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A Different Approach to Non-Linearities and Particle Tracking

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(Talk at RHIC Meeting, October 23, 1985

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Introduction

Earlier this year Ed Bleser organized a series of meetings on computers and accelerators. One purpose was to expose and discuss various views on that subject. At the time I had only good intentions but nothing to show. Now I do have some code that might be suitable and I should like to describe it to you.

I assume that we want to arrive at a computer model for a particular machine. A very important function of such a model is its description of linear behavior. Many existing programs perform this function very well. Nonlinear behavior poses a problem. Since nonlinear behavior is important our aim is to extend the linear optics programs to include nonlinear optics.

In all existing programs the machine under consideration is described as a string of optical elements. The action of each element is represented by a linear operation, i.e. by a matrix, which relates a particle's phase space coordinates at the exit of an element to those at its entrance. The effect of a sequence of elements is obtained by multiplication of their operators.

This approach can be extended to incorporate nonlinear behavior. Karl Browns "Transport" is an example. It has had a second order feature for more than a decade.

Second order optics may be not good enough however and the algebra associated with it is rather tedious.

The increasing power of computers may make it possible to go further if the required algebra can be put into the software. The code I am going to describe represents an attempt to do that. We again describe the system in terms of elements. Each element is characterized by its transfer function, which is specified as a table of coefficients. These coefficients are used in calculating a particle's phase space coordinates at the exit of an element from its coordinates at the entrance:

$$\bar{X} = \sum a_{xijk\ell}\, x^i x'^j y^k y'^\ell \qquad 0 \le i$$

$$\bar{X}' = \sum a_{x'ijh\ell}\, x^i x'^j y^h y'^\ell \qquad 0 \le j$$

$$\bar{Y} = \sum a_{yijk\ell}\, x^i x'^j y^k y'^\ell \qquad 0 \le k$$

$$\bar{Y}' = \sum a_{y'ijk\ell}\, x^i x'^j y^h y'^\ell \qquad 0 \le \ell$$

$$i + j + k + \ell = n$$

$$\sum = \sum^i \sum^j \sum^k \sum^\ell$$

The exponents i, j, k and ℓ are positive integers and the coefficients are grouped according to order (degree) n, e.g. n = 1 for the linear elements. In all cases the number of coefficients is limited by the storage capabilities of the computer used, thus the maximum value of n, n_{max} is restricted. The expansions must therefore be truncated and truncation errors are introduced.

Accepting these limitations for the moment we need a mechanism for generating the coefficients from the known physical properties of the element to be characterized. Code was written to assist in that task. We also need a mechanism that corresponds to, and reduces to, the matrix multiplication in the linear ease. A routine was written that performs that operation of "multiplication" or concatenation.

All routines contain the maximum order n_{max} as a free parameter. Choosing n_{max} low may be advantageous because the required processing time increases quickly with n_{max}.

Repeating the chaining operation for all elements around a ring one obtains a transfer or mapping function for that ring. It is obviously nonlinear for $n_{max} > 1$ and can be used for particle tracking, determination of fixed points, focussing functions, etc. It may also be useful for constructing a constant of the motion, akin to the emittance ellipses of linear optics.

It should be trivial to add $\delta = \Delta p/p$, the momentum deviation as an independent parameter, although this has not yet been done.

Generation of Coefficient Table

Let me consider the construction of the coefficient table for an arbitrary element.

One possibility is to search for, or to develop, a numerical integrator suitable for calculating particle trajectories and to use it to produce enough sets of exit/entrance coordinate pairs to calculate all coefficients required. The integrator used must observe the symplecticity of the motion of the particles.

Another possibility is to subdivide the element into a sequence of short sections with linear properties and thin nonlinear lenses and to calculate the overall transfer function from the transfer functions of the individual elements. These functions are well known for the short linear sections and can be written down straight forwardly for the (infinitely thin) nonlinear elements. In following this route the question arises of how to choose the number of subdivisions. Many subdivisions support high accuracy but require high order.

To illustrate this effect and to exercise and check the routines written, I calculated transfer functions for various descriptions of a thick sextupole, e.g. for the sextupole component of a long, otherwise perfect, dipole.

The system is represented by a string of thin lens sextupoles and drift spaces. The terminating sextupoles are half as strong as the others. The overall transfer function was calculated as function of the number of drift spaces while the overall length L and the integrated b_2 were kept constant. The calculations were done up to 7th order, i.e. $n_{max} = 7$. Some of the results are shown in Figs. 1, 2, and 3. Figure 3 shows the variation of the maxima and minima of some of the low order coefficients with the number of driftspaces.

The magnitudes of the coefficients are proportional to b_2^n with b_2 the integrated sextupole strength and n the order of the coefficient. The magnitude does not depend on the maximum order n_{max} of the expansion. Increasing n_{max} simply adds more coefficients, it does not change the old ones.

In all cases we considered only magnetic nonlinearities. In reality there are also the kinematic nonlinearities, due to the basic nonlinearity of the equation of motion, and those due to geometric effects.

The magnitudes of these nonlinearities may be small (in large machines) or not so small (in smaller ones) compared to those originating in the magnetic fields. Their incorporation in the transfer functions requires performing some algebra. Routines to do this have been written and tested.

Convergence Problems

Describing a quantity in terms of a series expansion raises questions about convergence and about the consequences of truncation. So far I have not addressed this matter in a serious fashion, partly because I believe that there is no problem.

We basically express transverse particle coordinates in terms of displacements along the system axis. In those terms the transverse coordinates are always small compared to 1, e.g. $\lesssim 10^{-3}$.

The coefficients of the transfer functions seen so far are always of the order of the element-length or smaller. This depends of course on the strengths of the nonlinearities in that element, but making them large would require very strong nonlinearities, e.g. for an n pole $b_n x^{n-1} >> b_1$, with b_1 a typical quadrupole strength and x a maximum particle excursion. Such strong nonlinearities do not usually occur in accelerators. This implies that the high order terms in the expansion will tend to be small compared to the low order ones by a factor of $\lesssim 10^{-3(n-1)}$, e.g. 10^{-18} for n = 7. This would fall beyond the resolution of any computer likely to be used for this work.

Chaining of Elements

Successive elements are chained by application of the multiplier or concatenator. This routine generates the coefficients of a "product" coefficient table from the tables for the two elements to be chained; e.g.:

$$\bar{\bar{x}} = \sum c_{xijk\ell}\, x^i x'^j y^k y'^\ell$$

$$= \sum b_{xijk\ell}\, \bar{x}^i \bar{x}'^j \bar{y}^k \bar{y}'^\ell$$

with

$$\bar{x} = \sum a_{xijk\ell}\, x^i x'^j y^k y'^\ell$$

$$\bar{x}' = \sum a_{x'ijk\ell}\, x^i x'^j y^k y'^\ell$$

$$\bar{y} = \sum a_{yijk\ell}\, x^i x'^j y^k y'^\ell$$

$$\bar{y}' = \sum a_{y'ijk\ell}\, x^i x'^j y^k y'^\ell$$

where a and b stand for the entries in the tables of the consecutive elements and c for those in the product table.

The practicality of this approach depends in part on the speed of execution. That speed decreases quickly with increasing order. A multiplication to 7th order takes about 10 seconds on the CDC 7600. The calculation of the transfer function for a single RHIC ring of some 240 independent elements might therefore require some 3 hours if we require 7th order. The various routines can be easily adapted to multi-processor computers, since the four coordinates are treated nearly independently, while the operations for each coordinate seem naturals for array processors.

Constant of the Motion

Once the transfer function of a complete ring is available one can ask whether there exist functions

$$\sum C_{ijk\ell}\, x'^i x'^j y^h y'^\ell = \sum C_{ijk\ell} \bar{x}^i \bar{x}'^j \bar{y}^k \bar{y}'^\ell$$

where

$$\bar{x} = \sum a_x x^i x'^j y^k y'^\ell$$

$$\bar{x}' = \sum a_{x'} x^i x'^j y^k y'^\ell$$

$$\bar{y} = \sum a_y x'^i x'^j y^k y'^\ell$$

$$\bar{y}' = \sum a_{y'} a^i a'^j y^k y'^\ell$$

with "a" the coefficients of the transfer function. This relation leads to a fairly standard eigen-value problem that can be written in the form

$$\big((m) - (1)\big)(c) = 0$$

$$[(m) - (1)](c) = 0$$

where (m) is a matrix whose elements are obtained from manipulations with the coefficients of the transfer function, (c) a vector whose components represent the coefficients c to be determined and (1) the unit matrix.

A nontrival solution for (c) requries that the matrix ((m) - (1)) have determinant zero, which is guaranteed for a conservative physical system. It can be transformed to a triangular matrix of the form shown by the application of standard operations

$$\begin{vmatrix} \delta_{11} & \bar{m}_{12} & \bar{m}_{1k} & \bar{m}_{1n} \\ 0 & \delta_{22} & \bar{m}_{2k} & \bar{m}_{2n} \\ 0 & 0 & \delta_{kk} & \bar{m}_{kn} \\ 0 & 0 & 0 & \delta_{nn} \end{vmatrix}$$

with $\delta_{kk} = 1$ or 0 and elements $\bar{m}_{k\ell} = 0$ if $\delta_{kk} = 0$. Code was written to do all this and to calculate the coefficients c.

It appears that there are several zero's on the diagonal, this is not too surprising. Execution for n = 2 should produce two independent emittances if the horizontal and vertical motions are uncoupled.

One recovers in that case in fact all the expected properties of linear transfer functions e.g. the α, the β and λ values, as well as the proper offsets in case of misalignment.

Comments

It is perhaps worth reemphasizing that the two functions discussed, the transfer function and the "constant of the motion" are approximations because they are described by means of arbitrarily truncated power series. The transfer functions can be described exactly by series with a manageable number of terms

only in the very simplest of cases. We mentioned already that, if truncation occurs, the terms in the discarded part are likely to be small. In accepting the multiplication of truncated series for the truncation of the multiplication of complete series we make use of the fact that any products with truncated parts will only appear in the truncated part:

$$\begin{aligned} S_1 S_2 &= (R_1 + T_1)(R_2 + T_2) \\ &= R_1 R_2 + (R_1 T_2 + R_2 T_1 + T_1 \cdot T_2) \\ &\curvearrowright R_1 R_2 \text{ for } T_1 \ll R_1,\ T_2 \ll R_2 \ . \end{aligned}$$

where S stands for complete series and R and T for retained and discarded parts respectively.

During the meeting the question was asked whether truncation violates the conservative nature of the physics described. This has not been considered yet.

In a discussion with E.D. Courant it became clear that the presence of any nonlinearity in the transfer function forces the number of the constants c, used for describing the associated so-called constant of the motion, to infinity. This makes truncation of that series a serious matter, particularly since convergence is not guaranteed, nor even that each of the discarded terms is small. The fact that the procedure used yields results (which are correct in the case of linear transfer functions) can therefore not be taken as a proof of the existence of such a constant. More work is needed.