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Developments on the Concept of the rf Wiggler

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This is a review of the interaction between the motion of a short and intense bunch of electrons and an electromagnetic wave, both propagating along the longitudinal axis of a waveguide. The interaction causes stimulation of radiation emission by having the electromagnetic wave to make the electron bunch oscillate coherently in a plane transverse to main direction of motion. The excitation has a maximum in correspondence of driving the waveguide at cut-off. To make realistic estimate of the amount of radiation emitted, the paper analyzes the case of a general rectangular cross-section of the waveguide with walls of the most general electro-magnetic properties. In particular the case of resistive walls is examined.

1. Introduction

In a previous note (AD/AP-35) we have seen that it is possible to convert electromagnetic power from one frequency to another by letting a short and intense electron bunch interact with an electromagnetic wave traveling along the axis of a square waveguide. The method to be effective requires that the waveguide is driven in proximity of the cut-off where the wave phase velocity is the largest.

In this note we reconsider the basic model of interaction and we take a waveguide with general rectangular cross-section. Moreover the electric and magnetic fields are estimated with the walls of the waveguide of the most general electromagnetic properties. In particular we considered the case of resistive walls.

We found that there is a maximum value of the phase velocity at cut-off due to the limited conductivity of the walls. We found also that the square geometry of the waveguide is still the optimum compared to a rectangular one. A new set of parameters for a demonstration of a frequency transformer is finally given with more realistic electron beam parameters.

2. Scalar and Vector Potentials in a Waveguide

We shall consider an infinitely long *waveguide*, straight, with rectangular cross-section of width w and height h. We shall introduce a rectangular coordinate system x, y and z; where x and y are the transverse distances from the upper left corner of the waveguide (see Fig. 1) and z is the longitudinal coordinate along the axis. In this section we describe the propagation of a TM traveling electromagnetic wave in the waveguide. If we use the Lorentz representation, the fields can be derived from a scalar V and vector potential $\mathbf{A}$ satisfying the following equations

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0 \tag{1}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \tag{2}$$

$$\text{div }\mathbf{A} + \frac{1}{c}\,\frac{\partial V}{\partial t} = 0 \tag{3}$$

In cartesian coordinates the explicitly form of Eq. (1) is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0 \tag{4}$$

A solution of Eq. (4) is

$$V = V_0\,(\sin\alpha_1 x + V_1\cos\alpha_1 x)\,(\sin\alpha_2 y + V_2\cos\alpha_2 y)\,e^{i(kz-\omega t)} \tag{5}$$

The nature of the traveling wave is described by the last factor where ω is the angular frequency and k the wave number which defines the longitudinal propagation mode. Insertion of Eq. (5) into Eq. (4) yields

$$k^2 = \frac{\omega^2}{c^2} - \alpha_1^2 - \alpha_2^2 \tag{6}$$

The horizontal and vertical propagation constants, respectively α_1 and α_2, are to be determined by specifying proper boundary conditions of the electric and magnetic fields at the walls of the waveguide. The same boundary conditions will be used to estimate V_1 and V_2 appearing at the right-hand side of Eq. (5).

According to the conventional waveguide terminology, a TM mode is defined as that traveling wave with vanishing magnetic field in the main direction of propagation, that is the z-axis of the waveguide. This mode is associated to solutions of the vector potential **A** which actually is completely directed along the z-axis and has vanishing components in the directions (x and y) perpendicular to the direction of the wave propagation, that is

$$\mathbf{A} \equiv (0, 0, A) \tag{7}$$

where A satisfies an equation similar to Eq. (4). Moreover, to satisfy the Lorentz condition represented by Eq. (3)

$$A = \beta_w V \tag{8}$$

where

$$\beta_w = \omega / kc \tag{9}$$

is the wave phase velocity, normalized to the speed of light. Thus the vector potential **A** is completely determined from the knowledge of the scalar potential V.

3. The Field Distribution

The electric **E** and magnetic **B** fields can be determined from the usual relations

$$\mathbf{E} = -\text{grad } V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{10}$$

$$\mathbf{B} = \text{rot } \mathbf{A} \tag{11}$$

We obtain after inserting Eqs. (5, 7 and 8)

$$E_x = -\alpha_1 V_0 \left(\cos \alpha_1 x - V_1 \sin \alpha_1 x\right) \left(\sin \alpha_2 y + V_2 \cos \alpha_2 y\right) e^{i\phi} \tag{12}$$

$$E_y = -\alpha_2 V_0 \left(\sin \alpha_1 x + V_1 \cos \alpha_1 x\right) \left(\cos \alpha_2 y - V_2 \sin \alpha_2 y\right) e^{i\phi} \tag{13}$$

$$E_z = ik \left(\beta_w^2 - 1\right) V \tag{14}$$

and

$$B_x = -\beta_w \, E_y \tag{15}$$

$$B_y = \beta_w \, E_x \tag{16}$$

$$B_z = 0 \tag{17}$$

where

$$\phi = kz - \omega t \tag{18}$$

is the wave phase function.

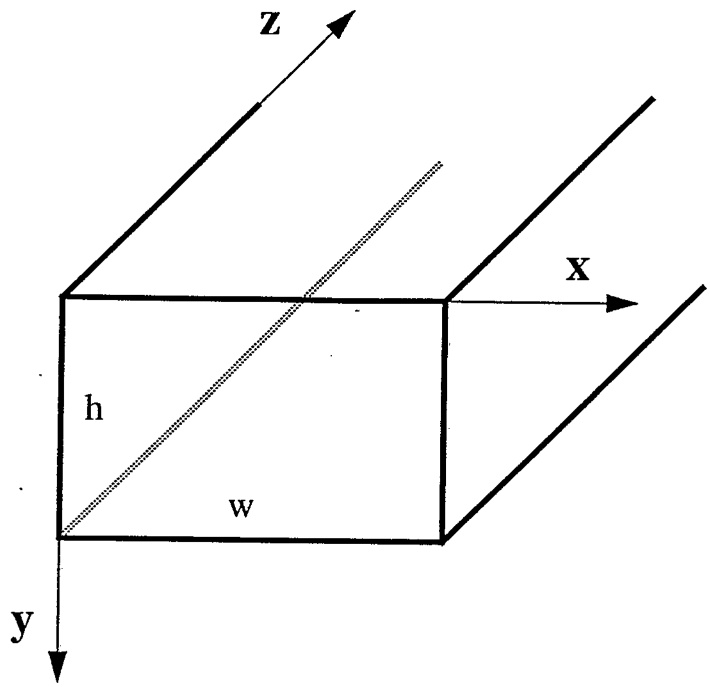

Fig. 1: Waveguide Geometry.

4. Boundary Conditions and Propagation Constants

We shall take the walls of the waveguide with the most general electromagnetic properties, described by the surface characteristic impedance ξ, a complex function of the angular frequency ω.

The following boundary conditions are to be satisfied at the walls of the waveguide

$$E_z = -\xi B_y \quad \text{at} \quad x = 0 \tag{19}$$

$$E_z = \xi B_y \quad \quad x = w \tag{20}$$

and

$$E_z = \xi B_x \quad \text{at} \quad y = 0 \tag{21}$$

$$E_z = -\xi B_x \quad \quad y = h \tag{22}$$

From Eqs. (19 and 21) we derive easily

$$V_j = -i\frac{\xi\beta_w\alpha_j}{k\left(\beta_w^2 - 1\right)} \tag{23}$$

where we let $j = 1$ or 2. The other two equations (20 and 22), after some manipulation, give

$$C_1 \sin\left(\alpha_1 w - \mu_1\right) = 0 \tag{24}$$

$$C_2 \sin\left(\alpha_2 h - \mu_2\right) = 0 \tag{25}$$

where

$$C_j = \frac{k^2\left(\beta_w^2 - 1\right)^2 - \xi^2\beta_w^2\alpha_j^2}{k\left(\beta_w^2 - 1\right)} \tag{26}$$

and

$$tg\ \mu_j = \frac{2i\xi\beta_w\alpha_j k\left(\beta_w^2 - 1\right)}{k^2\left(\beta_w^2 - 1\right)^2 + \xi^2\beta_w^2\alpha_j^2} \tag{27}$$

Eqs. (24 and 25) can be used to determine the eigenvalues of α_1 and α_2; that is

$$\alpha_1 w - \mu_1 = \pi n \tag{28}$$

$$\alpha_2 h - \mu_2 = \pi m \tag{29}$$

with n, m integer real numbers. Other possible propagation modes can also be obtained by letting either $C_1 = 0$ or $C_2 = 0$ or both at the same time. In turn, Eqs. (28 and 29) can be used in conjunction to Eq. (6) to calculate the propagation constant k. Because α_1 and α_2 depend on the phase shifts μ_1 and μ_2, in reality Eq. (27) is a nonlinear equation either in μ_1 or in μ_2 that ought to be solved for the determination of the phase shifts. Since μ_1 and μ_2 are in general complex, also the propagation constant k will be a complex quantity. The problem is thus completely solved. The only parameters left to be determined is the amplitude V_0 of the scalar potential function.

5. Perfectly Conductive Waveguide

A special case is a waveguide with perfectly conductive walls, that is $\xi = 0$. In this case it is easily seen that $V_1 = V_2 = 0$ and $\mu_1 = \mu_2 = 0$; moreover k^2 is real. Solving Eq. (6) gives the following dispersion relation

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{\omega_c^2}{c^2}} \tag{30}$$

where

$$\omega_c = \pi c \sqrt{\frac{n^2}{w^2} + \frac{m^2}{h^2}} \tag{31}$$

is the angular frequency at *cut-off*. It is convenient to introduce the form factor

$$q = \omega_c / \omega \tag{32}$$

It is seen from Eqs. (9 and 30) that

$$\beta_w = \frac{1}{\sqrt{1-q^2}} \tag{33}$$

The range of values of the form factor fulfilling the *condition of propagation*, which corresponds to k positive, is

$$0 < q < 1 \tag{34}$$

that is $\omega > \omega_c$. It is then seen that β_w is always real and larger than 1; that is the wave phase velocity is always larger than the speed of light.

An inspection of the dispersion relation, Eq. (30), shows that below the cut-off, $\omega < \omega_c$, there is no propagation, and k assumes no real values. For large values of ω, k increases about linearly. An interesting plot, shown in Fig. 2, is the display of the wave phase velocity β_w versus the form factor q as given by Eq. (33). Observe that approaching the cut-off from below, $q \rightarrow 1$, the phase velocity β_w becomes infinitely large.

6. Waveguide with Resistive Walls

In the following we consider the case of *resistive* walls. For this case the surface characteristic impedance is

$$\begin{aligned} \xi &= (1-i)\sqrt{\frac{\omega\mu}{8\pi\sigma}} \\ &= (1-i)\,\mathcal{R} \end{aligned} \tag{35}$$

where σ is the electric conductivity and μ the magnetic permeability of the wall material. The dispersion relation Eq. (6) can now be written

$$k^2 = \frac{\omega^2}{c^2} - \frac{\omega_c^2}{c^2} - \Delta^2 \tag{36}$$

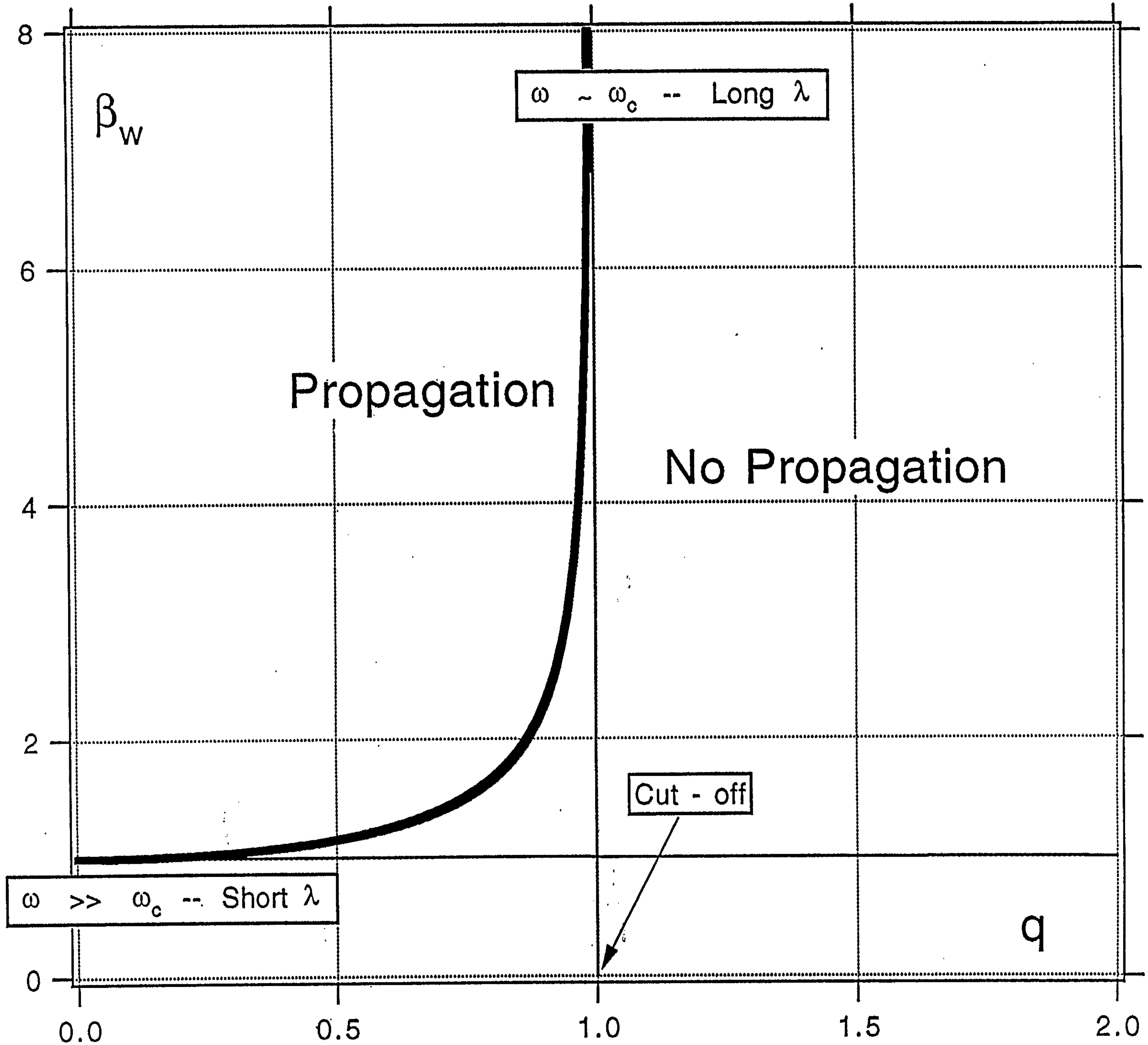

Fig. 2 Phase Velocity vs. Form Factor

where

$$\Delta^2 = \frac{\mu_1^2}{w^2} + \frac{\mu_2^2}{h^2} + 2\pi\left(\frac{n\mu_1}{w^2} + \frac{m\mu_2}{h^2}\right) \tag{37}$$

and k, Δ are complex quantities. Let us separate explicitly the real and imaginary parts; that is

$$k = k_r + i\ k_i \tag{38}$$

$$\Delta = \Delta_r + i\ \Delta_i \tag{39}$$

then

$$2k_r^2 = \frac{\omega^2 - \omega_c^2}{c^2} - \Delta_r^2 + \Delta_i^2 + \sqrt{\left(\frac{\omega^2 - \omega_c^2}{c^2} - \Delta_r^2 + \Delta_i^2\right)^2 + 4\Delta_r^2\Delta_i^2} \tag{40}$$

and

$$k_i = -\frac{\Delta_r\Delta_i}{k_r} \tag{41}$$

The last quantity k_i is the measure of the wave propagation attenuation per unit length, whereas k_r is the proper constant of propagation of the wave. The wave phase velocity is then given by

$$\beta_w = \frac{\omega}{ck_r} \tag{42}$$

from which we can derive the following relation to the form factor $q = \omega_c/\omega$

$$\beta_w^2 = \frac{2}{(1 - q_0^{-2}q^2) + \sqrt{(1 - q_0^{-2}q^2)^2 + 4c^4\frac{\Delta_r^2\Delta_i^2}{\omega_c^4}q^4}} \tag{43}$$

where

$$q_0^{-2} = 1 + \frac{c^2}{\omega_c^2}\left(\Delta_r^2 - \Delta_i^2\right). \tag{44}$$

Both Δ_r and Δ_i depend on the angular frequency ω and the propagation constant k; nevertheless in the case of a good conductor, for instance copper with $\mu = 1$ and $\sigma = 5\times10^{17}$ s^{-1} one can treat the contribution of the surface characteristic impedance $\mathcal{R}$, given by Eq. (35), as a perturbation to the field distribution. In proximity of the cut-off we can then let $\mathcal{R} = \mathcal{R}_c$ where $\mathcal{R}_c$ is $\mathcal{R}$ evaluated for $\omega = \omega_c$. Inspection of Eqs. (43 and 44) shows that β_w^2 is always a positive quantity for any value of q; the maximum occurs for $q = q_0$, which can be interpreted as a shift of the cut-off frequency. At cut-off the maximum is

$$\beta_{w\max}^2 \simeq \frac{\omega_c^2}{c^2\Delta_r\Delta_i} \tag{45}$$

where we have also approximated $q_0 \sim 1$.

From Eq. (37) we derive

$$\Delta_r \Delta_i = \pi \left(n \frac{\mu_{1i}}{w^2} + m \frac{\mu_{2i}}{h^2} \right) \tag{46}$$

provided $|\mu_j| << \pi$. In proximity of the cut-off $\beta_{w_{\max}} >> 1$, then from Eq. (27) in good approximation

$$\Delta_r \Delta_i \approx 2\pi^2 \frac{\mathcal{R}_c}{\omega_c / c} \left(\frac{n^2}{w^3} + \frac{m^2}{h^3} \right) \tag{47}$$

which inserted in Eq. (45) finally gives

$$\beta^2_{w_{\max}} \approx \frac{\omega_c^3}{2\pi^2 c^3 \mathcal{R}_c \left(\frac{n^2}{w^3} + \frac{m^2}{h^3} \right)} \tag{48}$$

7. Power Flow in the Waveguide

The constant V_0 in Eq. (5) determines the amplitude of the field potential and is related to the power flux in the waveguide. The flow of energy is described by the Poynting vector

$$\mathbf{S} = \frac{c}{8\pi} \mathbf{E} \times \mathbf{B} \tag{49}$$

To evaluate the total power flow P we integrate the axial component of $\mathbf{S}$ over the cross-section of the waveguide, that is:

$$P = \frac{c}{8\pi} \int_0^w \int_0^h (E_x B_y - E_y B_x)\, dy\; dx \tag{50}$$

Insertion of Eqs. (12, 13 and 15, 16) in Eq. (50) gives

$$P = \frac{c\beta_w}{64} \pi V_0^2 \left(\frac{k}{w} n^2 + \frac{w}{h} m^2 \right) \tag{51}$$

where we have ignored the resistivity of the walls since it gives only minor modifications. For a constant input of power P, the voltage amplitude V_0 is estimated when the dimensions of the waveguide and the mode of propagation are assigned.

8. The Equations of Motion

Consider now an electron with mass at rest m and electric charge e moving down the waveguide. The motion is relativistic and mainly along the axis of the waveguide with the same direction of the propagation of the electromagnetic wave. The components of the equations of motion can be derived from

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + e\frac{\mathbf{v}}{c} \times \mathbf{B} \tag{52}$$

where $\mathbf{p}$ is the electron vector momentum and $\mathbf{v} \equiv (\dot{x}, \dot{y}, \dot{z})$ the velocity vector. By inserting Eqs. (15-17) in Eq. (52) we obtain

$$\frac{dp_x}{dt} = -e\left(\frac{\dot{z}}{c}\beta_w - 1\right)E_x \tag{53}$$

$$\frac{dp_y}{dt} = -e\left(\frac{\dot{z}}{c}\beta_w - 1\right)E_y \tag{54}$$

$$\frac{dp_z}{dt} = e\left(\frac{\dot{x}}{c}\beta_w E_x + \frac{\dot{y}}{c}\beta_w E_y + E_z\right) \tag{55}$$

The equations of motion simplify considerably if we assume that the motion of the electron is confined in proximity of the $y = -h/2$ plane. In this case, if n is even and m is odd, $y = \dot{y} = 0$ is a solution of the equations of motion since $E_y = 0$. Moreover, if also x is very close to the $x = w/2$ axis, then in good approximation the equations of motion are

$$\frac{dp_x}{dt} \approx e\left(\frac{\dot{z}}{c}\beta_w - 1\right)V_0\alpha_1 \cos\phi \tag{56}$$

$$\frac{dp_y}{dt} \approx 0 \tag{57}$$

$$\frac{dp_z}{dt} \approx -e\frac{\dot{x}}{c}\beta_w V_0\alpha_1 \cos\phi \tag{58}$$

where we have also taken the real part of the traveling wave exponential factor.

It is seen that the perturbation to the longitudinal motion is of first order in $\dot{x}$ and it can thus be neglected. At the same time

$$\begin{aligned}\frac{d\phi}{dt} &= k\dot{z} - \omega \\ &\approx kv - \omega \\ &\approx ck\left(\beta - \beta_w\right) = -\Omega_0\end{aligned} \tag{59}$$

Since $\beta_w > \beta$ this quantity is always negative. A phase slippage occurs when the particle and the electromagnetic wave are traveling in the same direction. In proximity of the cut-off, the phase slippage $\Omega_0 \approx \omega$. It is reasonable to assume that during the interaction with the electromagnetic wave, the velocity β of the electron does not change considerably and it remains close to unit. In the approximation that the horizontal displacement remains small, that is $x << w/2$, Eq. (56) can be written as

$$\ddot{x} = \Omega^2 w \cos\phi \tag{60}$$

where

$$\Omega^2 = eV_0\alpha_1 \frac{\beta\beta_w - 1}{m_0\gamma w} \tag{61}$$

With a change of variables Eq. (60) becomes

$$\frac{d^2x}{d\phi^2} = \nu^2 w \cos\phi \tag{62}$$

with

$$\nu = \Omega/\Omega_0 \tag{63}$$

The solution of Eq. (62) can be easily derived to be

$$x = -a\cos\phi \tag{64}$$

that is an oscillation at the frequency equal to the phase slippage Ω_0 and amplitude

$$a = w\nu^2 \tag{65}$$

The solution given by Eq. (64) is correct only as long as the amplitude a of the oscillation is small compared to the width w of the waveguide, that is $\nu^2 << 1$, which sets a limit on the value of the voltage amplitude V_0.

9. Energy Loss by Radiation

Consider an electron which is moving at relativistic velocity along the z-axis and at the same time is performing small amplitude oscillations at the angular frequency Ω_0. It is well known that the electron will lose energy by radiating electromagnetic waves moving forward in the same direction of the motion of the particle, within an angular aperture of about $1/\gamma$. In the approximation that the oscillatory motion has been occurring for an infinitely long period of time, the spectrum of the radiation is made of only one line at the angular frequency

$$\omega_{\mathrm{rad}} = 2\gamma^2\Omega_0 \tag{66}$$

The radiated power at that frequency by one electron can also be calculated

$$P_0 = \frac{1}{3}\frac{e^2}{c^3}\gamma^4 a^2 \Omega_0^4 \tag{67}$$

where a is the amplitude of the oscillation which in our case may be given by Eq. (65) combined to Eq. (63) and Eqs. (59, 61).

The spatial distribution of the electrons in a short beam bunch all performing the same oscillatory motion is also important. In the extreme case where the beam bunch is much longer than the wavelength of the radiation $2\pi c/\omega_{\mathrm{rad}}$, each electron will radiate independently from the others and the total power radiated is $P_{\mathrm{rad}} = NP_0$ where P_0 is the power from a single electron, given by Eq. (67), and N the total number of electrons in the bunch. On the other hand, when the bunch length ℓ is considerably smaller than the radiated wavelength, that is

$$\ell << 2\pi c/\omega_{\mathrm{rad}} \tag{68}$$

it is conceivable that all the electrons are radiating *coherently* and in this case the total power is

$$P_{\mathrm{rad}} = N^2 P_0 \tag{69}$$

At the same time, though, in order to take advantage of this effect, it is also important that the transverse dimensions of the electron beam are made as small as possible. Indeed, they should not exceed the amplitude of the oscillations given by Eq. (65) and should be smaller than the bunch length itself.

10. Applications

It is convenient to define two parameters that best summarize the interaction between the electron motion and the field in the waveguide. One is the *frequency transformer ratio* $r = \omega_{\mathrm{rad}}/\omega$, that is the ratio of the *radiated* frequency to the *input* frequency to the waveguide. From Eqs. (59 and 66) we derive

$$r = 2\gamma^2 \frac{\beta_w - \beta}{\beta_w} \tag{70}$$

In proximity of the cut-off $\beta_w >> 1$ and with good approximation $r \sim 2\gamma^2$.

The second parameter is the *power amplification factor* $\eta = P_{\rm rad}/P$, that is the ratio of the power radiated by the beam bunch to the *input* power to the waveguide. Assuming that the condition of short bunches expressed by Eq. (68) is satisfied, we derive

$$\eta = \frac{64 N^2 r_0^2 \gamma^2 \pi n^2 \left(\beta\beta_w - 1\right)^2}{3hw^3 \left(\frac{n^2}{w^2} + \frac{m^2}{h^2}\right)\beta_w} \tag{71}$$

where $r_0 = 2.82 \times 10^{-15}$ m is the classical electron radius. An optimum case is given by a waveguide with a square cross-section, that is $h = w$, and by the lowest order of propagation, namely $m = 1$ and $n = 2$. In proximity of the cut-off then

$$\eta \simeq \frac{256}{15} \pi N^2 \gamma^2 \frac{r_0^2}{w^2} \beta_w \tag{72}$$

and from Eqs. (31 and 48)

$$\beta_w^2 \approx \frac{w\omega_c}{2c\,\mathcal{R}_c} \tag{73}$$

An application is a *frequency transformer.* In this mode of operation the power radiated by a short electron bunch is at a frequency larger than that used in input to the waveguide. In this case it is sufficient that the power gain $\eta \sim 1$. An example of frequency transformer is shown in Table 1, where the waveguide material is taken to be warm temperature copper and the waveguide itself is driven in proximity of the cut-off where the phase velocity β_w is the largest.

Table 1: An Example of Frequency Transformer

Kinetic Energy of Electrons	4 MeV
Number of Electrons	6×10^{10}
Bunch Length	1 mm
Input Frequency	1.3 GHz
Radiated Frequency	190 GHz
Frequency Transform Ratio	146
Power Amplification Factor	0.5
Phase Velocity, β_w	350
Cut-off Frequency	1.3 GHz
Waveguide Dimension, w	25.8 cm
Period of Oscillations	24.5 cm

11. Conclusion

We have shown in this paper that it is possible to convert electromagnetic power from one frequency to another with reasonable efficiency by letting a short electron bunch interact with a waveguide driven by an electromagnetic wave in proximity of the cut-off. We have estimated the maximum power gain and the required electron bunch and dimensions; they are within reach of present state of the art of electron sources. Our method to be effective relies on the coherent radiation by which, if the wavelength radiated is larger than the bunch length, the power radiated is proportional to the square of the number of electrons in the bunch.