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Luminosity for Unequal Beams

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RHIC-AP-10

LUMINOSITY FOR UNEQUAL BEAMS

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Luminosity For Unequal Beams: Do We Gain or Lose Luminosity if We have unequal Beam Sizes?

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(December 8, 1984)

Abstract:

For unequal colliding beams an approximate expression ~~are~~ is derived for the luminosity. This expression reduces to the one previously derived for equal beams in RHIC-4. The validity of this expression is briefly discussed and it is finally concluded ~~that~~ for all cases of physical relevance the luminosity will increase if one beam size is smaller.

Luminosity For Unequal Beams

Question to be answered:

Do we gain or lose luminosity

If we have unequal beam sizes?

Answer:

For all cases of physical relevance the luminosity will increase if one beam size grows smaller.

I Discussion:

In this talk I will show the motivation for the above assertion. To do so I will extend previous work that I've done on the luminosity to the case of unequal beams. An approximate formula will be developed for the luminosity which gives excellent agreement with the exact expression. This expression should reduce to one that was derived in RHIC Technical Note # 4 [1] for equal beams.

The expression will be valid for the most general case of a non-zero crossing angle.

II The Luminosity

Assumptions:

1) Both colliding beams are bunched.
2) The lattice has zero dispersion at the crossing point.

The most general expression for the luminosity is given by[2)3)]

$$L = N_1 N_2 f_{encounter} F \qquad (1)$$

where:

$f_{encounter} \equiv$ The frequency of encounter of the beams $= f_0 B$

$f_0 \equiv$ frequency of Revolution

$B \equiv$ # of bunches / ring

$N_1, N_2 \equiv$ # of Particles / bunch in beams 1 and 2 respectively and

$$F = \frac{2}{(2\pi)^{3/2}(\sigma_{\ell_1}^2+\sigma_{\ell_2}^2)^{1/2}} \int_{-\infty}^{+\infty} \frac{dS\, e^{-2S^2\left[\frac{1}{\sigma_{\ell_1}^2+\sigma_{\ell_2}^2} + \frac{\alpha^2/4}{\sigma_{x_1}^2+\sigma_{x_2}^2}\right]}}{(\sigma_{x_1}^2+\sigma_{x_2}^2)^{1/2}(\sigma_{z_1}^2+\sigma_{z_2}^2)^{1/2}} \quad (2)$$

In equation (2) the $\sigma_{\ell_i} \equiv$ r.m.s bunch lengths
the $\sigma_{x_i} \equiv$ horizontal beam sizes $= \sigma_{H_i}$
the $\sigma_{z_i} \equiv$ vertical beam sizes $= \sigma_{V_i}$

Define an effective bunch length by

$\sigma_\ell^{eff} \equiv \sqrt{\sigma_{\ell_1}^2 + \sigma_{\ell_2}^2}$ so that

$$F = \frac{2}{(2\pi)^{3/2}\sigma_\ell^{eff}} \int_{-\infty}^{+\infty} \frac{dS\, e^{-2S^2\left[\frac{1}{\sigma_\ell^{eff\,2}} + \frac{\alpha^2/4}{\sigma_{H_1}^2+\sigma_{H_2}^2}\right]}}{(\sigma_{H_1}^2+\sigma_{H_2}^2)^{1/2}(\sigma_{V_1}^2+\sigma_{V_2}^2)^{1/2}} \quad (3)$$

We make the usual assumption that both σ_{H_i} and σ_{V_i} depend on S in the following way:

$$\sigma_{H,V}^2 \equiv \sigma_{H,V}^{*2}(1 + S^2/B_{H,V}^{*2}) \quad (4)$$

So that if we make a change of variables $S \to \sigma_\ell^{eff} S'$ Consider the case

$\sigma_\ell^{eff} << \beta^*_{H_1}, \beta^*_{H_2}$ and drop terms of order s'^4 we immediately arrive at the following expression:

$$F \cong \frac{4}{(2\pi)^{3/2}(\sigma^{*2}_{H_1}+\sigma^{*2}_{H_2})^{1/2}(\sigma^{*2}_{V_1}+\sigma^{*2}_{V_2})^{1/2}} \int_0^\infty \frac{ds' \, e^{-2s'^2\left[1+\frac{\alpha^2 \sigma_\ell^{eff\,2}}{4(\sigma^{*2}_{H_1}+\sigma^{*2}_{H_2})}\right]}}{\sqrt{1+\rho^2 s'^2}} \qquad (5)$$

where
$$\rho^2 \equiv \frac{\sigma_\ell^{eff\,2}}{\sigma^{*2}_{H_1}+\sigma^{*2}_{H_2}}\left(\frac{\sigma^{*2}_{H_1}}{\beta^{*2}_{H_1}}+\frac{\sigma^{*2}_{H_2}}{\beta^{*2}_{H_2}}\right)+\frac{\sigma_\ell^{eff\,2}}{\sigma^{*2}_{V_1}+\sigma^{*2}_{V_2}}\left(\frac{\sigma^{*2}_{V_1}}{\beta^{*2}_{V_1}}+\frac{\sigma^{*2}_{V_2}}{\beta^{*2}_{V_2}}\right). \qquad (6)$$

The dominant contribution comes from the region about the origin where we can make the approximation

$$\frac{1}{\sqrt{1+\rho^2 s'^2}} \cong e^{-s'^2 \rho^2/2} \qquad (7)$$

so that immediately

$$F \cong \frac{1}{\pi(\sigma^{*2}_{H_1}+\sigma^{*2}_{H_2})^{1/2}(\sigma^{*2}_{V_1}+\sigma^{*2}_{V_2})^{1/2}} \frac{1}{\sqrt{1+\rho^2/4+\alpha^2\sigma_\ell^{eff\,2}/4(\sigma^{*2}_{H_1}+\sigma^{*2}_{H_2})}} \qquad (8)$$

writing this expression out fully

$$F_{gen} \cong \frac{1}{\pi(\sigma_{H_1}^{*2}+\sigma_{H_2}^{*2})^{1/2}(\sigma_{V_1}^{*2}+\sigma_{V_2}^{*2})^{1/2}} \frac{1}{\sqrt{1+\frac{\sigma_\ell^{eff\,2}}{4(\sigma_{H_1}^{*2}+\sigma_{H_2}^{*2})}\left(\frac{\sigma_{H_1}^{*2}}{\beta_{H_1}^{*2}}+\frac{\sigma_{H_2}^{*2}}{\beta_{H_2}^{*2}}\right)+\frac{\sigma_\ell^{eff\,2}}{4(\sigma_{V_1}^{*2}+\sigma_{V_2}^{*2})}\left(\frac{\sigma_{V_1}^{*2}}{\beta_{V_1}^{*2}}+\frac{\sigma_{V_2}^{*2}}{\beta_{V_2}^{*2}}\right)+\frac{\alpha^2\sigma_\ell^{eff\,2}}{4(\sigma_{H_1}^{*2}+\sigma_{H_2}^{*2})}}}. \quad (9)$$

Now for equal beams

$\sigma_{H_1}^* = \sigma_{H_2}^* \equiv \sigma_H^*$, $\sigma_{V_1}^* = \sigma_{V_2}^* \equiv \sigma_V^*$, $\beta_{H_1}^* = \beta_{H_2}^* \equiv \beta_H^*$, $\sigma_{\ell_1} = \sigma_{\ell_2} \equiv \sigma_\ell$ so that $\sigma_\ell^{eff\,2} = 2\sigma_\ell^2$ and $\beta_{V_1}^* = \beta_{V_2}^* \equiv \beta_V^*$ so that

$$F_{eq} \cong \frac{1}{2\pi\,\sigma_H^*\,\sigma_V^*}\;\frac{1}{\sqrt{1+\frac{1}{2}\left(\frac{\sigma_\ell}{\beta_H^*}\right)^2+\frac{1}{2}\left(\frac{\sigma_\ell}{\beta_V^*}\right)^2+\left(\frac{\alpha\sigma_\ell}{2\sigma_H^*}\right)^2}} \quad (10)$$

which was obtained in RHIC-4.

Before we compare equations (9) and (10) we want to see how well they approximate the exact expressions. Figures 1 thru 4 show that (9) & (10) are quite good approximations to the exact expressions.

Now we wish to let beam 2 become less in size than beam 1 and compare equations (9) and (10) which will determine whether the luminosity increases or diminishes. To do this let $\sigma^*_{H_1} = \sigma_H$, $\sigma^*_{V_1} = \sigma_V$. We consider 2 cases:

a) β's are tuned at the crossing point in such a way as to be equal

$$\beta^*_{H_1} = \beta^*_{H_2} = \beta^*_H \;,\; \beta^*_{V_1} = \beta^*_{V_2} = \beta^*_V \;,\; \sigma_{\ell_1} = \sigma_{\ell_2} \equiv \sigma_\ell$$

$$F_{gen} \cong \frac{1}{\pi(\sigma_H^{*2}+\sigma_{H_2}^{*2})^{1/2}(\sigma_V^{*2}+\sigma_{V_2}^{*2})^{1/2}} \frac{1}{\sqrt{1+\sigma_\ell^2/2\beta_H^{*2}+\sigma_\ell^2/2\beta_V^{*2}+\alpha^2\sigma_\ell^2/2(\sigma_H^{*2}+\sigma_{H_2}^{*2})}} \qquad (11)$$

which immediately shows that for $\sigma^*_{H_2}, \sigma^*_{V_2} < \sigma^*_H, \sigma^*_V$ and for small crossing angle LUMINOSITY INCREASES.

b) β's are not tuned at the crossing point and bunch lengths are equal:

$$F_{gen} \cong \frac{1}{\pi(\sigma_H^{*2}+\sigma_{H_2}^{*2})^{1/2}(\sigma_V^{*2}+\sigma_{V_2}^{*2})^{1/2}} \frac{1}{\sqrt{1+\frac{\sigma_\ell^2}{2(\sigma_H^{*2}+\sigma_{H_2}^{*2})}\left(\frac{\sigma_H^{*2}}{\beta_H^{*2}}+\frac{\sigma_{H_2}^{*2}}{\beta_{H_2}^{*2}}\right)+\frac{\sigma_\ell^2}{2(\sigma_V^{*2}+\sigma_{V_2}^{*2})}\left(\frac{\sigma_V^{*2}}{\beta_V^{*2}}+\frac{\sigma_{V_2}^{*2}}{\beta_{V_2}^{*2}}\right)+\frac{\alpha^2\sigma_\ell^2}{2(\sigma_H^{*2}+\sigma_{H_2}^{*2})}}} \qquad (12)$$

take limit as $\epsilon_{H_2}, \epsilon_{V_2} \to 0$ since the β's are fixed:

then

$$F_{gen} \longrightarrow \frac{1}{\pi\sigma_H^*\sigma_V^*} \frac{1}{\sqrt{1+\sigma_\ell^2/2\beta_H^{*2} + \sigma_\ell^2/2\beta_V^{*2} + \alpha^2\sigma_\ell^2/2\sigma_H^{*2}}} \quad . \qquad (13)$$

Upon Comparing equations (13) and (10) we see that for this Case the LUMINOSITY INCREASES also.

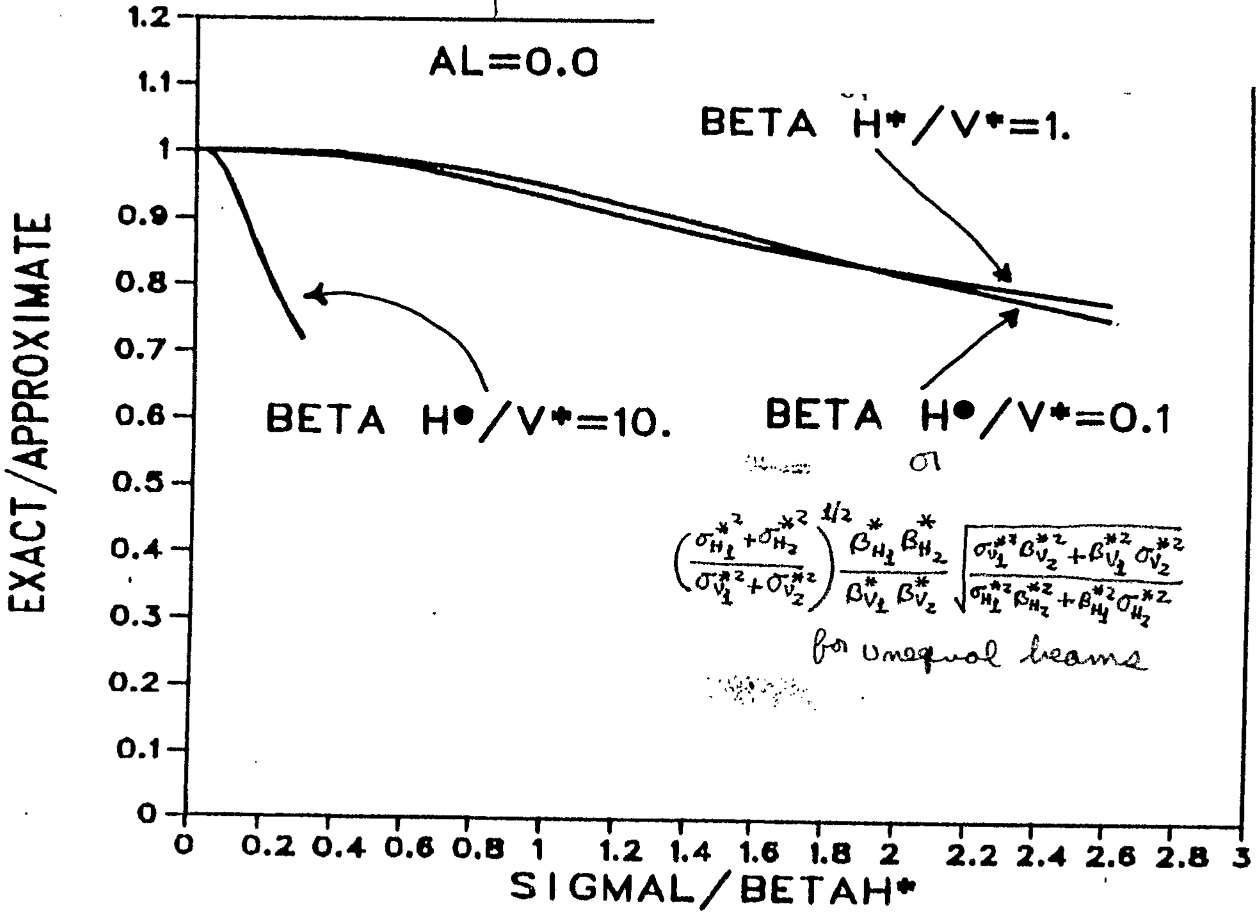
References!

1) L.E. Roberts, RHIC-4, August 29, 1984.

2) Landau, Lifshitz, "Classical Theory of Fields," Addison-Wesley, 1971, Pages 34-36.

3) L. Smith, PEP-Note-20, Lawrence Berkeley Radiation Laboratory, 1972.

$$
AL = \begin{cases} \dfrac{\alpha \sigma_\ell}{2\sigma_H^*} & \text{for equal beams} \\[2ex] \dfrac{\alpha \sigma_\ell^{eff}}{2\sqrt{\sigma_{H_1}^{*2} + \sigma_{H_2}^{*2}}} & \text{for unequal beams} \end{cases}
$$

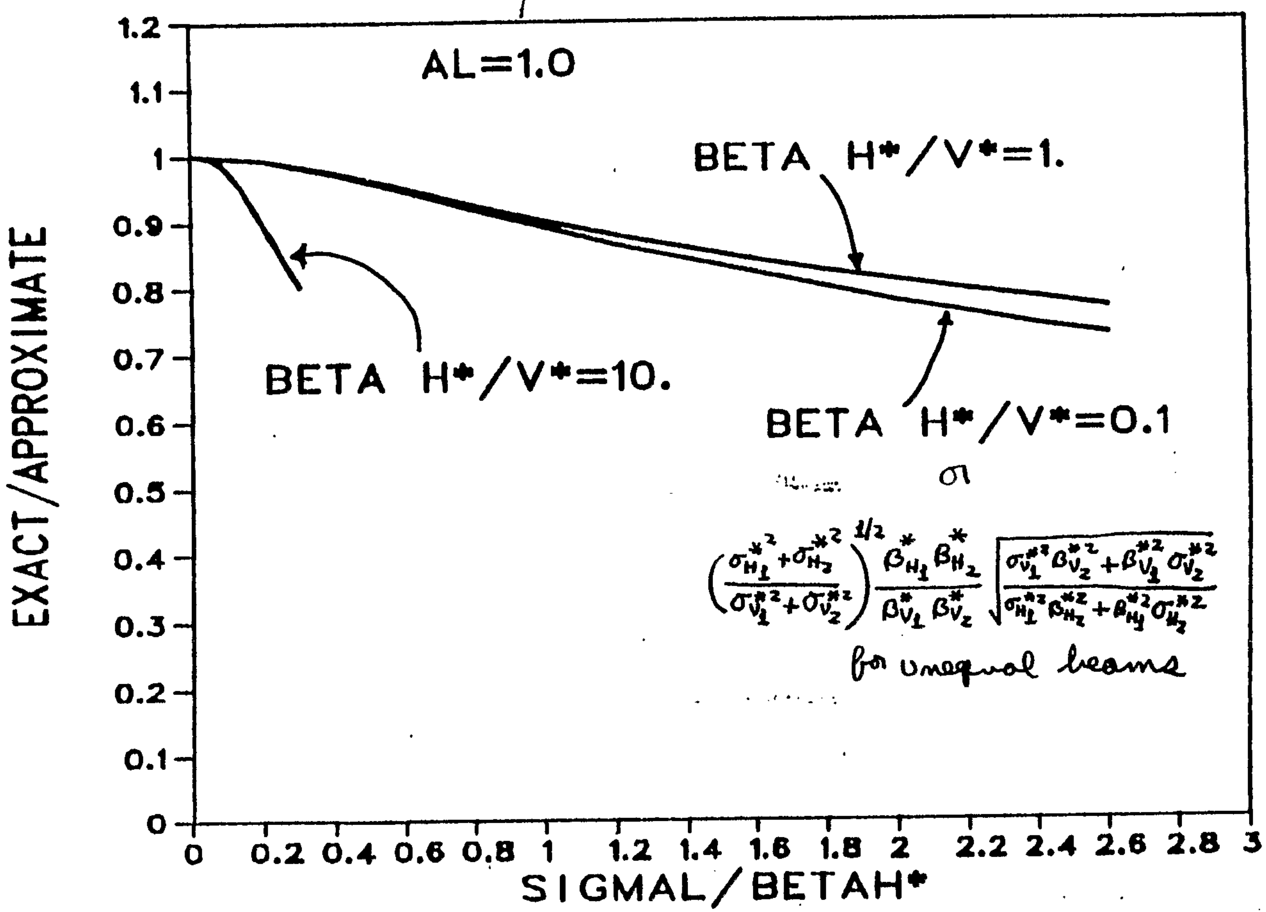


OR

$$
\frac{\sigma_\ell^{eff}}{\sqrt{2(\sigma_{H_1}^{*2} + \sigma_{H_2}^{*2})}} \left(\frac{\sigma_{H_1}^{*2}}{\beta_{H_1}^{*2}} + \frac{\sigma_{H_2}^{*2}}{\beta_{H_2}^{*2}} \right)^{1/2} \quad \text{for unequal Beams}
$$

Fig. 1

$$AL = \begin{cases} \dfrac{\alpha\sigma_\ell}{2\sigma_H^*} & \text{for equal beams} \\[2ex] \dfrac{\alpha\sigma_\ell^{eff}}{2\sqrt{\sigma_{H_1}^{*2}+\sigma_{H_2}^{*2}}} & \text{for unequal beams} \end{cases}$$

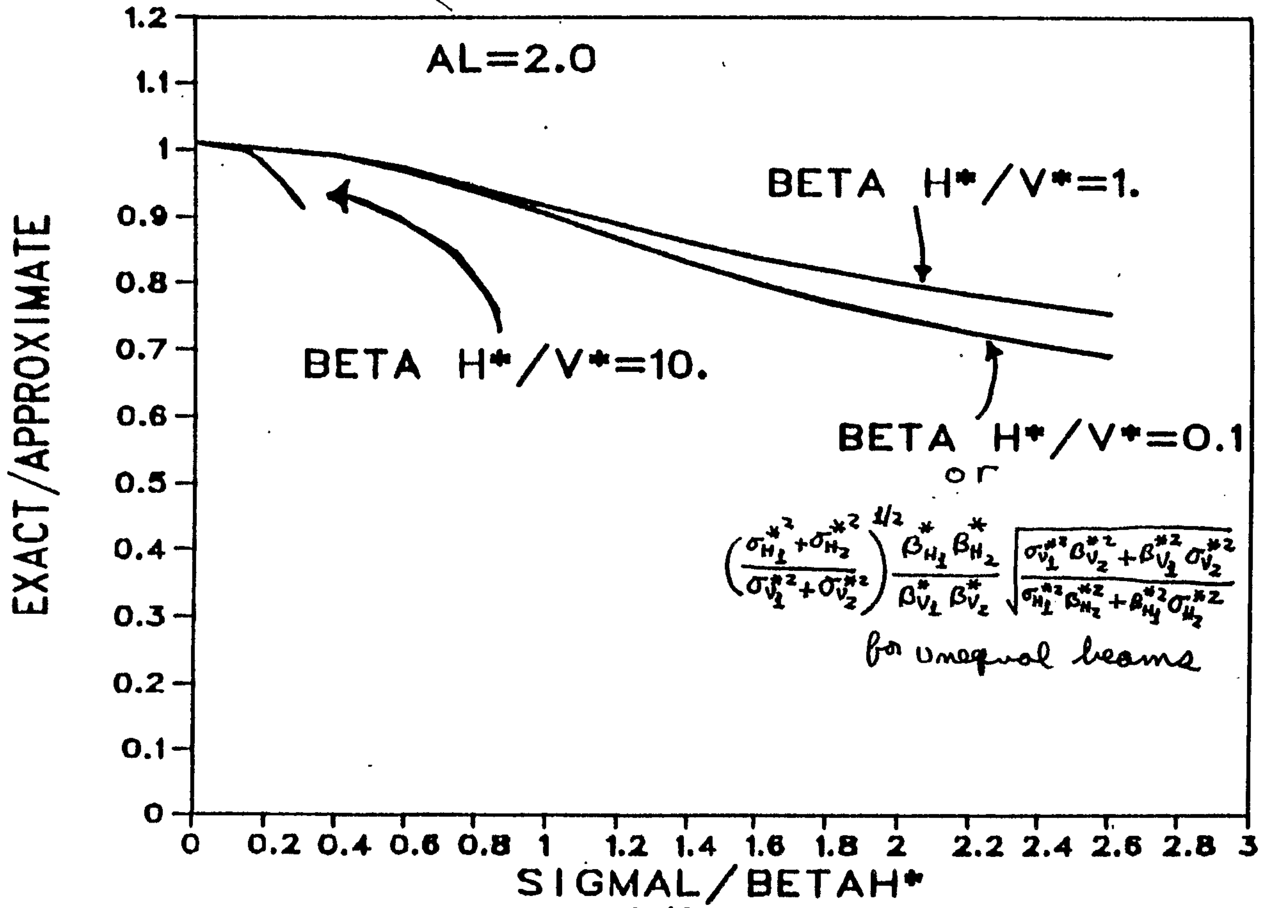


OR

$$\frac{\sigma_\ell^{eff}}{\sqrt{2(\sigma_{H_1}^{*2}+\sigma_{H_2}^{*2})}}\left(\frac{\sigma_{H_1}^{*2}}{\beta_{H_1}^{*2}}+\frac{\sigma_{H_2}^{*2}}{\beta_{H_2}^{*2}}\right)^{1/2} \quad \text{for unequal Beams}$$

FIG. 1

$$AL = \begin{cases} \dfrac{\alpha\sigma_l}{2\sigma_H^*} & \text{for equal beams} \\[2ex] \dfrac{\alpha\sigma_l^{eff}}{2\sqrt{\sigma_{H_1}^{*2}+\sigma_{H_2}^{*2}}} & \text{for unequal beams} \end{cases}$$

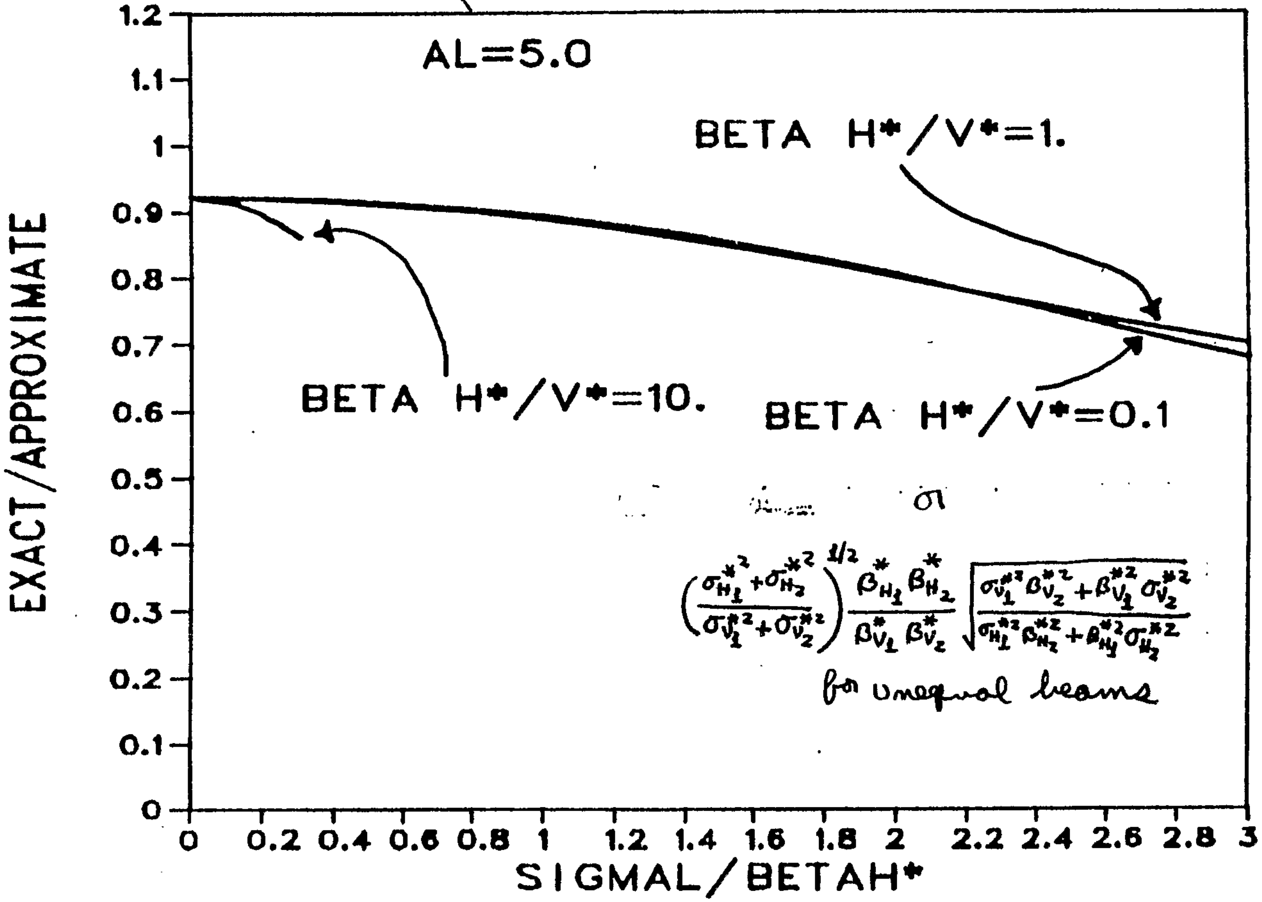


or

$$\frac{\sigma_l^{eff}}{\sqrt{2(\sigma_{H_1}^{*2}+\sigma_{H_2}^{*2})}} \left(\frac{\sigma_{H_1}^{*2}}{\beta_{H_1}^{*2}} + \frac{\sigma_{H_2}^{*2}}{\beta_{H_2}^{*2}}\right)^{1/2}$$ for unequal Beams

Fig. 3

$$AL = \begin{cases} \dfrac{\alpha\sigma_\ell}{2\sigma_H^*} & \text{for equal beams} \\ \dfrac{\alpha\sigma_\ell^{eff}}{2\sqrt{\sigma_{H_1}^{*2}+\sigma_{H_2}^{*2}}} & \text{for unequal beams} \end{cases}$$



or

$$\left(\frac{\sigma_{H_1}^{*2}+\sigma_{H_2}^{*2}}{\sigma_{V_1}^{*2}+\sigma_{V_2}^{*2}}\right)^{1/2}\frac{\beta_{H_1}^*\beta_{H_2}^*}{\beta_{V_1}^*\beta_{V_2}^*}\sqrt{\frac{\sigma_{V_1}^{*2}\beta_{V_2}^{*2}+\beta_{V_1}^{*2}\sigma_{V_2}^{*2}}{\sigma_{H_1}^{*2}\beta_{H_2}^{*2}+\beta_{H_1}^{*2}\sigma_{H_2}^{*2}}}$$

for unequal beams

OR

$$\frac{\sigma_\ell^{eff}}{\sqrt{2(\sigma_{H_1}^{*2}+\sigma_{H_2}^{*2})}}\left(\frac{\sigma_{H_1}^{*2}}{\beta_{H_1}^{*2}}+\frac{\sigma_{H_2}^{*2}}{\beta_{H_2}^{*2}}\right)^{1/2}$$

for unequal Beams