilon$$

$$\leq 6\pi \text{ mm.mrad}$$

Usually in tracking studies $x' = y' = 0$

$$\epsilon_T = \frac{x^2}{\beta_H} + \frac{y^2}{\beta_V} \leq 6\epsilon = 6\pi \text{ mm.mrad}$$

CONCLUSION (coupled beam):

- Intra beam scattering, space charge determined by maximum density in core of beam
i.e. $N_4 = N_B / 0.8$

- aperture requirement determined by $\epsilon_T = 6\epsilon$