BNL-101529-2014-TECH

AD/AP 39;BNL-101529-2013-IR

Effects Due to Linear Coupling, to the Second-Order in the Skew-Quadrupole Strengths

V. Garczynski

April 1992

Collider Accelerator Department

Brookhaven National Laboratory

U.S. Department of Energy

USDOE Office of Science (SC)

DISCLAIMER

AD/AP-39

Accelerator Development Department
Accelerator Physics Division
BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, NY 11973

Accelerator Physics Technical Note No. 39

Effects Due to Linear Coupling, to the Second-Order in the Skew-Quadrupole Strengths

V. Garczynski

April 1992

Effects Due to Linear Coupling, to the Second-Order in the Skew-Quadrupole Strengths

V. Garczynski

Accelerator Development Department
Brookhaven National Laboratory
April 1992

The Thin Lens Model is extended to higher-orders in the skew-quadrupole strengths. Its applications are made to describe a variety of effects due to linear coupling in circular accelerators. The tune-splitting, the tune-shift, the beta-function distortions, the emittance change and the Thick Ellipse Effect are calculated, up to the second-order.

1. Introduction

The advent of accelerating rings made of superconducting magnets, which are prone to larger errors, motivates an extension of the Thin Lens Model (TLM) to higher-orders in the skew-quadrupole strengths.[1–8] In RHIC, for example, a residual tune-splitting, quadratic in skew-quadrupole errors, was found in computer simulations.[9] This revives an old subject of the linear coupling problem and gives him a new life.

In the paper we describe the application of the TLM, extended to the second-order,[10–16] to various effects due to linear coupling, (the tune-splitting, the tune-shift, the beta-function distortions, the emittance growth and the Thick Ellipse Effect).

A local tune-splitting correction scheme is described which is complementary to a global correction scheme, in terms of minimizing of some positive-definite quadratic form (called "badness") in the transversal coordinates.[4]

2. The TLM in the Second-Order

Consider a ring, of circumference-C, containing N thin skew-quadrupoles of strengths $q_1, \ldots, q_N$ and locations $0 < s_1 < \ldots < s_N < C$. Assume that a transfer matrix of an ideal ring, that is a ring without the skew-quadrupole errors, is known and is of the (decoupled) form

$$T_0\left(s'', s'\right) = \begin{bmatrix} T_{0x}\left(s'', s'\right) & \mathbf{0} \\ \mathbf{0} & T_{0y}\left(s'', s'\right) \end{bmatrix}, \tag{2.1}$$

where $T_{0x,y}$ are the usual 2×2 symplectic transfer matrices written in terms of the Courant-Snyder parameters. Passing to the circular representation (normalized coordinates) we get, (see Appendix)

$$\overset{\circ}{T}_0\left(s'', s'\right) = \mathcal{B}\left(s''\right) T_0\left(s'', s'\right) \mathcal{B}^{-1}\left(s'\right) = \begin{bmatrix} R\left[\psi_x\left(s'', s'\right)\right] & \mathbf{0} \\ \mathbf{0} & R\left[\psi_y\left(s'', s'\right)\right] \end{bmatrix}, \tag{2.2}$$

where $R\left(\psi_{x,y}\right)$ are rotations

$$R\left(\psi\right) = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}, \tag{2.3}$$

and $\psi_{x,y}$ are the phase-advances

$$\psi_{x,y}\left(s'', s'\right) = \int_{s'}^{s''} \frac{ds}{\beta_{x,y}}. \tag{2.4}$$

The single- turn transfer matrix of total ring, skew-quads including, at the reference point $s = 0$, can be written as a polynomial

$$\overset{\circ}{T} = \begin{bmatrix} \overset{\circ}{M} & \overset{\circ}{n} \\ \overset{\circ}{m} & \overset{\circ}{N} \end{bmatrix} = \sum_{k=0}^{N} \overset{\circ}{T}{}^{(k)}, \tag{2.5}$$

where $\overset{\circ}{T}{}^{(k)}$ is of the k-th order homogeneous polynomial in the skew-quadrupole strengths. More specifically, its elements can be expressed through the first $d^{(1)}$ and the second-order $d^{(2)}$ driving terms as follows, (see Appendix A1-5):

$$\overset{\circ}{M}_{11} = \cos\mu_x - d^{(2)}_{SC}\cos\mu_x + d^{(2)}_{CC}\sin\mu_x + 0\left(q^4\right), \tag{2.6}$$

$$\overset{\circ}{M}_{12} = \sin\mu_x - d^{(2)}_{SS}\cos\mu_x + d^{(2)}_{CS}\sin\mu_x + 0\left(q^4\right), \tag{2.7}$$

$$\overset{\circ}{M}_{21} = -\sin\mu_x + d^{(2)}_{CC}\cos\mu_x + d^{(2)}_{SC}\sin\mu_x + 0\left(q^4\right), \tag{2.8}$$

$$\overset{\circ}{M}_{22} = \cos\mu_x + d^{(2)}_{CS}\cos\mu_x + d^{(2)}_{SS}\sin\mu_x + 0\left(q^4\right), \tag{2.9}$$

and

$$\overset{\circ}{N}_{kl} = \left(\overset{\circ}{M}_{kl}\right)^{\vee}, \quad k,l = 1,2, \tag{2.10}$$

and

$$\overset{\circ}{n}_{11} = -d_{SC}^{(1)} \cos\mu_x + d_{CC}^{(1)} \sin\mu_x + 0\left(q^3\right), \tag{2.11}$$

$$\overset{\circ}{n}_{12} = -d_{SS}^{(1)} \cos\mu_x + d_{CS}^{(1)} \sin\mu_x + 0\left(q^3\right), \tag{2.12}$$

$$\overset{\circ}{n}_{21} = d_{CC}^{(1)} \cos\mu_x + d_{SC}^{(1)} \sin\mu_x + 0\left(q^3\right), \tag{2.13}$$

$$\overset{\circ}{n}_{22} = d_{CS}^{(1)} \cos\mu_x + d_{SS}^{(1)} \sin\mu_x + 0\left(q^3\right), \tag{2.14}$$

and

$$\overset{\circ}{m}_{kl} = \left(\overset{\circ}{n}_{kl}\right)^{\vee}, \quad k,l = 1,2. \tag{2.15}$$

Here the notations are:

$$\begin{bmatrix} d_{SS}^{(1)} \\ d_{SC}^{(1)} \\ d_{CS}^{(1)} \\ d_{CC}^{(1)} \end{bmatrix} = \sum_{r=1}^{N} q_r \begin{bmatrix} \sin\mu_x^r \sin\mu_y^r \\ \sin\mu_x^r \cos\mu_y^r \\ \cos\mu_x^r \sin\mu_y^r \\ \cos\mu_x^r \cos\mu_y^r \end{bmatrix}, \tag{2.16}$$

and for the second-order driving terms

$$\begin{bmatrix} d_{SS}^{(2)} \\ d_{SC}^{(2)} \\ d_{CS}^{(2)} \\ d_{CC}^{(2)} \end{bmatrix} = \sum_{1\le r<s\le N} q_r q_s \sin\left(\mu_y^s - \mu_y^r\right) \begin{bmatrix} \sin\mu_x^s \sin\mu_x^r \\ \sin\mu_x^s \cos\mu_x^r \\ \cos\mu_x^s \sin\mu_x^r \\ \cos\mu_x^s \cos\mu_x^r \end{bmatrix}, \tag{2.17}$$

where μ_x^r, μ_y^r are phase advances

$$\mu_x^r = \psi_x\left(s_r, 0\right), \tag{2.18}$$

and similar for the μ_y^r.

The thin skew-quadrupole strengths are

$$q_k = \left(\beta_x\beta_y\right)^{1/2} f_k^{-1}\Big|_{s_k}, \quad k = 1,\ldots,N. \tag{2.19}$$

The "v" operation replaces x with y and x' and y'.

For example, for the first-order driving terms we get

$$\begin{aligned} \left(d^{(1)}_{CC}\right)^{\vee} &= d^{(1)}_{CC}, \\ \left(d^{(1)}_{SS}\right)^{\vee} &= d^{(1)}_{SS}, \\ \left(d^{(1)}_{CS}\right)^{\vee} &= d^{(1)}_{SC}, \\ \left(d^{(1)}_{SC}\right)^{\vee} &= d^{(1)}_{CS}. \end{aligned} \tag{2.20}$$

Similar but less symmetric results follow for the second-order driving terms. In particular, the relations hold

$$d^{(1)}_{SS}d^{(1)}_{CC} - d^{(1)}_{SC}d^{(1)}_{CS} = \det n \equiv |n|, \tag{2.21}$$

and

$$\left[\left(d^{(1)}_{CC} + d^{(1)}_{SS}\right)^2 + \left(d^{(1)}_{SC} - d^{(1)}_{CS}\right)^2\right]^{1/2} = \left|\sum_{k=1}^{N} q_k e^{i\left(\mu_x^k - \mu_y^k\right)}\right|. \tag{2.22}$$

In order to estimate a magnitude of an effect we will assume that the skew-quadrupole errors $q_r, r = 1, \ldots, N$ are normally distributed random variables, i.e., that

$$\langle q_r \rangle = 0, \quad \langle q_r q_s \rangle = \delta_{rs} G_0^2/N, \tag{2.23}$$

and the phase-advances are such that, for both x and y directions

$$\begin{aligned} \langle \sin \mu^r \rangle &= \langle \cos \mu^r \rangle = 0, \\ \langle \sin^2 \mu^r \rangle &= \langle \cos^2 \mu^r \rangle = 1/2, \end{aligned} \tag{2.24}$$

while the averages of mixed products assumed to vanish. In this case we get for the averages of the driving terms

$$\langle d^{(1)}_{\ldots} \rangle = \langle d^{(2)}_{\ldots} \rangle = 0, \tag{2.25}$$

and

$$\langle d^{(1)^2}_{\ldots} \rangle = 1/4\ G_0^2, \tag{2.26}$$

and similar for the $\overset{\vee}{d}_{\ldots}$-driving terms. As the result one gets the estimates

$$\langle |n| \rangle = 0 + \cdots, \quad \langle |n|^2 \rangle = 1/8 G_0^4 + \cdots \quad , \tag{2.27}$$

where

$$\begin{aligned} G_0 &\simeq 0.25, && \text{for RHIC,} \\ G_0 &\simeq 0.5 - 1.0, && \text{for SSC.} \end{aligned} \tag{2.28}$$

3. Applications of TLM to Some Effects Due to Linear Coupling

3.1 The Stability Problem

If $\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}$, are eigenvalues of the single-turn transfer matrix T then their sums $\Lambda_1 = \lambda_1 + \lambda_1^{-1} = 2\cos\mu_1$ and $\Lambda_2 = \lambda_2 + \lambda_2^{-1} = 2\cos\mu_2$, where μ_1 and μ_2 are, so called, new tunes, are given by the well known formula[1]

$$\Lambda_{1,2} = \frac{1}{2} Tr\,(M+N) \pm \left(\left[\frac{1}{2} Tr\,(M-N) \right]^2 + |\overline{m} + n| \right)^{1/2} . \tag{3.1}$$

All the elements appearing here can be easily expressed through the driving terms (see Appendix). The stability conditions

$$\begin{aligned} &1^o \quad \Lambda_k - \text{real}, \\ &2^o \quad |\Lambda_k| \le 2, \quad k = 1,2, \end{aligned} \tag{3.2}$$

can be most easily satisfied on the resonance, $\mu_x = \mu_y$, since the determinant $|\overline{m} + n|$ is positive, in this case.

3.2 The Tune-Splitting

Let the new tunes $\mu_{1,2}$ differ slightly from the old ones:

$$\mu_1 = \mu_x + 2\pi\Delta\nu_1, \quad \mu_2 = \mu_y + 2\pi\Delta\nu_2, \quad (\mu_x > \mu_y)\,, \tag{3.3}$$

then from the formula (3.1) it follows that

$$\begin{aligned} \Delta\nu_1 = \frac{1}{2\pi}\cot\mu_x - \frac{1}{8\pi\sin\mu_x} Tr\,(M+N) - \\ - \frac{1}{4\pi\sin\mu_x} \left(\left[\frac{1}{2} Tr\,(M-N) \right]^2 + |\overline{m} + n| \right)^{1/2} + \cdots, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \Delta\nu_2 = \frac{1}{2\pi}\cot\mu_y - \frac{1}{8\pi\sin\mu_y} Tr\,(M+N) + \\ + \frac{1}{4\pi\sin\mu_y} \left(\left[\frac{1}{2} Tr\,(M-N) \right]^2 + |\overline{m} + n| \right)^{1/2} + \cdots . \end{aligned} \tag{3.5}$$

The leading terms, on the resonance $\mu_x = \mu_y$, are

$$\Delta\nu_1 = -\text{sgn}\,(\sin\mu_x)\,\frac{1}{4\pi} \left| \sum_{k=1}^{N} q_k e^{i(\mu_x^k - \mu_y^k)} \right| + \cdots, \tag{3.6}$$

and

$$\Delta\nu_2 = -\Delta\nu_1. \tag{3.7}$$

The higher-order terms in the expansions of $\frac{1}{2}Tr\,(M \pm N)$ contribute to, so called, the residual tune-splitting which persists after all the first-order driving terms are corrected to zero,

$$\Delta\nu_1\Big|_{\text{resid}} = -a - \text{sgn}\,(\sin\mu_x)\,|b|, \tag{3.8}$$

and

$$\Delta\nu_2\Big|_{\text{resid}} = -a + \text{sgn}\,(\sin\mu_x)\,|b|, \tag{3.9}$$

where a, b are expressed through the second-order driving terms as follows

$$8\pi a \equiv d^{(2)}_{CC} + d^{(2)}_{SS} + \check{d}^{(2)}_{CC} + \check{d}^{(2)}_{SS}, \tag{3.10}$$

and

$$8\pi b \equiv d^{(2)}_{CC} + d^{(2)}_{SS} - \check{d}^{(2)}_{CC} - \check{d}^{(2)}_{SS}\,. \tag{3.11}$$

In order to correct the tune-splitting, up to the second-order, one requires that, at the reference point $s = 0$, the following conditions hold:

$$d^{(1)}_{SS} = d^{(1)}_{SC} = d^{(1)}_{CS} = d^{(1)}_{CC} = 0, \tag{3.12}$$

and

$$d^{(2)}_{CC} + d^{(2)}_{SS} - \check{d}^{(2)}_{CC} - \check{d}^{(2)}_{SS} = \sum_{r<s} q_r q_s \sin(\delta_r - \delta_s) = 0, \tag{3.13}$$

and

$$d^{(2)}_{CC} + d^{(2)}_{SS} + \check{d}^{(2)}_{CC} + \check{d}^{(2)}_{SS} = -\sum_{r<s} q_r q_s \sin(\sigma_r - \sigma_s) = 0, \tag{3.14}$$

where

$$\delta_r \equiv \mu^r_x - \mu^r_y, \quad \sigma_r \equiv \mu^r_x + \mu^r_y. \tag{3.15}$$

Notice that the last condition (3.14), which corrects the coefficient a to zero, can be abandoned without affecting the total tune-splitting: $\Delta\nu = \frac{1}{2}(\Delta\nu_1 - \Delta\nu_2)$ simply because this term cancels. Thus the minimal local correction scheme for the tune-splitting consists of the five conditions as given by (3.12) and (3.13).

3.3 The Tune-Shift

From the basic formula (2.6)-(2.15) one finds for the traces of the submatrices M and N

$$\frac{1}{2}TrM = \cos\left(\mu_x + \Delta\mu_x\right) = \left(1 - \frac{1}{2}|n|\right)\cos\mu_y + \frac{1}{2}\left(d^{(2)}_{CC} + d^{(2)}_{SS}\right)\sin\mu_x + \cdots, \quad (3.16)$$

and

$$\frac{1}{2}TrN = \cos\left(\mu_y + \Delta\mu_y\right) = \left(1 - \frac{1}{2}|n|\right)\cos\mu_y + \frac{1}{2}\left(\check{d}^{(2)}_{CC} + \check{d}^{(2)}_{SS}\right)\sin\mu_y + \cdots. \quad (3.17)$$

Hence, for small tune-shifts $\Delta\mu_x, \Delta\mu_y$ we get

$$\Delta\mu_x = \frac{1}{2}|n|\cot\mu_x - \frac{1}{2}\left(d^{(2)}_{CC} + d^{(2)}_{SS}\right) + \cdots, \quad (3.18)$$

and

$$\Delta\mu_y = \frac{1}{2}|n|\cot\mu_y - \frac{1}{2}\left(\check{d}^{(2)}_{CC} + \check{d}^{(2)}_{SS}\right) + \cdots. \quad (3.19)$$

The tune-shift vanishes, at the point where the full tune-splitting correction was done.

3.4 The Beta-Function Distortions

The new beta-functions are given by (cf. Appendix B)

$$\beta_1 = \beta_x + \Delta\beta_x = \left(\sin\mu_1\right)^{-1} A_{12}, \quad (3.20)$$

and

$$\beta_2 = \beta_y + \Delta\beta_y = \left(\sin\mu_2\right)^{-1} B_{12}, \quad (3.21)$$

where $\Delta\beta_{x,y}$ are the beta-function distortions. Taking into account the formulae for the A and B matrices one gets the results

$$\frac{\Delta\beta_x}{\beta_x} = -1 + \left(\beta_x\sin\mu_x\right)^{-1} M_{12} - 2\pi\Delta\nu_1\cot\mu_x + \left[\beta_x\sin\mu_x\left(t+\delta\right)\right]^{-1}\left[\left(\overline{m}+n\right)m\right]_{12} + \cdots (3.22)$$

and

$$\frac{\Delta\beta_y}{\beta_y} = -1 + \left(\beta_y\sin\mu_y\right)^{-1} N_{12} - 2\pi\Delta\nu_2\cot\mu_y - \left[\beta_y\sin\mu_y\left(t+\delta\right)\right]^{-1}\left[\left(m+\overline{n}\right)n\right]_{12} + \cdots. (3.23)$$

The leading terms, on the resonance $\mu_x = \mu_y$, are

$$\frac{\Delta\beta_x}{\beta_x} = \frac{1}{2}\mathrm{sgn}\left(\sin\mu_x\right)\cot\mu_x\left|\sum_{k=1}^{N} q_k e^{i\left(\mu_x^k - \mu_y^k\right)}\right| + \cdots, \quad (3.24)$$

and

$$\frac{\Delta\beta_y}{\beta_y} = -\frac{\Delta\beta_x}{\beta_x}. \tag{3.25}$$

There are residual beta-function distortions, coming from the M_{12} and N_{12} terms, after the tune-splitting correction is locally performed. One notices also, that if one reverses the order of actions and goes on the resonance $\mu_x = \mu_y$ before the tune-splitting correction, the beta-function distortions could be large. This is because the quantity $(t+\delta)^{-1}$ can be large when on the resonance.

3.5 The Emittance Change Due to Linear Coupling

When the linear coupling is present one considers, instead of two separate invariant ellipses, a single 4-dimensional ellipsoid, at a point of a ring,[13,17]

$$\tilde{z}\,\sigma^{-1} z = 1, \tag{3.26}$$

where

$$\sigma = \begin{bmatrix} \sigma_x & t \\ \tilde{t} & \sigma_y \end{bmatrix},$$

is a symmetric and positive definite matrix while σ_x, σ_y are symmetric, positive-definite submatrices describing projected emittance and t represents the linear coupling. When passing from a point s_0 to another s_1 in a ring the σ matrix transforms as follows

$$\sigma_1 = T \sigma_0 \tilde{T}\,. \tag{3.27}$$

Assuming that the initial beam is decoupled, $(t_0 = 0)$ one gets the relations

$$\sigma_{x1} = M\sigma_{x0}\,\tilde{M} + n\sigma_{y0}\,\tilde{n}, \tag{3.28}$$

and

$$\sigma_{y1} = N\sigma_{y0}\,\tilde{N} + m\sigma_{x0}\,\tilde{m}\,. \tag{3.29}$$

Denoting the initial projected emittances as ϵ_{x0}, ϵ_{y0} we have the point s_0

$$\epsilon_{x0}^2 = |\sigma_{x0}|, \quad \epsilon_{y0}^2 = |\sigma_{y0}|, \tag{3.30}$$

and at the point s_1

$$\epsilon_{x1}^2 = \left|M\sigma_{x0}\,\tilde{M} + n\sigma_{y0}\,\tilde{n}\right|, \quad \epsilon_{y1}^2 = \left|N\sigma_{y0}\,\tilde{N} + m\sigma_{x0}\,\tilde{m}\right|. \tag{3.31}$$

Assuming for simplicity that the initial beam ellipse are upright and that they coincide with the machine ellipses (perfect match), we get the results[13]

$$\epsilon_{x1}^2 = (1 - |n|)^2 \epsilon_{x0}^2 + |n|^2 \epsilon_{y0}^2 + \Delta, \tag{3.32}$$

and

$$\epsilon_{y1}^2 = (1 - |n|)^2 \epsilon_{y0}^2 + |n|^2 \epsilon_{x0}^2 + \Delta, \tag{3.33}$$

and where the positive quantity Δ is given by the expression

$$\Delta = \epsilon_{x0}\epsilon_{y0} \left[\left(d_{CC}^{(1)}\right)^2 + \left(d_{CS}^{(1)}\right)^2 + \left(d_{SC}^{(1)}\right)^2 + \left(d_{SS}^{(1)}\right)^2 \right] + 0\left(q^4\right). \tag{3.34}$$

We have used here the formulae which follow from the symplecticity of the transfer matrix[13,17]

$$|M| = |N| = 1 - |n|, \tag{3.35}$$

and

$$|m| = |n| = d_{CC}^{(1)} d_{SS}^{(1)} - d_{SC}^{(1)} d_{CS}^{(1)} + 0\left(q^4\right). \tag{3.36}$$

It is clear that the projected emittance stays unchanged when the first-order driving terms vanish. This happens when the tune-splitting is locally corrected, at the reference point $s = 0$. The emittance changes from point to point if the linear coupling as represented by the determinant $|n|$ and the quantity Δ varies around a ring.

At the end we would like to collect some estimates of magnitudes of the various effects using (2.23) - (2.27). One has, for example, the relations

$$\langle \Delta\mu_x \rangle = \langle \Delta\mu_y \rangle = 0 + \cdots , \tag{3.37}$$

and

$$\left(\frac{\Delta\beta_x}{\beta_x}\right)_{\text{rms}} = \left(\frac{\Delta\beta_y}{\beta_y}\right)_{\text{rms}} = 1/2G_0|\cot\mu_x| + \cdots , \tag{3.38}$$

and

$$\langle \Delta \rangle = G_0^2 \epsilon_{x0}\epsilon_{y0} + \cdots \geq 0, \tag{3.39}$$

and

$$\langle \epsilon_{x1}^2 \rangle = \epsilon_{x0}^2 + G_0^2 \epsilon_{x0}\epsilon_{y0} + G_0^4/8\left(\epsilon_{x0}^2 + \epsilon_{y0}^2\right) + \cdots, \tag{3.40}$$

$$\langle \epsilon_{y1}^2 \rangle = \epsilon_{y0}^2 + G_0^2 \epsilon_{x0}\epsilon_{y0} + G_0^4/8\left(\epsilon x0^2 + \epsilon_{y0}^2\right) + \cdots. \tag{3.41}$$

Appendix A. Derivation of the Basic Formulae (2.6)-(2.15)

To extend the TLM beyond the first-order one uses so called "projection approach"[6,7] which yields the following basic formula for the single-turn transfer matrix

$$\overset{\circ}{T} = \overset{\circ}{T}_0 \, P_N \cdots P_1, \tag{A.1}$$

where the "projection" on the k-th skew-quadrupole is

$$P_k = \begin{bmatrix} \mathbf{1}_2 & F_k \\ G_k & \mathbf{1}_2 \end{bmatrix}, \quad k = 1, \ldots, N. \tag{A.2}$$

and where

$$F_k = 1/2 q_k R\left(-\frac{\pi}{2}\right)\left[R\left(-\mu_x^k + \mu_y^k\right) + R\left(-\mu_x^k - \mu_y^k\right) J\right], \tag{A.3}$$

and

$$G_k = \overset{\vee}{F}_k, \tag{A.4}$$

and

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A.5}$$

Performing the multiplications of the projections leads to the expansion (2.5), and to the basic formulae (2.6)-(2.15).

Expressions of the traces $\frac{1}{2}Tr\,(M \pm N)$, and determinant $|\overline{m} + n|$ through the driving terms

Using the basic formulae (2.6)-(2.15) one gets the following results

$$\begin{aligned} \frac{1}{2}Tr\,(M+N) &= 2\left(1 - \frac{1}{2}|n|\right)\cos\left[\pi\left(\nu_x + \nu_y\right)\right]\cos\left[\pi\left(\nu_x - \nu_y\right)\right] + \\ &+ \frac{1}{2}\left(d_{CC}^{(2)} + d_{SS}^{(2)} + \overset{\vee(2)}{d_{CC}} + \overset{\vee(2)}{d_{SS}}\right)\sin\left[\pi\left(\nu_x + \nu_y\right)\right]\cos\left[\pi\left(\nu_x - \nu_y\right)\right] + \\ &+ \frac{1}{2}\left(d_{CC}^{(2)} + d_{SS}^{(2)} - \overset{\vee(2)}{d_{CC}} - \overset{\vee(2)}{d_{SS}}\right)\cos\left[\pi\left(\nu_x + \nu_y\right)\right]\sin\left[\pi\left(\nu_x - \nu_y\right)\right] + 0\left(q^4\right), \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} \frac{1}{2}Tr\,(M-N) &= -2\left(1 - \frac{1}{2}|n|\right)\sin\left[\pi\left(\nu_x + \nu_y\right)\right]\sin\left[\pi\left(\nu_x - \nu_y\right)\right] + \\ &+ \frac{1}{2}\left(d_{CC}^{(2)} + d_{SS}^{(2)} + \overset{\vee(2)}{d_{CC}} + \overset{\vee(2)}{d_{SS}}\right)\cos\left[\pi\left(\nu_x + \nu_y\right)\right]\cos\left[\pi\left(\nu_x - \nu_y\right)\right] + \\ &+ \frac{1}{2}\left(d_{CC}^{(2)} + d_{SS}^{(2)} - \overset{\vee(2)}{d_{CC}} - \overset{\vee(2)}{d_{SS}}\right)\sin\left[\pi\left(\nu_x + \nu_y\right)\right]\sin\left[\pi\left(\nu_x - \nu_y\right)\right] + 0\left(q^4\right), \end{aligned} \tag{A.7}$$

and

$$|\overline{m}+n| = \left|\sum_{k=1}^{N} q_k e^{i\left(\mu_x^k-\mu_y^k\right)}\right|^2 \sin^2\left[\pi\left(\nu_x+\nu_y\right)\right] - \left|\sum_{k=1}^{N} q_k e^{i\left(\mu_x^k+\mu_y^k\right)}\right|^2 \sin^2\left[\pi\left(\nu_x-\nu_y\right)\right] + 0\left(q^4\right). \tag{A.8}$$

Owing to the definitions (2.16) of the first-order driving terms one has the equalities

$$\left|\sum_{k=1}^{N} q_k e^{i\left(\mu_x^k-\mu_y^k\right)}\right|^2 = \left(d_{CC}^{(1)} + d_{SS}^{(1)}\right)^2 + \left(d_{SC}^{(1)} - d_{CS}^{(1)}\right)^2, \tag{A.9}$$

and

$$\left|\sum_{k=1}^{N} q_k e^{i\left(\mu_x^k+\mu_y^k\right)}\right|^2 = \left(d_{CC}^{(1)} - d_{SS}^{(1)}\right)^2 + \left(d_{SC}^{(1)} + d_{CS}^{(1)}\right)^2. \tag{A.10}$$

Appendix B. The Universal Parameterization of the Single-Turn Transfer Matrix

It was shown by Edwards and Teng,[2] and by Talman,[4] that the single-turn transfer matrix T can be brought to a quasidiagonal form as follows: If

$$T = \begin{bmatrix} M & n \\ m & N \end{bmatrix} \tag{B.1}$$

is a 4×4 real, C-periodic and symplectic, single-turn transfer matrix, then

$$U = R^{-1}\, T\, R = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}, \tag{B.2}$$

where A, B and R are symplectic and

$$A = M + (t+\delta)^{-1}\left(\overline{m}+n\right)m = \begin{bmatrix} \cos\mu_1 + \alpha_1 \sin\mu_1 & \beta_1 \sin\mu_1 \\ -\gamma_1 \sin\mu_1 & \cos\mu_1 - \alpha_1 \sin\mu_1 \end{bmatrix}, \tag{B.3}$$

and

$$B = N - (t+\delta)^{-1}\left(m+\overline{n}\right)n = \begin{bmatrix} \cos\mu_2 + \alpha_2 \sin\mu_2 & \beta_2 \sin\mu_2 \\ -\gamma_2 \sin\mu_2 & \cos\mu_2 - \alpha_2 \sin\mu_2 \end{bmatrix}, \tag{B.4}$$

and

$$t = \frac{1}{2} Tr\left(M-N\right), \tag{B.5}$$

$$\delta = \frac{1}{2} Tr\left(A-B\right) = \left(t^2 + |\overline{m}+n|\right)^{1/2}. \tag{B.6}$$

The diagonalizing matrix R can also be expressed through the submatrices of T (cf [4], for example).

Acknowledgments

Author would like to thank Dr. A.G. Ruggiero for suggesting the linear coupling problem in higher-orders, and for discussions during early stage of the work. Also author would like to acknowledge the many constructive comments made by members of the Accelerator Development Department of BNL during his work on the linear coupling problem.

References

1. E.D. Courant and H.S. Snyder, Ann. Phys. 3 (1958) 1.
2. D.A. Edwards and L.C. Teng, IEEE Trans. Nucl. Sci., Vol. NS-20, 3 (1973) 885.
3. L.C. Teng, "Concerning N-Dimensional Coupled Motions", Fermilab Report FN-229 (1971), and "Coupled Transverse Motion", FNAL-TM-1566 (1989).
4. R. Talman, "Single Particle Motion", in Frontiers of Particle Beams, Observation, Diagnosis and Corrections", Proceedings, Anacapri, Italy, 1988, Eds. M. Month and S. Turner, Springer Lecture Notes in Physics, No. 343, pp. 4-45.
5. L. Schachinger, T. Sun and R. Talman, Manual for the TEAPOT Program, SSC Central Design Group (1990).
6. Y. Kobayashi, Nucl. Instrum. Methods, 83 (1970) 77.
7. S. Peggs, Particle Accelerators, 12 (1982) 219.
8. H. Zyngier, IEEE Trans. Nucl. Sci., Vol. NS-32, No. 5 (1985) 2282.
9. G. Parzen, BNL Report, AD/RHIC-102 (July 1991).
10. A.G. Ruggiero, Proceedings of the "5th ICFA Advanced Beam Dynamics Workshop," Corpus Christi, Texas (October 1991).
11. V. Garczynski, Beta-Function Distortions Due to Linear Coupling, BNL AD/AP Technical Note No. 24 (August 1991).
12. V. Garczynski, The Tune Shift Due to Linear Coupling, BNL AD/AP Technical Note No. 25 (August 1991).
13. V. Garczynski, Emittance Change Due to Linear Coupling. Possible Correction Scheme of Emittance Growth, BNL AD/AP Technical Note No. 28 (September 1991).

14. V. Garczynski, The Thick Ellipse Effect Due to Linear Coupling, and Projections of the Invariant 4-Ellipsoid onto the (x, x'), (y, y') and the (x, y)-Planes, BNL AD/AP Technical Note No. 33 (December 1991).
15. V. Garczynski, A General Scheme of Solenoid Compensation Using Antisolenoids. An Example: Decoupling of the STAR Detector in RHIC, BNL AD/AP Technical Note No. 37 (February 1992).
16. V. Garczynski, A General Scheme of Solenoid Compensation Using Skew-Quadrupoles. An Example: Decoupling of the STAR Detector in RHIC, BNL AD/AP Technical Note No. 38 (February 1992).
17. K.L. Brown and R.V. Servranckx, Cross-Plane Coupling and its Effects on Projected Emittance, SLAC-PUB-4679 (1989).
18. K.L. Brown, A First- and Second-Order Matrix Theory for the Design of Beam Transport Systems and Charged Particle Spectrometers, SLAC Report No. 75, or Advances in Particle Physics 1 (1967) 71-134.
19. K.L. Brown, D.C. Carey, Ch. Iselin and F. Rothacker, TRANSPORT, A Computer Program for Designing Charged Particle Beam Transport Systems, SLAC 91 (1973 Rev.), NAL91 and CERN 80-04.
20. K.L. Brown and R.V. Servranckx, First- and Second-Order Charged Particle Optics, SLAC-PUB-3381.