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Emittance and Beam Size Distortion Due to Linear Coupling

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1. Introduction

At injection, the presence of linear coupling may result in an increased beam emittance and in increased beam dimensions. Results for the emittance in the presence of linear coupling will be found. These results for the emittance distortion show that the harmonics of the skew quadrupole field close to $\nu_x + \nu_y$ are the important harmonics. Results will be found for the important driving terms for the emittance distortion. It will be shown that if these driving terms are corrected, then the total emittance is unchanged, $\epsilon_x + \epsilon_y = \epsilon_1 + \epsilon_2$. Also, the increase in the beam dimensions will be limited to a factor which is less than 1.414. If the correction is good enough, see below for details, one can achieve $\epsilon_1 = \epsilon_x$, $\epsilon_2 = \epsilon_y$, where ϵ_1, ϵ_2 are the emittances in the presence of coupling, and the beam dimensions are unchanged. Global correction of the emittance and beam size distortion appears possible.

2. The Emittance for Coupled Motion

One definition for the emittances when the particle motion is coupled was given by Edwards and Teng.[1] In four dimensions, one can go from the coordinates x, p_x, y, p_y to an uncoupled set of coordinates v, p_v, u, p_u by the transformation[1]

$$x = R\, v$$

$$x = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \qquad v = \begin{pmatrix} v \\ p_v \\ u \\ p_u \end{pmatrix} \tag{2.1}$$

$$R = \begin{pmatrix} I\cos\varphi & \overline{D}\sin\varphi \\ -D\sin\varphi & I\cos\varphi \end{pmatrix}.$$

I and D are 2×2 matrices. I is the 2×2 identity matrix. $\overline{D} = D^{-1}$ and $|D| = 1$. R is a symplectic matrix

$$\begin{aligned} \overline{R}\, R &= I \\ \overline{R} &= \tilde{S} \tilde{R}\, S \end{aligned} \tag{2.2}$$

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} .$$

$\tilde{R}$ is the transpose of R.

v, p_v and u, p_u are uncoupled. Thus v, p_v satisfy differential equations with periodic coefficients whose solutions have the form

$$\begin{aligned} v &= \beta_1^{\frac{1}{2}} \exp\left(i\psi_1\right) \\ p_v &= \beta_1^{-\frac{1}{2}} \left(-\alpha_1 + i\right) \exp\left(i\psi_1\right) . \end{aligned} \tag{2.3}$$

A second solution exists with $\psi_1, \beta_1, \alpha_1$ replaced by $\psi_2, \beta_2, \alpha_2$. As in the case of 2 dimensional motion

$$\epsilon_1 = \gamma_1 v^2 + 2\alpha_1 v p_v + \beta_1 p_v^2 \tag{2.4a}$$

is an invariant. $\gamma_1 = \left(1 + \alpha_1^2\right) / \beta_1$. Similarly, ϵ_2 is an invariant,

$$\epsilon_2 = \gamma_2 u^2 + 2\alpha_2 u p_u + \beta_2 p_u^2 . \tag{2.4b}$$

For two dimensional motion, one can find α, β from the one turn transfer matrix $M\left(s + L, s\right)$.

In 4 dimensions, α_1, β_1 and α_2, β_2 can be found from the one turn transfer matrix. The process is quite involved[1], and using Eq. (2.4) to find ϵ_1, ϵ_2 when the transfer matrix is known is also involved.

A second definition of the emittance was suggested by A. Piwinski[2] which seems easier to apply. The emittance ϵ_1 is defined by

$$\epsilon_1 = \left| \tilde{x}_1^* \, S \, x \right|^2 \tag{2.5a}$$

x_1 is the 4 vector for the eigenfunction of the transfer matrix, which are assumed to be $x_1, x_2 = x_1^*$, $x_3, x_4 = x_3^*$.

Since $\tilde{x}_1^* Sx$ has the form of the Lagrange invariant[3] ϵ_1 is an invariant. It will be shown below that ϵ_1 defined by Eq. (2.5) and ϵ_1 defined by Eq. (2.4) are the same. In a similar way, ϵ_2 is defined by

$$\epsilon_2 = \left| \tilde{x}_3^* \; S \; x \right|^2 \tag{2.5b}$$

Note that x_1 and x_3 have to be normalized so that

$$\tilde{x}_1^* \; S \; x_1 = \tilde{x}_3^* \; S \; x_3 = 2i \tag{2.6}$$

Analytic expressions for x_1, x_3 were given in a previous paper.[4] These results for x_1, x_3 when put in Eq. (2.5) give an analytic expression for ϵ_1 and ϵ_2.

To show that ϵ_1, ϵ_2 defined by Eqs. (2.4) and Eqs. (2.5) are equal, one may note that since v, p_v, u, p_u are uncoupled coordinates, the eigenfunctions in this coordinate system may be written as

$$v_1 = \begin{bmatrix} \beta_1^{\frac{1}{2}} \\ \beta_1^{-\frac{1}{2}} (-\alpha_1 + i) \\ 0 \\ 0 \end{bmatrix} \exp(i\psi_1), \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ \beta_2^{\frac{1}{2}} \\ \beta_1^{-\frac{1}{2}} (-\alpha_2 + i) \end{bmatrix} \exp(i\psi_2) \tag{2.7}$$

One can then show that

$$\tilde{v}_1^* \; s \; v_1 = \tilde{v}_3^* \; s \; v_3 = 2i,$$

and

$$\left| \tilde{v}_1^* \; s \; v \right|^2 = \gamma_1 v^2 + 2\alpha_1 v p_v + p_v^2,$$

which is ϵ_1 according to Eq. (2.4).

One can show that since $x = Rv$ and R is symplectic, that

$$\left| \tilde{x}_1^* \; s \; x \right|^2 = \left| \tilde{v}_1^* \; s \; v \right|^2, \tag{2.8}$$

and thus the ϵ_1 defined by Eq. (2.5) is the same as ϵ_1 defined by Eq. (2.4). One may note that $x_1 = R \; v_1$.

It also can be shown that

$$\int dx dp_x dy dp_y = \epsilon_1 \epsilon_2, \tag{2.9}$$

where the integral is over the region of 4-space which lies inside the two surfaces

$$\begin{aligned} \epsilon_1 \left(x, p_x, y, p_y\right) &= \epsilon_1 \\ \epsilon_2 \left(x, p_x, y, p_y\right) &= \epsilon_2 \end{aligned} \tag{2.10}$$

This can be shown by transforming the integral in Eq. (2.10) from the x coordinates to the v coordinates and using the result $|R| = 1$.

3. Analytical Results for the Emittance Distortion and its Correction

Analytical results for the eigenfunctions of the 4×4 transfer matrix were found in Ref. 4. These are summarized in the following:

$$\begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} = G \begin{bmatrix} \eta_x \\ p_{\eta x} \\ \eta_y \\ p_{\eta y} \end{bmatrix} \tag{3.1}$$

$$G = \begin{bmatrix} G_x & 0 \\ 0 & G_y \end{bmatrix}$$

$$G_x = \begin{bmatrix} \beta_x^{\frac{1}{2}} & 0 \\ -\alpha_x \beta_x^{-\frac{1}{2}} & \beta_x^{-\frac{1}{2}} \end{bmatrix}, \quad G_y = \begin{bmatrix} \beta_y^{\frac{1}{2}} & 0 \\ -\alpha_y \beta_y^{-\frac{1}{2}} & \beta_y^{-\frac{1}{2}} \end{bmatrix}$$

$$\eta_x = A \exp\left(i\nu_{xs}\theta_x\right) \left[1 + \sum_{n \neq -p} f_n\right]$$

$$\eta_y = B \exp\left(i\nu_{ys}\theta_y\right) \left[1 + \sum_{n \neq p} g_n\right] \tag{3.2}$$

$$f_n = \frac{\nu_{xs} - \nu_x}{\Delta\nu} \quad \frac{2\nu_x b_n \exp\left[-i\left(n+p\right)\theta_x\right]}{\left(n - \nu_x - \nu_y\right)\left(n+p\right)}$$

$$g_n = \frac{\nu_{ys} - \nu_y}{\Delta\nu^*} \quad \frac{2\nu_y c_n \exp\left[-i\left(n-p\right)\theta_y\right]}{\left(n - \nu_x - \nu_y\right)\left(n-p\right)}$$

$$\Delta\nu = \left(1/4\pi\rho\right) \int ds \left(\beta_x\beta_y\right)^{\frac{1}{2}} a_1 \quad \exp\left[i\left(-\nu_{xs}\theta_x + \nu_{ys}\theta_y\right)\right]$$

$$b_n = \frac{1}{4\pi\rho} \int ds \left(\beta_x\beta_y\right)^{\frac{1}{2}} a_1 \quad \exp\left[i\left(n - \nu_y\right)\theta_x + \nu_y\theta_y\right]$$

$$c_n = \frac{1}{4\pi\rho} \int ds \left(\beta_x\beta_y\right)^{\frac{1}{2}} a_1 \quad \exp\left[i\nu_x\theta_x + \left(n - \nu_x\right)\theta_y\right]$$

$$\theta_x = \psi_x/\nu_x \quad , \quad \theta_y = \psi_y/\nu_y$$

ν_{xs}, ν_{ys} are the solutions of

$$\nu_x = \nu_{ys} + p, \quad \left(\nu_{xs} - \nu_x\right)\left(\nu_{ys} - \nu_y\right) = \left|\Delta\nu\right|^2 \tag{3.3}$$

ν_x, ν_y are assumed to be close to the resonance line $\nu_x = \nu_y + p$. $p_{\eta x}$ and $p_{\eta y}$ can be found using

$$p_{\eta x} = (1/\nu_x)\, d\eta_x/d\theta_x, \quad p_{\eta y} = (1/\nu_y)\, d\eta_y/d\theta_y \tag{3.4}$$

The A and B coefficients are determined by the condition on the eigenfunctions

$$\overset{\sim *}{x}\ S\ x = 2i \tag{3.5}$$

This gives the relationship[4]

$$|A|^2 (\nu_{xs}/\nu_x) + |B|^2 (\nu_{ys}/\nu_y) = 1 \tag{3.6}$$

There are two solutions of Eq. (3.3) corresponding to the two normal modes. For the mode for which $\nu_{sx} \to \nu_x$ when $a_1 \to 0$, we will put $\nu_{xs} = \nu_1$, $\nu_{ys} = \nu_1 - p$. For the mode for which $\nu_{ys} \to \nu_y$ when $a_1 \to 0$, we will put $\nu_{ys} = \nu_2$, $\nu_{xs} = \nu_2 + p$.

For the ν_1 mode

$$\begin{aligned} B_1 &= -\frac{\nu_1 - \nu_x}{\Delta\nu} A_1 \\ |A_1|^2 &\left(\frac{\nu_1}{\nu_x} + \frac{(\nu_1 - p)}{\nu_y} \left| \frac{\nu_1 - \nu_x}{\Delta\nu} \right|^2 \right) = 1. \end{aligned} \tag{3.7a}$$

For the ν_2 mode

$$\begin{aligned} A_2 &= -\frac{\nu_2 - \nu_y}{\Delta\nu^*} B_2 \\ |B_2|^2 &\left(\frac{\nu_2}{\nu_y} + \frac{(\nu_2 + p)}{\nu_x} \left| \frac{\nu_2 - \nu_y}{\Delta\nu} \right|^2 \right) = 1. \end{aligned} \tag{3.7b}$$

The eigenfunctions being known, one can now compute ϵ_1 and ϵ_2

$$\epsilon_1 = \left| \overset{\sim *}{x_1}\ S\ x \right|^2 = |\eta_1^* S\ \eta|^2 \tag{3.8}$$

since G is symplectic.

$$\eta_1 = \begin{bmatrix} \eta_{x1} \\ p_{\eta x1} \\ \eta_{y1} \\ p_{\eta y1} \end{bmatrix} \tag{3.9}$$

one finds

$$
\begin{aligned}
\epsilon_1 = &\left|\eta_{x1}\right|^2 p_{\eta x}^2 + \left|p_{\eta x1}\right|^2 \eta_x^2 - \eta_x p_{\eta x}\left(p_{\eta x1}^* \eta_{x1} + \text{c.c.}\right) \\
&+ \left|\eta_{y1}\right|^2 p_{\eta y}^2 + \left|p_{\eta y1}\right|^2 \eta_y^2 - \eta_y p_{\eta y}\left(p_{\eta y1}^* \eta_{y1} + \text{c.c.}\right) \\
&+ p_{\eta x} p_{\eta y}\left(p_{\eta x1}^* p_{\eta y1}^* + \text{c.c.}\right) \\
&+ \eta_x \eta_y\left(p_{\eta x1}^* p_{\eta y1} + \text{c.c.}\right) \\
&- p_{\eta x} \eta_y\left(\eta_{x1}^* p_{\eta y1} + \text{c.c.}\right) \\
&- \eta_x p_{\eta y}\left(p_{\eta x1}^* \eta_{y1}^* + \text{c.c.}\right)
\end{aligned}
\tag{3.10}
$$

One can now find analytic expressions for ϵ_1 by substituting for η_1 from Eqs. (3.1) to (3.7) into Eq. (3.10). This result is usually quite complicated. One interesting case is when a correction system has been used to cancel the b_n and c_n for $n \simeq \nu_x + \nu_y$, which generate the larger terms in the expressions for the eigenfunctions. Let us assume that enough b_n, c_n have been corrected so that, from Eq. (3.2), the eigenfunctions can be written as

$$
\begin{aligned}
\eta_x &= A \;\exp\left(i\nu_{xs}\theta_x\right) \\
\eta_y &= B \;\exp\left(i\nu_{ys}\theta_y\right) \\
p_{\eta x} &= iA \;\exp\left(i\nu_{xs}\theta_x\right) \\
p_{\eta y} &= iB \;\exp\left(i\nu_{ys}\theta_y\right)
\end{aligned}
\tag{3.11}
$$

It has been assumed that the different resonance has also been corrected, and that ν_x, ν_y is very close to the nearby difference resonance $\nu_x - \nu_y = p$, so that $\nu_{xs}/\nu_x \simeq 1$ and $\nu_{ys}/\nu_y \simeq 1$. It will be seen that correcting the b_n, c_n for $n \simeq \nu_x + \nu_y$ and the nearby different resonance will essentially correct the emittance distortion and the beam size distortion.

Putting the corrected results for the eigenfunctions Eq. (3.11) into the emittance result Eq. (3.10) one finds

$$
\begin{aligned}
\epsilon = &\left|A\right|^2\left(p_{\eta x}^2 + \eta_x^2\right) + \left|B\right|^2\left(p_{\eta y}^2 + \eta_y^2\right) \\
&+ p_{\eta x} p_{\eta y}\left(A^* B + \text{c.c.}\right) \\
&+ \eta_x \eta_y\left(A^* B + \text{c.c.}\right) \\
&- p_x \eta_y\left(-iA^* B + \text{c.c.}\right) \\
&- \eta_x p_{\eta y}\left(-iA^* B + \text{c.c.}\right)
\end{aligned}
\tag{3.12}
$$

There are two solutions of interest corresponding to how well one can correct $\Delta\nu$,

$$\begin{aligned} &\text{Case 1.} \quad |\Delta\nu| \ll |\nu_x - \nu_y - p| \\ &\text{Case 2.} \quad |\nu_x - \nu_y - p| \ll |\Delta\nu| \end{aligned} \tag{3.13}$$

For the first case, $|\Delta\nu| \ll |\nu_x - \nu_y - p|$, then the coefficients A, B in the eigenfunctions satisfy[4]

$$\begin{aligned} |A_1| = 1 \qquad B_1 = 0 \\ |B_2| = 1 \qquad A_2 = 0 \end{aligned} \tag{3.14}$$

Then for case (1) Eq. (3.12) gives

$$\begin{aligned} \epsilon_1 = \epsilon_x \\ \epsilon_2 = \epsilon_y \end{aligned} \tag{3.15}$$

where use has been made of the results

$$\begin{aligned} \eta_x + p_{\eta x}^2 = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2 = \epsilon_x \\ \eta_y^2 + p_{\eta y}^2 = \gamma_y y^2 + 2\alpha_y y_\gamma p_y + \beta_y p_y^2 = \epsilon_y \end{aligned} \tag{3.16}$$

Thus in case 1, ϵ_1, ϵ_2 are the same as ϵ_x, ϵ_y.

For case (2), $|\nu_x - \nu_y - p| \ll |\Delta\nu|$ then[4]

$$\begin{aligned} |A_1| = |B_1| = 1/\sqrt{2} \\ |A_2| = |B_2| = 1/\sqrt{2} \\ A_1^* B_1 + A_2^* B_2 = 0 \end{aligned} \tag{3.17}$$

Then for case (2), Eq. (3.12) gives

$$\epsilon_t = \epsilon_1 + \epsilon_2 = \epsilon_x + \epsilon_y \tag{3.18}$$

We no longer have $\epsilon_1 = \epsilon_x$, $\epsilon_2 = \epsilon_y$ as in case (1) however ϵ_t is not increased by the linear coupling.

Thus, if one corrects enough of the b_n, c_n for $n \simeq \nu_y + \nu_x$ and also corrects $\Delta\nu$, the driving term of the nearby difference resonance, $\nu_x - \nu_y = p$, then the emittance distortion has also been corrected. We will either obtain $\epsilon_1 = \epsilon_x$, $\epsilon_2 = \epsilon_y$ or $\epsilon_1 + \epsilon_2 = \epsilon_x + \epsilon_y$ depending on how well $\Delta\nu$ has been corrected.

4. Analytical Results for the Beam Size Distortion and its Correction

In the previous section, results were found for the emittance distortion, and it was found that if the b_n, c_n for $n \simeq \nu_x + \nu_y$ and $\Delta\nu$ are corrected, then the emittance distortion is also largely corrected. For 4 dimensional motion, the connection between the beam size and the emittance is not as simple as it is in the 2 dimensional uncoupled case. In this section the maximum beam size will be computed when the b_n, c_n and $\Delta\nu$ are corrected. It will be shown that the beam size distortion is also largely corrected, although in one case it may be increased by a factor which is ≤ 1.414.

The particle motion can be written in terms of the eigenfunctions x_1, x_2, x_3, x_4 as

$$x = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 \tag{4.1}$$

where

$$x = \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} \tag{4.2}$$

$x_2 = x_1^*$, $x_4 = x_3^*$ and $\tilde{x}_i^* \; S \; x_i = 2i$, $x_i^* s x_y = 0$, $i \neq j$

The a_i can then be found

$$\begin{aligned} a_1 &= (1/2i) \, \tilde{x}_1^* \; S \; x \\ a_3 &= (1/2i) \, \tilde{x}_3^* \; S \; x \end{aligned} \tag{4.3}$$

and thus

$$|a_1| = \frac{1}{2}\epsilon_1^{1/2}, \quad |a_3| = \frac{1}{2}\epsilon_2^{1/2} \tag{4.4}$$

If the b_n, c_n and $\Delta\nu$ have been corrected so that the eigenfunctions are given by Eq. (3.11), then x and y of the eigenfunctions are given by

$$\begin{aligned} x &= \beta_x^{1/2} A \exp\left[i\left(\nu_{xs}\theta_x\right)\right] \\ y &= \beta_y^{1/2} B \exp\left[i\nu_{ys}\theta_y\right]. \end{aligned} \tag{4.5}$$

x and y are then given by

$$\begin{aligned} x &= (\beta_x \epsilon_1)^{\frac{1}{2}} |A_1| \cos\left[\nu_1\theta_x + \delta_1\right] + (\beta_x \epsilon_2)^{\frac{1}{2}} |A_2| \cos\left[(\nu_2 + p)\,\theta_x + \delta_2\right] \\ y &= (\beta_y \epsilon_1)^{\frac{1}{2}} |B_1| \cos\left[(\nu_1 - p)\,\theta_y + \delta_1\right] + (\beta_y \epsilon_2) |B_2| \cos\left[\nu_2\theta_y + \delta_2\right] \end{aligned} \tag{4.6}$$

δ_1, δ_2 are the phases of a_1 and a_3.

$x_{\max}$ and $y_{\max}$ are then

$$
\begin{aligned}
x_{\max} &= (\beta_x \epsilon_1)^{\frac{1}{2}} |A_1| + (\beta_x \epsilon_2)^{\frac{1}{2}} |A_2| \\
y_{\max} &= (\beta_y \epsilon_1)^{\frac{1}{2}} |B_1| + (\beta_y \epsilon_2) |B_2|
\end{aligned}
\tag{4.7}
$$

As was done for the emittance, we will find $x_{\max}$ for the two cases given by Eq. (3.13).

For case 1, $|\Delta\nu| \ll |\nu_x - \nu_y - p|$ then

$$
\begin{aligned}
|A_1| &= 1, \qquad |B_1| = 0 \\
|A_2| &= 0, \qquad |B_2| = 1 \\
\epsilon_1 &= \epsilon_x, \qquad \epsilon_2 = \epsilon_y
\end{aligned}
\tag{4.8}
$$

Then Eq. (4.7) gives

$$
\begin{aligned}
x_{\max} &= \sqrt{\beta_x \epsilon_x} \\
y_{\max} &= \sqrt{\beta_y \epsilon_y}
\end{aligned}
\tag{4.9}
$$

and there is no growth in beam size.

For case 2, $|\nu_x - \nu_y - p| \ll |\Delta\nu|$ then

$$
\begin{aligned}
|A_1| &= |B_1| = 1/\sqrt{2} \\
|A_2| &= |B_2| = 1/\sqrt{2} \\
\epsilon_t &= \epsilon_1 + \epsilon_2 = \epsilon_x + \epsilon_y
\end{aligned}
\tag{4.10}
$$

Eq. (4.6) then gives for $x_{\max}$, $y_{\max}$

$$
\begin{aligned}
x_{\max} &= (\beta_x/2)^{\frac{1}{2}} \left(\epsilon_1^{\frac{1}{2}} + \epsilon_2^{\frac{1}{2}}\right) \\
y_{\max} &= (\beta_y/2)^{\frac{1}{2}} \left(\epsilon_1^{\frac{1}{2}} + \epsilon_2^{\frac{1}{2}}\right)
\end{aligned}
\tag{4.11}
$$

Since $\epsilon_2 = \epsilon_t - \epsilon_1$, then as one varies ϵ_1 from $\epsilon_1 = 0$ to $\epsilon_1 = \epsilon_t$, $x_{\max}$ reaches its maximum at $\epsilon_1 = \epsilon_2 = \epsilon_t/2$. Thus

$$
\begin{aligned}
x_{\max} &\leq (\beta_x (\epsilon_x + \epsilon_y))^{\frac{1}{2}} \\
y_{\max} &\leq (\beta_y (\epsilon_x + \epsilon_y))^{\frac{1}{2}}
\end{aligned}
\tag{4.12}
$$

For the case where $\epsilon_x = \epsilon_y$, then $x_{\max} \leq 1.4 (\beta_x \epsilon_x)^{\frac{1}{2}}$ and the coupling may increase $x_{\max}$ by the factor 1.414. So in case (2) $|\nu_x - \nu_y - p| \ll |\Delta\nu|$, then when the b_n, c_n and $\Delta\nu$ are corrected one may still have a beam size increase of the factor 1.414.

5. References

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