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Longitudinal Stochastic Cooling in RHIC

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Abstract

The transport equation for a bunched beam in longitudinal stochastic cooling under a sinusoidal rf voltage has been derived. A computer program has been developed to numerically solve this equation. The analysis is applied to the cooling of high charge-state ions in the RHIC collider where the Schottky noise is dominant. It has been shown that the agreement between the coasting- and the bunched-beam theory is satisfactory provided that the bunch occupies a large amount ($\geq$75%) of the rf bucket.

I. Introduction

Stochastic cooling[1–5] in the RHIC[6] aims at compensating for the emittance growth of heavy-ion beams, primarily due to intrabeam scattering. In order to reduce the growth of the bunch spread during the 10 hour period of storage, the currently adopted scenario requires blowing-up the initial normalized transverse emittance of $^{197}Au^{79+}$ beams from 10 to 60πmm-mrad. The final peak voltage required for the 160MHz rf system is 4.5MV. Compared with the case that the initial emittance is kept at 10πmm-mrad, this scenario results in a loss of at least a factor of 3 in the integrated colliding-beam luminosity. Nevertheless, without the initial emittance blow-up, a peak rf voltage of 11.5MV is required to compensate the growth in momentum spread. Stochastic cooling in longitudinal phase space is therefore of particular interest in reducing the amount of rf voltage. Indeed, previous studies[7] indicate that with a constant momentum-cooling rate of about 1/9 hour^{-1} and the same 4.5MV rf voltage, the integrated luminosity can be increased by more than 3 times even without the initial blow-up.

Previous estimates[8] based on coasting-beam theory indicate that a reasonable cooling rate can be achieved with a cooling system of frequency bandwidth 4–8 GHz. However, detailed analysis based on bunched-beam theory has to be performed to both understand the cooling mechanism and optimize the performance.

This paper summarizes our study on longitudinal stochastic cooling of hadron bunches in the absence of intrabeam scattering. The discussion is restricted to the so-called "slow cooling" regime, where the characteristic cooling time τ is long compared with the period of synchrotron oscillation. Section II introduces the transport equation which, in terms of action variable J, describes the evolution of longitudinal particle density distribution under the influence of stochastic cooling. Section III essentially addresses the numerical solution of this equation. The analysis is applied to the cooling of high charge-state ions in the RHIC collider in section IV, where the required amplifier power is also estimated. A comparison on the cooling results obtained from the bunched-beam and the coasting-beam theory is given in section V.

II. Theoretical Approaches

Longitudinal (momentum) stochastic cooling of hadron bunches can be classified into two categories: "fast" and "slow" cooling. The former refers to the case where the relative change of the particle distribution is large during one synchrotron-oscillation period, while the later refers to the opposite case.

The fast-cooling process can be investigated using the longitudinal transfer matrices.[9] Due to the fast (non-adiabatic) variation of the particle motion, the momentum spread in the bunch is often reduced faster than the phase spread. The resulting bunch shape is then continuously mismatched to the rf bucket.

In reality, due to system bandwidth and amplifier power limitations, it is often of practical interest to consider slow-cooling process. The essential formulation applicable to this regime will be presented in the following subsections.

A. Single-particle equations of motion

Longitudinal motion of the particles can be described by two variables, the phase $\phi_s + \phi$ of the rf field at the moment the particle passes the cavity, and $W = \Delta E / h\omega_0$, with the energy deviation ΔE from the synchronous value, as

$$\left\{ \begin{array}{lcl} \dot{W} & = & \dfrac{qe\hat{V}}{2\pi h} \left[\sin(\phi_s + \phi) - \sin\phi_s\right] + U_W \\ \\ \dot{\phi} & = & \dfrac{h^2\omega_0^2\eta}{E\beta^2} \, W \, , \end{array} \right. \tag{1}$$

where h is the harmonic number, $\eta = 1/\gamma_T^2 - 1/\gamma^2$, γ_T is the transition energy, $\omega_0 = 2\pi f_0$, $E = Am_0c^2\gamma$, and βc are the synchronous revolution frequency, energy, and velocity, and q and A are the charge and the atomic number of the particles, respectively. For simplicity, the synchronous phase will be taken as $\phi_s = \pi$, which represents the storage mode above transition in the RHIC.

The energy increment U_W of a "test particle" due to the cooling contains a deterministic (coherent) part U_W^C resulting from the signal generated by itself, and a fluctuating (incoherent) part U_W^I resulting from the signals generated by all the other particles. It can be expressed in terms of the voltage V_K that the particle experiences at the kickers,

$$U_W(t) = \frac{qe}{h\omega_0} \sum_{n=-\infty}^{\infty} V_K(t)\, \delta\left(t - \frac{2\pi n}{\omega_0} - \frac{\phi}{h\omega_0} - \frac{\theta_K}{\omega_0}\right) = U_W^C + U_W^I, \tag{2}$$

where ϕ is the phase deviation of the test particle, θ_K is the azimuthal location of the kickers on the ring,

$$V_K(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) I_{PU}(\omega) e^{i\omega t} d\omega, \tag{3}$$

and the subscripts PU and K denote the pick-up and the kickers, respectively. The causality condition requires that the gain function $G(\omega)$ of the cooling system satisfies the condition

$$G(-\omega) = G^*(\omega). \tag{4}$$

Furthermore, the Fourier transform $I_{PU}(\omega)$ of the beam current $I_{PU}(t)$ measured at the puck-up can be expressed as

$$I_{PU}(\omega) = \int_{-\infty}^{\infty} I_{PU}(t) e^{-i\omega t} dt = qe \sum_{m=-\infty}^{\infty} \sum_{j=1}^{N} \exp\left[-i\omega\left(\frac{2\pi m}{\omega_0} + \frac{\phi_j}{h\omega_0} + \frac{\theta_{PU}}{\omega_0}\right)\right]. \tag{5}$$

The first summation represents revolutions; the second summation represents N particles in each bunch. Since signals pertaining to different bunches are not correlated, it will suffice to consider only an individual bunch.

Regarding the cooling contribution U_W as a perturbation, the unperturbed particle motion can be derived from an Hamiltonian

$$H(\phi, W) = C_W W^2 + C_\phi \sin^2 \frac{\phi}{2}, \tag{6}$$

where the coefficients

$$C_W = \frac{h^2 \omega_0^2 \eta}{2E\beta^2}, \quad \text{and} \quad C_\phi = \frac{qe\hat{V}}{\pi h}$$

may be time dependent. The incoherent noise term U_W^I in eq. 1 is a function of location ϕ only. It therefore can be derived from a perturbative Hamiltonian

$$H_1(\phi, W) = -\int^{\phi} U_W^I(\phi')d\phi'. \tag{7}$$

On the other hand, the coherent cooling term U_W^C is a function of both location ϕ and momentum deviation W; it is non-Hamiltonian.[5]

In the absence of the perturbation, the action invariant J and the canonically conjugate angle Q can be obtained by using the Hamilton-Jacobi theory,[10] e.g.

$$J = \oint W d\phi = 8\sqrt{\frac{C_\phi}{C_W}}\left[(k^2-1)\mathrm{K}(k) + \mathrm{E}(k)\right], \quad k = \sqrt{\frac{H}{C_\phi}} \leq 1, \tag{8}$$

where

$$\mathrm{K}(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2\sin^2 t}}, \text{ and } \mathrm{E}(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2 t}dt \tag{9}$$

are the complete elliptical integrals[11] of first and second kind. J represents the phase-space area enclosed by the trajectory of the particle performing synchrotron oscillation. The oscillation frequency is given by

$$\Omega_s = 2\pi\dot{Q} = \frac{\pi\sqrt{C_W C_\phi}}{2\mathrm{K}(k)}, \quad k \leq 1. \tag{10}$$

It is seen that Ω_s is a monotonic function of k. At the boundary (separatrix) of the stable region, $J_{max} = 8\sqrt{\frac{C_\phi}{C_W}}$, and $\Omega_s(J_{max}) = 0$.

B. Equations of motion in the action-angle variables

The study in the slow-cooling regime can be simplified be averaging the particle motion over one synchrotron-oscillation period. First, the stochastic equation (1) has to be written in terms of the action-angle variables Q and J. This transformation can be achieved by means of a generating function of Goldstein's[10] second type,

$$F_2(\phi, J, t) = \int^{\phi} \sqrt{\frac{1}{2\pi C_W}\int^{J} \Omega_s(J')dJ' - \frac{C_\phi}{C_W}\sin^2\frac{\phi'}{2}}d\phi'. \tag{11}$$

If $K(Q,J,t)$ is the transformed unperturbed Hamiltonian, eq. 1 may be expressed in terms of Q and J as

$$\begin{cases} \dot{Q} &= \dfrac{\partial K}{\partial J} + U_Q \\ \dot{J} &= U_J. \end{cases} \qquad (12)$$

By expressing W and Q in terms of ϕ and J, and by using[12] the identities

$$dW(\phi, J, t) = \left.\frac{\partial W}{\partial \phi}\right|_J \left.\frac{\partial H}{\partial W}\right|_\phi dt + \left.\frac{\partial W}{\partial J}\right|_\phi \dot{U}_J dt = \left(-\left.\frac{\partial H}{\partial \phi}\right|_W + U_W\right) dt, \qquad (13)$$

and

$$dQ(\phi, J, t) = \left.\frac{\partial Q}{\partial \phi}\right|_J \left.\frac{\partial H}{\partial W}\right|_\phi dt + \left.\frac{\partial Q}{\partial J}\right|_\phi \dot{U}_J dt = \left(\left.\frac{\partial K}{\partial J}\right|_Q + U_Q\right) dt, \qquad (14)$$

the stochastic quantities U_Q and U_J can be related to U_W as

$$U_Q = \left.\frac{\partial Q}{\partial J}\right|_\phi \left.\frac{\partial W}{\partial J}\right|_\phi^{-1} U_W, \quad \text{and} \quad U_J = \left.\frac{\partial W}{\partial J}\right|_\phi^{-1} U_W. \qquad (15)$$

Using eq. 8 and the canonical relations, the transformation coefficients can be simplified, e.g.

$$\left.\frac{\partial W}{\partial J}\right|_\phi^{-1} = \frac{4\pi C_W}{\Omega_s} W. \qquad (16)$$

Equivalent to eq. 1, the transformed equations (12) thus describe the stochastic motion of each individual particle.

In order to obtain an explicit expression of the transformed stochastic equations, the variable W in eq. 16 has to be written in terms of Q and J. Define polar coordinates[13] k and φ such that

$$W = \sqrt{\frac{C_\phi}{C_W}}\, k \cos\varphi, \quad \text{and} \quad \sin\frac{\phi}{2} = k \sin\varphi. \qquad (17)$$

It is straightforward[14] to expand W as a series of the periodic functions of Q,

$$W = \sqrt{\frac{C_\phi}{C_W}} \frac{2\pi}{\mathrm{K}(k)} \sum_{n=1}^{\infty} \frac{\xi^{n-\frac{1}{2}}}{1+\xi^{2n-1}} \cos\left[2(2n-1)\pi Q\right]. \qquad (18)$$

As shown in figure 1, the order parameter

$$\xi = \exp\left[-\pi \mathrm{K}'(k)/\mathrm{K}(k)\right],$$

with $K'(k) = K(\sqrt{1-k^2})$, becomes significant only when it is near the separatrix J_{max}. Hence, eq. 16 can be written as a series expansion of ξ,

$$\left.\frac{\partial W}{\partial J}\right|_{\phi}^{-1} = 8k\ K(k)\cos 2\pi Q\ \left[1 - 4\xi \sin^2 2\pi Q + O(\xi^2)\right]. \tag{19}$$

Note that k is a function of J only (eq. 8).

C. The transport equation

Longitudinal distribution of N particles in a bunch may in general be described by a two-dimensional density distribution function $\tilde{\Psi}(Q,J,t)$, so defined that

$$N\tilde{\Psi}(Q,J,t)\ dJd\phi$$

is the number of particles which, at time t, have synchrotron phase Q lying within an element dQ about Q and action J lying within an element dJ about J. Assuming that the stochastic process is Markovian, the time evolution of $\tilde{\Psi}$ satisfies the Fokker-Planck equation[15]

$$\begin{aligned}\frac{\partial\tilde{\Psi}}{\partial t} &= -\frac{\Omega_s}{2\pi}\frac{\partial\tilde{\Psi}}{\partial Q} - \frac{\partial}{\partial Q}\left(\tilde{F}_Q\tilde{\Psi}\right) - \frac{\partial}{\partial J}\left(\tilde{F}_J\tilde{\Psi}\right) \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial Q^2}\left(\tilde{D}_{QQ}\tilde{\Psi}\right) + \frac{1}{2}\frac{\partial^2}{\partial J^2}\left(\tilde{D}_{JJ}\tilde{\Psi}\right) + \frac{\partial^2}{\partial Q\partial J}\left(\tilde{D}_{QJ}\tilde{\Psi}\right),\end{aligned} \tag{20}$$

where

$$\begin{aligned}\tilde{F}_M &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle\int_0^{\Delta t}\left[U_M^C(t) + U_M^I(t)\right]\ dt\rangle, \\ \tilde{D}_{MN} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle\int_0^{\Delta t}\int_0^{\Delta t} U_M^I(t)U_N^I(t')\ dtdt'\rangle\ , \quad M,N = Q,J,\end{aligned} \tag{21}$$

and $U_{Q,J}^C$ and $U_{Q,J}^I$ are the coherent and the incoherent part of $U_{Q,J}$, respectively. The limit $\Delta t \to 0$ implies that Δt is long as compared to the correlation time, but still short as compared to the time interval within which $\tilde{\Psi}$ changes appreciably.

As mentioned earlier, eq. 20 can be reduced by averaging Q over one synchrotron-oscillation period. Using the periodic conditions of $\tilde{\Psi}$, $\partial\tilde{\Psi}/\partial J$, and $\partial\tilde{\Psi}/\partial Q$ in Q, eq. 20

becomes

$$\frac{\partial \Psi}{\partial t} = -\frac{\partial}{\partial J}\left(\tilde{F}_J \Psi\right) + \frac{1}{2}\frac{\partial^2}{\partial J^2}\left(\tilde{D}_{JJ}\Psi\right), \tag{22}$$

where Ψ is the phase-averaged distribution which depends on J and t only. Since the incoherent part is derivable from an Hamiltonian (eq. 7), eq. 22 can be simplified as[16,17]

$$\frac{\partial \Psi}{\partial t} = -\frac{\partial}{\partial J}(F\Psi) + \frac{1}{2}\frac{\partial}{\partial J}\left[(D_{SH} + D_T)\frac{\partial \Psi}{\partial J}\right]. \tag{23}$$

The boundary conditions of eq. 23 is that Ψ vanishes at the separatrix J_{max}, and the flux vanishes at $J = 0$, i.e.

$$\begin{cases} J = 0: & -F\Psi + \frac{1}{2}(D_{SH} + D_T)\frac{\partial \Psi}{\partial J} = 0 \ , \\ J = J_{max}: & \Psi = 0 \ . \end{cases} \tag{24}$$

Note that the drifting coefficient F in eq. 23 contains only the coherent contribution. With the causality condition (eq. 4), it is straightforward to find

$$\begin{aligned} F(J) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\langle \int_0^{\Delta t} U_J^C(t)\ dt \rangle \\ &= \frac{q^2 e^2 \omega_0}{\pi} \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \frac{l}{m} \mathrm{J}_l^2\left[m\omega_0 \tau(J)\right] Re\left\{G_F\left[m\omega_0 + l\Omega_s(J)\right] - G_F\left[m\omega_0 - l\Omega_s(J)\right]\right\} \\ &\quad + O(\xi), \end{aligned} \tag{25}$$

where $O(\xi)$ represents terms whose magnitudes are of order ξ or higher when compared with the first term, Δt represents a time interval that is long compared to the synchrotron-oscillation period, Re denotes the real part of a complex quantity, and

$$G_F\left[m\omega_0 + l\Omega_s(J)\right] = G\left[m\omega_0 - l\Omega_s(J)\right]\ e^{-im(\theta_{PU}-\theta_K)}\ e^{il\Omega_s(J)(\theta_{PU}-\theta_K)/\omega_0}\ . \tag{26}$$

J_l is the Bessel function of lth order, and $\tau(J)$ is the amplitude of synchrotron oscillation in time space,

$$\tau(J) = \frac{1}{h\omega_0}\arccos\left(1 - 2k^2\right), \quad k \le 1. \tag{27}$$

The summation on the revolution bands in eq. 25 is performed over the range of system bandwidth, while the summation on the synchrotron side-bands is actually performed over the range from $l = 1$ to $m\omega_0\tau(\hat{J})$, with $\hat{J}$ the bunch area, corresponding to m times the revolution-frequency spread in the bunch. The factor $e^{il\Omega_s(J)(\theta_{PU}-\theta_K)/\omega_0}$ represents the phase slip that non-synchronous particles experience during their passage from the pick-up to the kickers. In order to minimize this undesirable "mixing", the distance between the pick-up and the kickers should be chosen such that

$$(\theta_{PU} - \theta_K)n_{av}\hat{\tau}\Omega_s(0) \ll 1, \qquad \hat{\tau} \equiv \tau(\hat{J}), \tag{28}$$

where n_{av} is the average of the revolution harmonic numbers of the cooling system bandwidth.

The diffusion is contributed from the beam Schottky noise (D_{SH}) and the system (thermal) noise (D_T). If the revolution bands are not overlapping within the system bandwidth, the Schottky coefficient D_{SH} at location J can be shown to depend both on the particle density $\Psi(J)$, and the density $\Psi(J')$ at location J' that is correlated to J by synchrotron side-band overlapping, e.g. $k\Omega_s(J) = l\Omega_s(J')$ between the kth and the lth side-band, as

$$\begin{aligned} D_{SH}(J) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle\int_0^{\Delta t}\int_0^{\Delta t} U_J^I(t)U_J^I(t')\,dtdt'\rangle \\ &= \frac{Nq^4e^4\omega_0^2}{\pi}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\left.\frac{k^2\Psi(J')}{l\left|\frac{d\Omega_s(J')}{dJ'}\right|}\right|_{\Omega_s(J')=k\Omega_s(J)/l} \\ &\quad \left\{|G_{SH}(k,l)|^2 + |G_{SH}(-k,-l)|^2 - 2Re\left[G_{SH}(k,l)G_{SH}(-k,-l)\right]\right\} + O(\xi), \end{aligned} \tag{29}$$

where

$$G_{SH}(k,l) = \sum_{m=1}^{\infty}\frac{1}{m}G\left[m\omega_0 + l\Omega_s(J')\right]e^{-im(\theta_{PU}-\theta_K)}\,\mathrm{J}_k\left[m\omega_0\tau(J)\right]\mathrm{J}_l\left[m\omega_0\tau(J')\right]. \tag{30}$$

The quantity $\left|\frac{d\Omega_s(J)}{dJ}\right|$ appeared in eq. 29 is a monotonically increasing function of J,

$$\frac{d\Omega_s(J)}{dJ} = -\frac{\pi C_W}{16}\frac{1}{k^2\mathrm{K}^3(k)}\left[\frac{\mathrm{E}(k)}{1-k^2} - \mathrm{K}(k)\right], \quad k \le 1. \tag{31}$$

If it is near the separatrix, $k \to 1$, then $\left|\frac{d\Omega_s}{dJ}\right| \to \infty$. This implies that the enhancement of the noise density due to synchrotron oscillation monotonically approaches 0 near the separatrix, while it is the largest at the center of the bucket.

The thermal coefficient D_T is expressed in terms of the thermal temperature T_{PU} at the pick-up. Similar to D_{SH}, it can be obtained using eqs. 1–5,

$$\begin{aligned} D_T(J) &= 2k_B T_{PU} q^2 e^2 \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \frac{l^2}{m^2} \mathrm{J}_l^2\left[m\omega_0 \tau(J)\right] \cdot \\ & \left\{\left|G_T\left[m\omega_0 + l\Omega_s(J)\right]\right|^2 + \left|G_T\left[m\omega_0 - l\Omega_s(J)\right]\right|^2\right\} + O(\xi), \end{aligned} \tag{32}$$

where G_T is the gain in the cooling system excluding the pick-up. Since D_T is proportional to q^2 while D_{SH} is proportional to q^4, the thermal noise contribution often appears less important for the cooling of intense beams of high charge-state ions.

D. Power limitation

The average power required from the amplifier that applies on an array of N_K kickers, each with an equivalent impedance R_K, can be expressed in terms of the total voltage V_K on the kickers,

$$P = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle \int_0^{\Delta t} V_K^2(t) dt \rangle / R_K, \tag{33}$$

where V_K can be split into a Schottky part and a thermal part. The corresponding average Schottky power is proportional both to the number of bunches N_b inside the ring and to the number of particles N inside each bunch,

$$\begin{aligned} \overline{P_{SH}} &= \frac{N_b N q^2 e^2 \omega_0^2}{2\pi^2} \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \int_0^{\hat{J}} dJ \Psi(J) \mathrm{J}_l^2\left[m\omega_0 \tau(J)\right] \\ & \left\{\left|G\left[m\omega_0 + l\Omega_s(J)\right]\right|^2 + \left|G\left[m\omega_0 - l\Omega_s(J)\right]\right|^2\right\}. \end{aligned} \tag{34}$$

The peak Schottky power $\hat{P}_{SH}$ needed from the amplifier is thus $\overline{P_{SH}}$ divided by the bunching factor. As an example, for a bunch of Gaussian distribution,

$$\hat{P}_{SH} = \frac{\sqrt{2\pi} h}{N_b \sigma_\phi} \overline{P_{SH}}, \tag{35}$$

which is independent of N_b. The thermal power P_T can again be expressed in terms of T_{PU},

$$P_T = \frac{k_B T_{PU}}{\pi} \int_0^\infty |G_T(\omega)|^2 d\omega. \tag{36}$$

If the cooling system has N_{PU} pick-ups, each with an equivalent impedance R_{PU}, the gain function G and G_T can be written as

$$G = \sqrt{N_K N_{PU} R_K R_{PU}}\, \tilde{G}, \quad \text{and} \quad G_T = \sqrt{R_K R_{PU}}\, \tilde{G}, \tag{37}$$

where $\tilde{G}$ is the gain in the circuit between the pick-up and the kickers. Obviously, increasing the number of pick-ups will improve the signal-to-noise ratio, while increasing the number of kickers will decrease the amplifier power.

III. Computer Techniques

A computer program has been developed to solve the transport equation (eq. 23) numerically. First, the J-space $(0, J_{max})$ is equally divided into N_J bins of width ΔJ. An initial discrete distribution $\Psi(J_i, t = 0)$, $i = 1, \cdots, N_J$, is then generated. The transport equation (23), written in a difference form, is iterated according to the boundary conditions eq. 24. At each time step, the three-point differentiation formulae[18] are used to evaluate the density-flux difference at each J_i using eqs. 25, 29, and 32. The change in bunch area (first moment in J, i.e. $\sum_{i=1}^{N_J} J_i \Psi(J_i)\Delta J$) is used to obtain the cooling rate.

The accuracy of this numerical approach depends on the choices of the time step and the J step, the calculation of the coefficients, the approximation method of the differentiations, and the treatment of the boundary conditions. In the case of poor modeling, artificial wiggling in $\Psi(J)$ is often observed. Various methods can be used to evaluate the accuracy. The most straightforward one is to evaluate the summation $A_0 = \sum_{i=1}^{N_J} \Psi(J_i)\Delta J$ (zeroth moment in J) and compare it to the expected value, which is equal to 1 in the case of proper cooling without particle loss.

Results obtained from this approach will be discussed in section IV.

Another approach of computer modeling is to perform revolution-by-revolution simulation of the cooling process. However, this is practically limited by the available computer capacity. A relatively small number of representative particles has to be chosen for simulation, and a scaling rule has to be used to interpret the simulation result. Furthermore, the drastic difference between a slow- and a fast-cooling process has to be considered.[19]

IV. Longitudinal Cooling in the RHIC

The heavy-ion beams will be stored in the RHIC collider using a 160 MHz, $h = 2052$ rf system. Consider the beam of one of the highest charge-state ions $^{197}Au^{79+}$ in the RHIC from which intrabeam scattering is expected to be the severest. Each of the 57 or 114 bunches contains 10^9 particles. Due to the high charge state, Schottky noise dominates over the room-temperature thermal noise.

Define the cooling rate τ^{-1} as, within the unit time, the relative reduction in the average phase-space area occupied by the particles in the bunch. Using a cooling system of frequency bandwidth 4–8 GHz, figure 2 shows the cooling rate as a function of the initial bunch area, calculated from the numerical solution of the transport equation (23). In each case, the rf voltage is chosen so that initially the bunch occupies about 75% of the bucket area. The phase spread in the bunch is thus the same. The solid line represents the optimal cooling rate, which is linearly proportional to the initial bunch area.

Assuming a total of 64 kickers, each with an equivalent impedance 100 ohms, figure 3 shows the corresponding peak amplifier power P_{opt} needed to achieve the optimal cooling rate. As the bunch area (or the momentum spread) increases, the required P_{opt} increases to its 4th power. The dash lines in figure 2 indicate the cooling rates that can possibly be achieved with a maximum peak amplifier power no more than 2 kW, 1 kW, and 0.5 kW, respectively.

As shown by the solid line in figure 4, the initial particle distribution is assumed to be a Gaussian distribution in longitudinal phase space, which is modified such that the density smoothly approaches zero at $\hat{J}$ ($\hat{J} = 0.3$eV·sec/amu). The time evolution of the distribution is illustrated by the subsequent dash lines. Correspondingly, figure 5 shows the reduction in the bunch spread as a function of time. If the amplifier gain is fixed, the cooling will soon saturate (the short-dash line); if the gain is constantly decreased according to the bunch spread, the cooling will be continued although with a decreasing cooling rate, $\tau^{-1} \sim \left[\hat{J}(t)\right]^2$ (the long-dash line). Furthermore, if the rf voltage is lowered according to the decreasing momentum spread so that the phase spread (i.e. the filling factor defined in the next section) is kept constant, the cooling will be improved further ($\tau^{-1} \sim \hat{J}(t)$, the solid line).

It is assumed that the cooling system has a single pick-up station consisting of two pairs of stripline located, side by side, in a large dispersion region ($x_p \sim 1$m). By taking the difference of the sum signals from these two pairs, one obtains a signal which is proportional to the horizontal displacement and thus the momentum deviation of the beam sample. The pick-up and the kickers are located at two nearby sextants along the ring, allowing a delay time of about 90 ns. With a 4–8 GHz system, the mixing between the pick-up and the kickers is small for bunch area below 1eV·sec/amu. Since thermal noise is not important, one pick-up should give satisfactory signal-to-noise ratio. The number of kickers, however, has to be much larger than unity to reduce the necessary power.

V. Comparison with the Coasting-Beam Theory

In the case that the number of significant synchrotron side-bands is much larger than unity ($n_{av}\omega_0\hat{\tau} \gg 1$), and that the bunch occupies the entire rf bucket, coasting-beam theory is expected to provide comparable results to that of the bunched-beam theory. In particular, when the phase spread in the bunch is kept constant, the scaling behaviour of the cooling rate and the power on the momentum spread predicted from the coasting-beam theory, should also apply to the bunched beam. This analogy however disappears when the bunch area becomes small compared with the bucket area.

A. Scaling behaviour from the coasting-beam theory

Let $W = \Delta n\ f_0$ be the frequency bandwidth of the cooling system. The momentum-cooling rate obtained from the coasting-beam theory can be expressed as[2]

$$\tau^{-1} = \frac{W}{N}\left[2g - g^2(M+U)\right], \tag{38}$$

where g is the effective gain, $M \approx (2\eta n_{av}\sigma_{\Delta p/p})^{-1}$ is the "mixing factor", U is the noise-to-signal ratio, and $\sigma_{\Delta p/p}$ is the rms momentum spread in the beam. The optimal cooling rate can be readily obtained in the case $U \ll M$, as

$$\tau_{opt}^{-1} = \frac{\Delta n\ f_0}{N\ (M+U)} \approx \frac{2n_{av}\Delta n\ \eta f_0 \sigma_{\Delta p/p}}{N}. \tag{39}$$

The power required to achieve this cooling rate is estimated as

$$P_{opt} \approx \frac{1}{q^2 N_K R_K}\frac{E^2\sigma^2_{\Delta p/p}}{\beta^4 \tau_{opt} f_0 (M+U)}, \tag{40}$$

which, in the case that revolution bands are not overlapping (i.e. "bad mixing" case), is proportional to the momentum spread to the 4th power. The optimal rate is expected only when this P_{opt} is achieved. With a practical power $P \le P_{opt}$, the cooling rate becomes

$$\tau^{-1} = \tilde{g}\tau_{opt}^{-1}, \tag{41}$$

where $\tilde{g} = \sqrt{P/P_{opt}} \le 1$. With a given power, the achievable cooling rate is inversely proportional to the momentum spread in the case of bad mixing. This scaling behaviour is shown by the dash lines in figs. 2 and 6.

The original program (DBFP) using the coasting-beam theory[20] has been modified to obtain an order-of-magnitude estimate of the momentum cooling of the bunched beam in the RHIC. This program essentially solves the Fokker-Planck equation for a coasting beam using numerical method. The peak current of the bunch is taken as the coasting-beam current $\bar{I}$. The initial momentum distribution is assumed to be a Gaussian distribution with its rms value the same as the bunched beam. Figure 6 shows the cooling rate as a function of the momentum spread that corresponds to figure 2. Taking into account the numerical accuracy and the approximations, the agreement between these two theories is within a factor of 2. The fact that the cooling rate calculated from the coasting-beam theory is less than that from the bunched-beam theory, is because that the peak current, instead of a properly averaged current, has been used for coasting-beam calculation.

B. Comparison and discussion

It has been shown above that the agreement between the coasting- and the bunched-beam theory is satisfactory provided that the bunch occupies a large amount (≥75%) of the rf bucket. On the contrary, the prediction from these two theories are drastically different if the relative bunch area is small.

Suppose that a bunch of initial area $\hat{J} = 0.3$ eV·sec/amu is stored using a peak rf voltage $\hat{V}$. Define the rf-bucket filling factor R as the ratio of the bunch area $\hat{J}$ to the bucket area J_{max}. When the voltage $\hat{V}$ is raised, R decreases as $\hat{V}^{-1/2}$ for a given bunch area. The solid line in figure 7 shows the optimal cooling rate as a function of R, calculated from the bunch-beam theory. The scaling rule can be obtained by inspecting the transport equation (23): The synchrotron-oscillation frequency $\Omega_s \sim R^{-1}$; the number of synchrotron side-bands of consideration $n_{av}\omega_0\hat{\tau} \sim R^{1/2}$; therefore, the drifting coefficient $F \sim R^{-1/2}$, while the Schottky coefficient $D_{SH} \sim R^{-3/2}$. Hence, the optimal gain and, consequently, the optimal cooling rate are proportional to R, as shown by the solid line in figure 7. In the case that R approaches zero, the bunch length becomes comparable to the sampling width of the cooling system; stochastic cooling is essentially not possible.

On the other hand, the coasting-beam theory predicts a different scaling behaviour: the peak current $\hat{I} \sim R^{-1}$; the momentum spread $\Delta p/p \sim R^{-1}$; therefore, the optimal cooling rate is independent of R (eq. 39) in the case of bad mixing, as shown by the dash line in figure 7.

The other reason that stochastic cooling for a short bunch is difficult is that the synchrotron-frequency spread is small for particles of small synchrotron-oscillation amplitude (J). In the transport equation, this is taken into account by the factor $d\Omega_s/dJ$.

It has been assumed in the bunched-beam calculation that $\xi \ll 1$, which is valid when particles are away from the separatrix (figure 1). On the other hand, it has also been assumed that the particle density in longitudinal phase space vanishes near the separatrix (eq. 24). In the case that the number of particles near the separatrix is not negligible, the bunch behaves more like a coasting beam. Because particles near the separatrix have larger momentum deviations compared with those near the center of the bucket, they are more strongly affected by the stochastic cooling.

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REFERENCES

1) D. Möhl, G. Petrucci, L. Thorndahl and S. van de Meer, CERN/PS/AA 79-23.

2) D. Möhl, CERN 84-15, 97 (1984).

3) F. Sacherer, CERN-ISR-TH/78-11.

4) J. Bisognano and C. Leeman, AIP Conf. Proc. No. 87, 583 (1982).

5) S. Chattopadhyay, LBL-14826, 1982.

6) Brookhaven National Laboratory, *Conceptual Design of the RHIC*, BNL-52195 (1989).

7) G. Parzen, Nucl. Instrum. Methods A251, 220 (1986).

8) S. Van de Meer, RHIC-AP-9 (1984).

9) J. Claus, private communication.

10) H. Goldstein, *Classical Mechanics* (Addison-Wesley, New York, 1953).

11) I.S. Gradshteyn and I.M. Ryshik, *Table of Integrals, Series, and Products* (Academic, New York, 1965).

12) J.M. Jowett, AIP Conf. Proc. No. 153, 864 (1987).

13) G. Dôme, CERN 84-15 (Geneva, 1984), p.215.

14) E. Jahnke and F. Emde, *Tables of Functions with formulae and curves* (Dover, New York, 1945), p92.

15) M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945).

16) N. Wax, *Selected papers on noise and Stochastic Processes* (Dover, New York, 1954).

17) S. Krinsky and J. Wang, Particle Accelerators, **12**, 107 (1982).

18) J. Wei, S.Y. Lee, and A.G. Ruggiero, Particle Accelerators **24**, 211 (1989).

19) J. Wei, et. al., unpublished.

20) A.G. Ruggiero, *Design of the Stochastic Cooling for the Experimental Storage Ring at GSI*, Oct. 1985.

Appendix. Longitudinal Schottky Signal of a Bunched Beam

The current of the circulating beam can be written as

$$I(t) = qe \sum_{m=-\infty}^{\infty} \sum_{j=1}^{N} \delta\left(t - \frac{2\pi m}{\omega_0} - \frac{\phi_j}{h\omega_0}\right). \tag{42}$$

Substituting into eq. 42 the expression of synchrotron oscillation

$$\phi_j = h\omega_0\tau_j \cos\left(\frac{2\pi n\Omega_{s,j}}{\omega_0} + \varphi_j^0\right), \tag{43}$$

where φ_j^0 is the initial phase of synchrotron oscillation of the jth particle, the Fourier transform of $I(t)$ becomes

$$I(\omega) = qe\omega_0 \sum_{m=-\infty}^{\infty} \sum_{j=1}^{N} \sum_{k=-\infty}^{\infty} i^k \mathrm{J}_k(-\omega\tau_j) e^{ik\varphi_j^0} \delta(\omega - m\omega_0 + k\Omega_{s,j}). \tag{44}$$

Obviously, the width of the frequency spread of the kth synchrotron side-band is $|k|$ times that of the synchrotron-frequency spread of the bunch. The number of significant synchrotron side-bands of the nth revolution band is $2n\omega_0\hat{\tau}$.

Assume that initially particles are uniformly distributed in phase with $\langle\varphi_j^0\rangle = 0$, and that different revolution bands are not overlapping. The power of the beam current at the nth revolution band becomes

$$\langle I^2(t)\rangle = (qef_0)^2 \left\{ \left[\sum_{j=1}^{N} J_0(n\omega_0\tau_j)\right]^2 + \sum_{j=1}^{N} \left[1 - J_0^2(n\omega_0\tau_j)\right] \right\}. \tag{45}$$

The first term in eq. 45 corresponds to the coherent signal at the multiple of the synchronous revolution frequency nf_0. At high frequency ($n\omega_0\hat{\tau} \gg 1$), this term is contributed only from the particles of small oscillation amplitude,

$$n\omega_0\tau_j \leq 2.4, \qquad j = 1, \cdots, N. \tag{46}$$

Nevertheless, because this coherent part is proportional to N^2, it may be comparable to the Schottky signal even at frequencies as high as the microwave cutoff frequency of the vacuum pipe. The second term in eq. 45 corresponds to the so-called Schottky power. In contrast to that of a coasting beam, the Schottky power at different revolution band n is not the same. Nevertheless, at high frequency ($n\omega_0\hat{\tau} \gg 1$) it is very nearly a constant.

FIGURE CAPTIONS

Figure 1. The parameter ξ as a function of relative phase-space area J/J_{max}.

Figure 2. The bunch-area reduction rate as a function of initial bunch area, evaluated by numerically solving the bunched-beam transport equation.

Figure 3. The amplifier power required to achieve the optimal cooling rate, as a function of the initial bunch area to the 4th power.

Figure 4. The time evolution of the particle density distribution in J during the longitudinal stochastic-cooling process.

Figure 5. The reduction in bunch area as a function of time that corresponds to figure 3.

Figure 6. The momentum-spread reduction rate as a function of the equivalent initial bunch area, calculated by numerically solving the coasting-beam transport equation.

Figure 7. A comparison of the optimal cooling rate obtained from the bunched-beam and the coasting-beam theory, respectively.

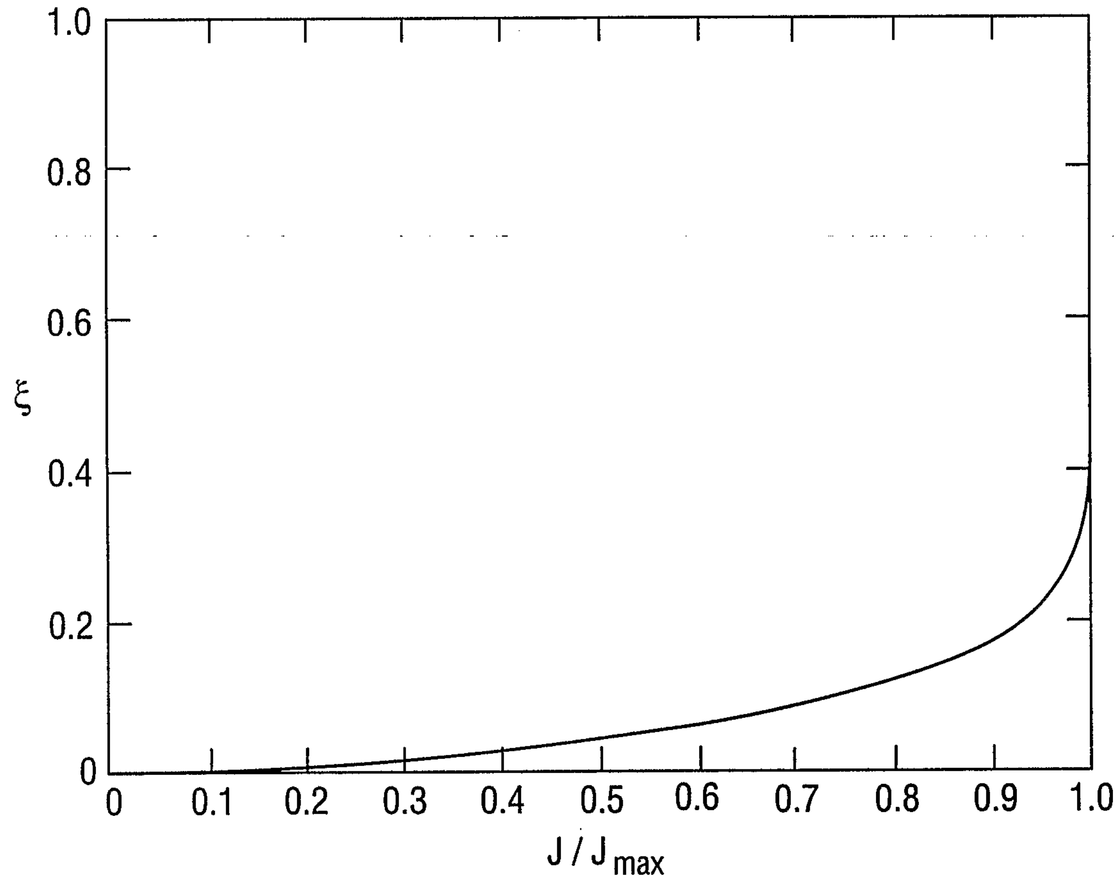
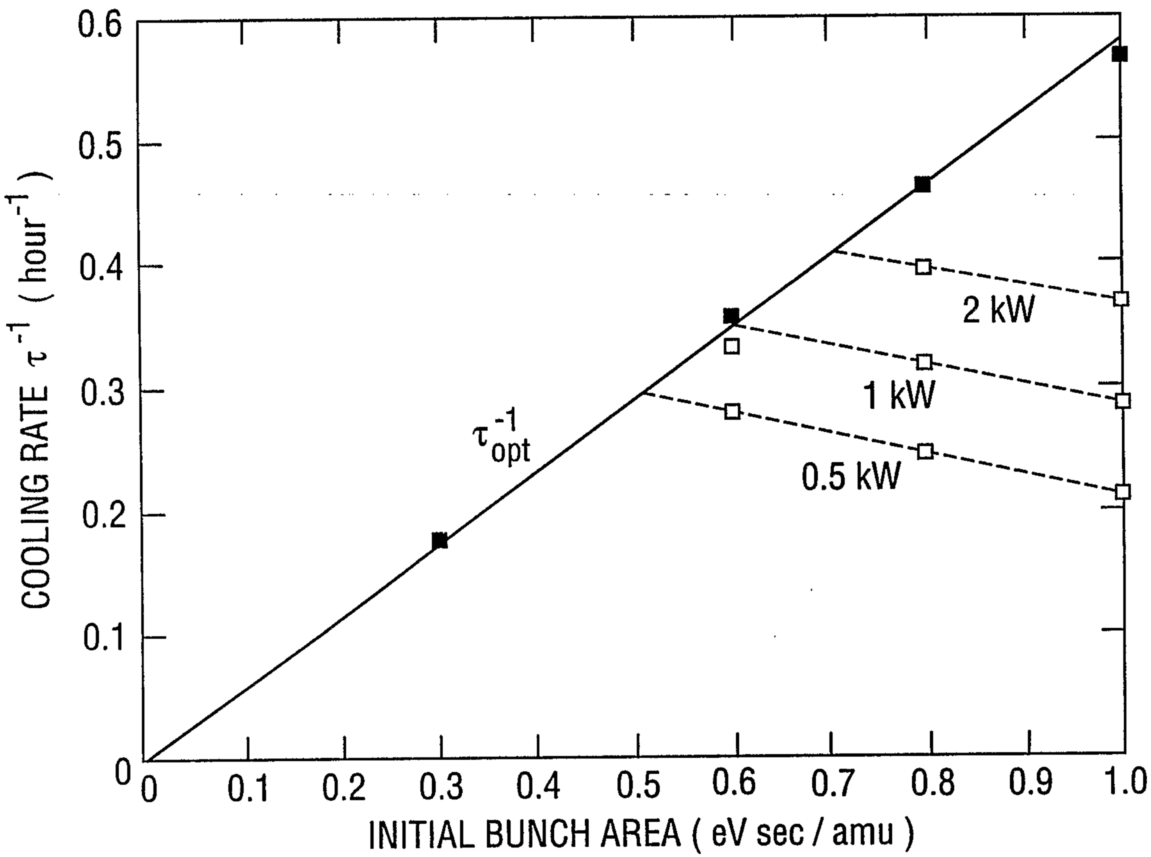


Figure 1.



COOLING RATE τ⁻¹ (hour⁻¹)
INITIAL BUNCH AREA (eV sec / amu)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
τ⁻¹opt
2 kW
1 kW
0.5 kW

Figure 2.

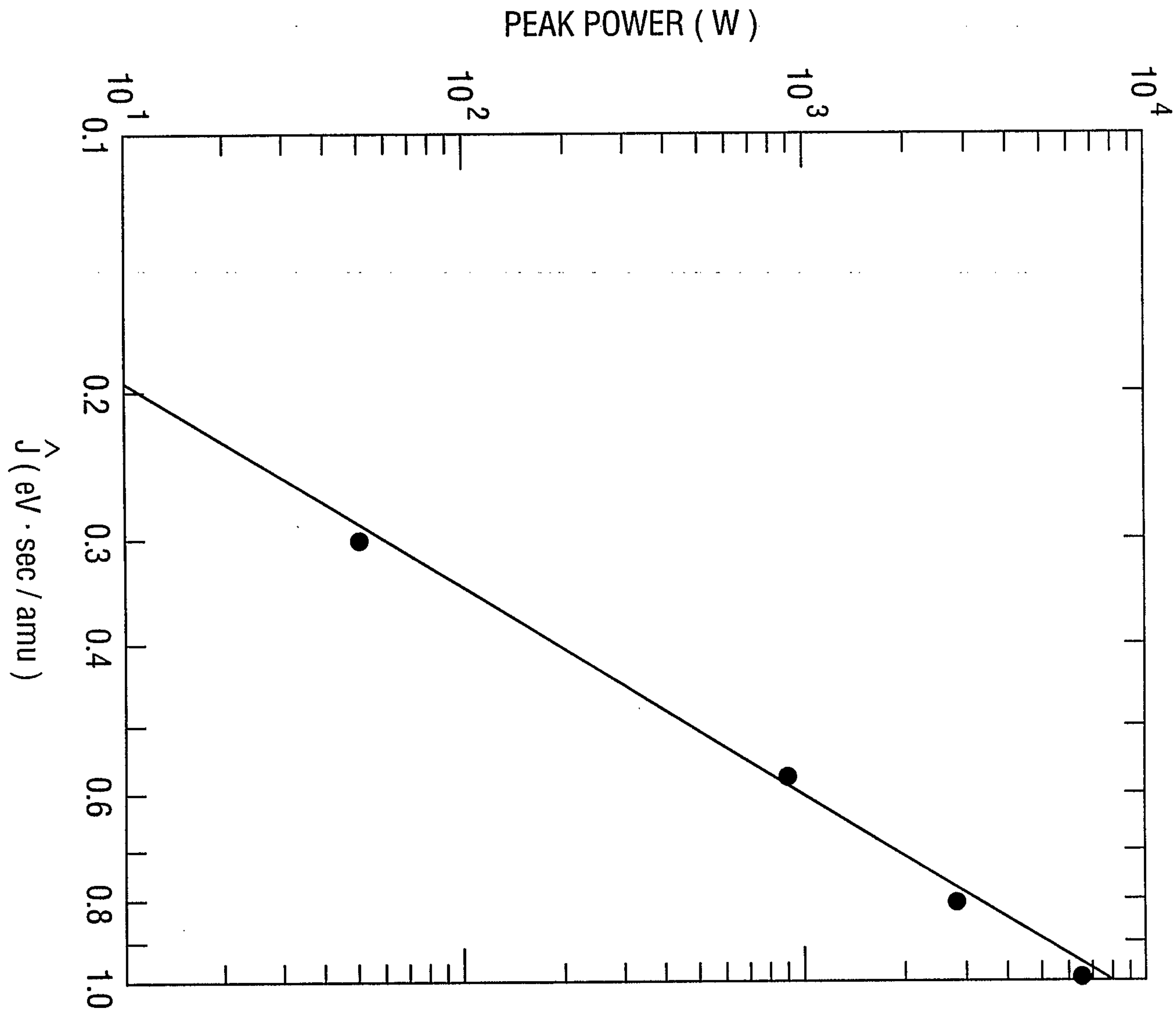


Figure 3.

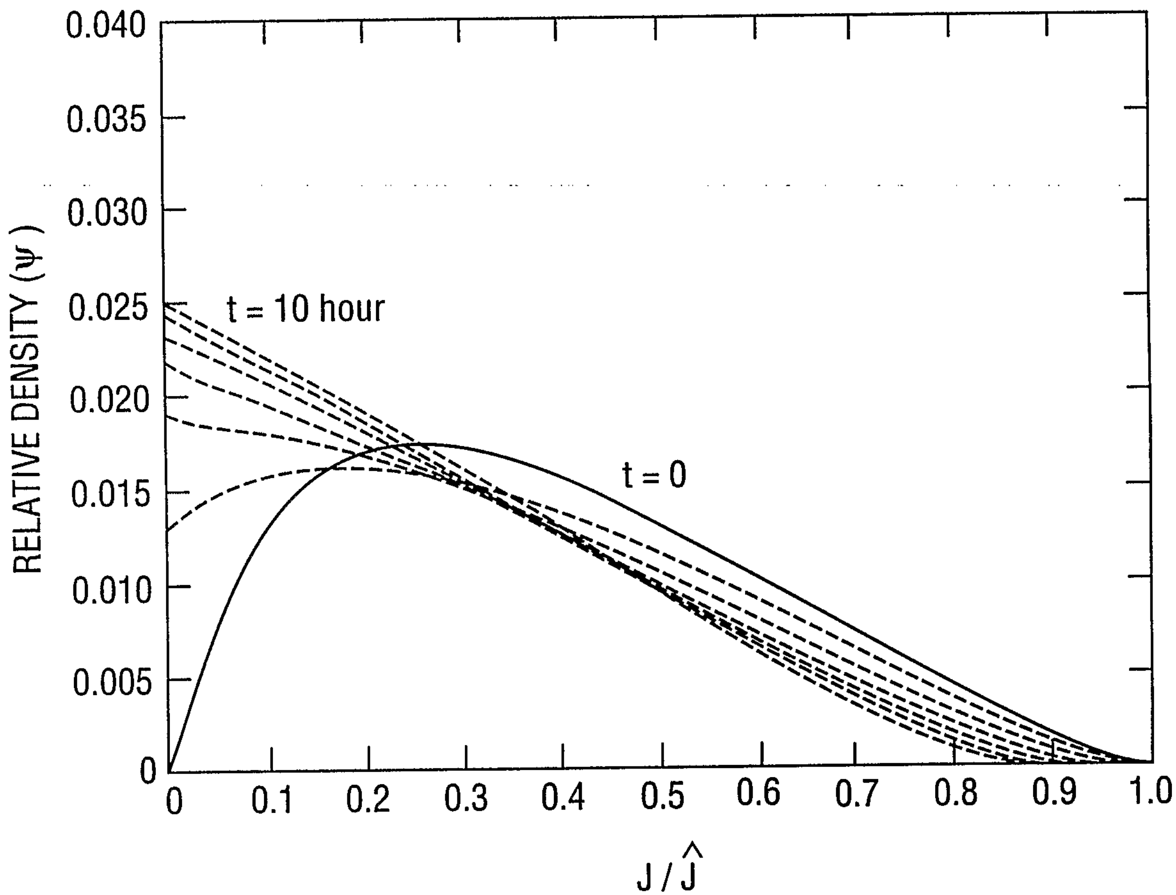
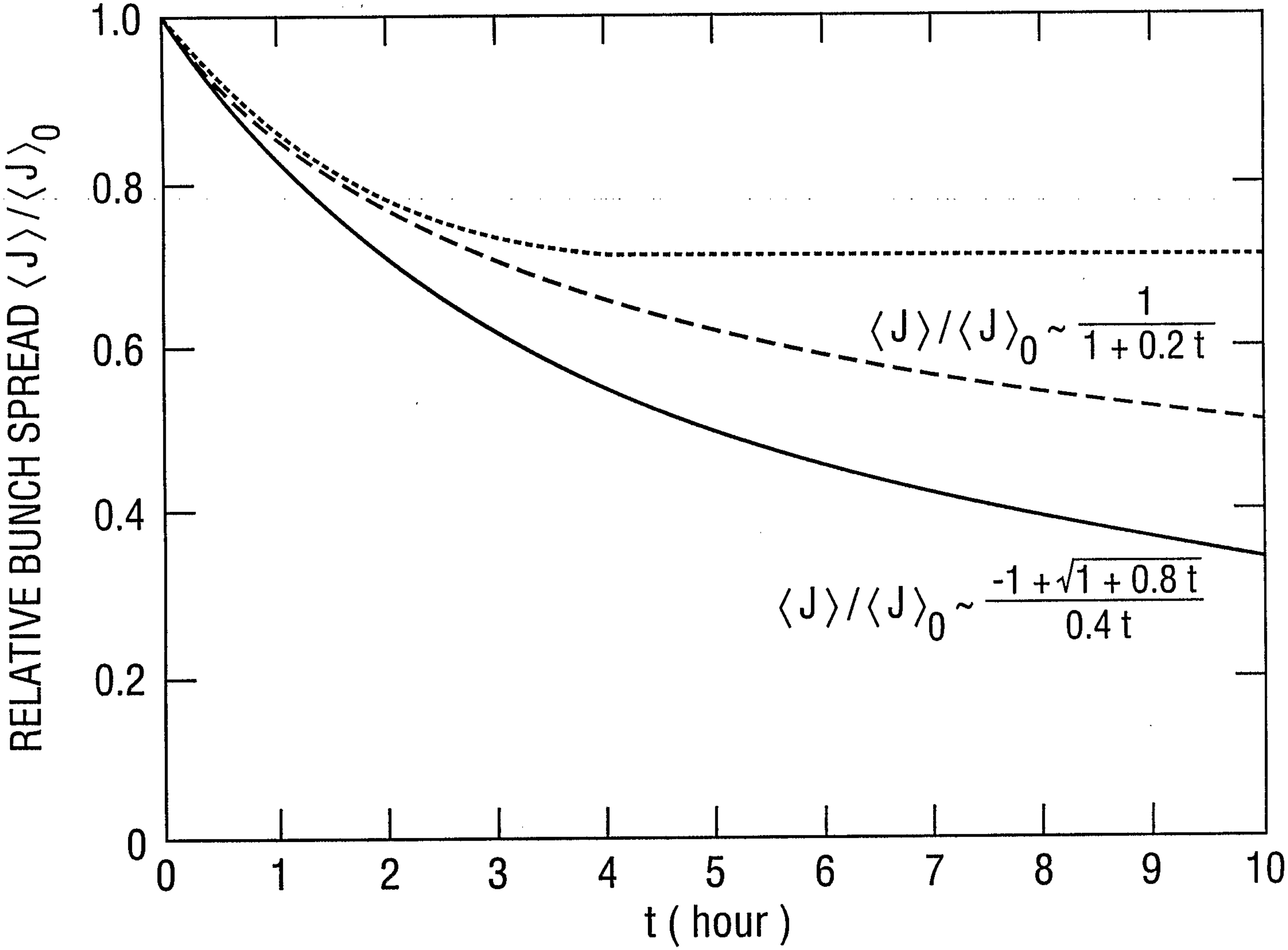
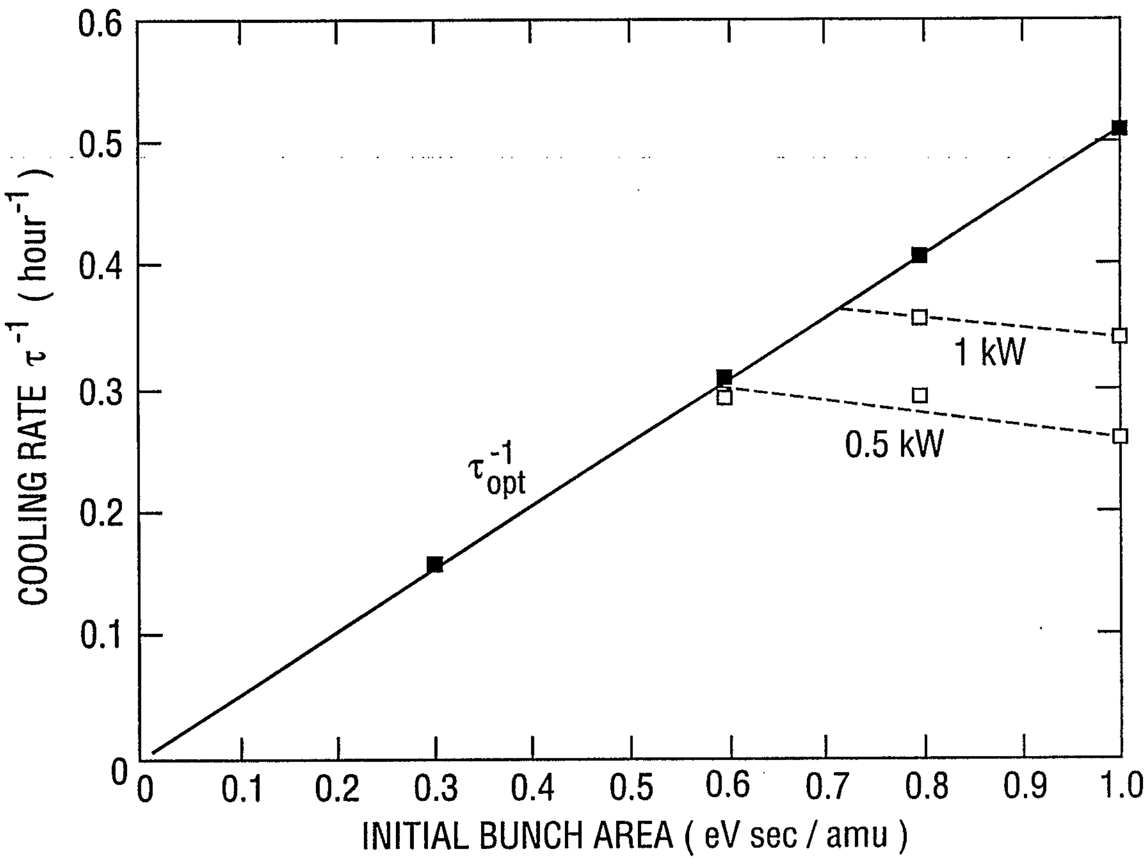

0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0
RELATIVE DENSITY (ψ)
t = 10 hour
t = 0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
J / Ĵ

Figure 4.



RELATIVE BUNCH SPREAD ⟨J⟩/⟨J⟩$_0$
1.0
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
9
10
t (hour)
$\langle J \rangle / \langle J \rangle_0 \sim \frac{1}{1 + 0.2\,t}$
$\langle J \rangle / \langle J \rangle_0 \sim \frac{-1 + \sqrt{1 + 0.8\,t}}{0.4\,t}$

Figure 5.



0.6
0.5
0.4
0.3
0.2
0.1
0
COOLING RATE τ^-1 (hour^-1)
τ^-1_opt
1 kW
0.5 kW
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
INITIAL BUNCH AREA (eV sec / amu)

Figure 6.

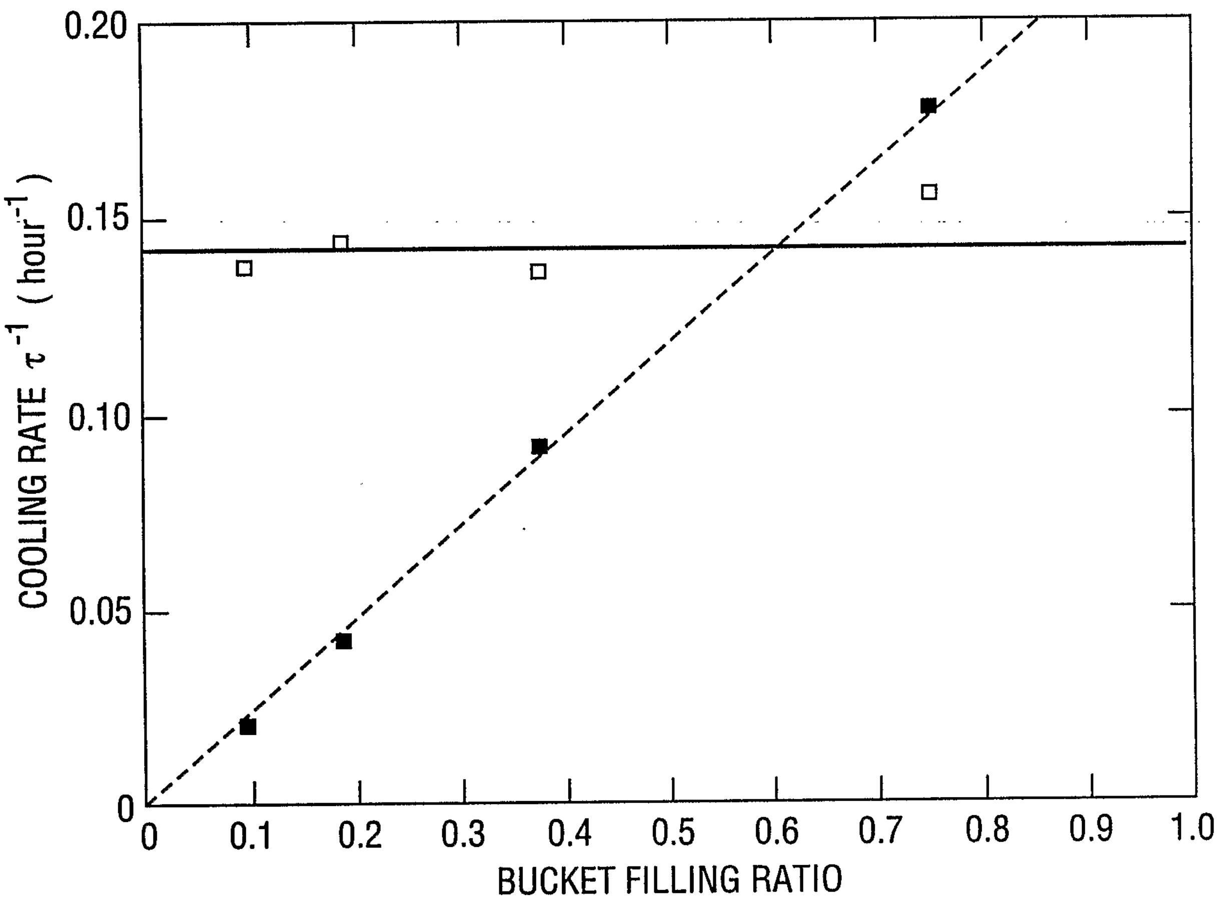

0.20
0.15
0.10
0.05
0
COOLING RATE τ⁻¹ (hour⁻¹)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
BUCKET FILLING RATIO

Figure 7.