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The RF Wiggler (A novel concept of RF power generator)

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1. Introduction

It is well-known that there are no rf power sources of significant amount for frequencies larger than 3 GHz. Yet, rf sources in the centimeter/millimeter wavelength range would be very useful to drive, for example, high-gradient accelerating linacs for electron-positron linear colliders.

Ordinary conceived methods to produce radiation with short wavelengths is to let a bunch of electrons to travel down a magnetic structure with alternating field direction, up and down, like in the *wigglers* or in the *undulators*, as it is done in circular synchrotron radiation facilities. Nevertheless, an application of these devices to generate radiation in the centimeter-to-millimeter wavelength range, though often proposed, has still to be demonstrated. These devices are ordinarly used to generate radiation of much shorter wavelength.

We would like to propose an alternative, novel method to produce such radiation, which still makes use of a short electron bunch, traveling this time down a waveguide which is at the same time excited by a lower TM propagating electromagnetic wave. It is indeed well known that radiation can be obtained by *wiggling* the motion of the electrons in a direction perpendicular to the main one. The wiggling action can be induced by electromagnetic fields in a fashion similar to the one caused by wiggler magnets, as we shall indeed show in this paper. We found that an interesting mode of operation is to drive the waveguide with an excitation of frequency very close to the cut off. Indeed for such excitation, the corresponding e.m. wave travels with a very large phase velocity which in turn has the effort to increase the wiggling action on the electron bunch.

Our method, to be effective, relies also on the *coherence* of the radiation from electron-to-electron; that is the bunch length is taken to be considerably shorter than the radiated wavelength. In this case, the total power radiated should be proportional to the square of the total number of electrons in the bunch, as opposed to the case where the bunch is longer and the power is linearly proportional to the number of particles. This is, of course, a theoretical speculation, easily explained, which nevertheless requires an experimental demonstration.

In summary we investigate here the interaction of a short bunch of electrons with a TM electromagnetic wave, both traveling along the axis of a waveguide. This structure can be driven in a variety of modes. For reasons that will be clear to the reader, we have chosen a TM mode with an antisymmetric voltage distribution on the plane where the electrons are to oscillate. We have also taken the lowest order of propagation of such a mode. Other cases may certainly be possible and are left to future analysis.

2. Scalar and Vector Potentials in a Waveguide

We shall consider an infinitely long *waveguide*, straight, with square cross-section. Let the walls be perfectly conductive and the inner side w. We shall introduce a rectangular cartesian coordinate system x, y and z; where x and y are the transverse distances from the axis of the waveguide (see Fig. 1) and z is the longitudinal coordinate along the axis itself. In this section we describe the propagation of a TM traveling electromagnetic wave in the waveguide. If we use the Lorentz representation, the fields can be derived from a scalar V and vector potential $\mathbf{A}$ satisfying the following equations

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0 \tag{1}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \tag{2}$$

$$\mathrm{div}\mathbf{A} + \frac{1}{c}\ \frac{\partial V}{\partial t} = 0 \tag{3}$$

In cartesian coordinates the explicit form of Eq. (1) is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0 \tag{4}$$

A solution of Eq. (4), which is the one next to the lowest order of propagation, is

$$V = V_0 \sin\left(2\pi\frac{x}{w}\right) \cos\left(\pi\frac{y}{w}\right) \cos\left(kz - \omega t\right) \tag{5}$$

which satisfies the condition $V = 0$ at the wall surfaces $x = \pm w/2$ and $y = \pm w/2$. The nature of the traveling wave is described by the last factor where ω is the angular frequency and k the wave number which defines the longitudinal propagation mode. Insertion of Eq. (5) in Eq. (4) yields the following *dispersion relation*

$$k = \sqrt{\frac{\omega^2}{c^2} - 5\left(\frac{\pi}{w}\right)^2} \tag{6}$$

The condition for the *propagation* of the wave in the waveguide is that the wave number k is real, that is

$$\omega > \omega_c \tag{7}$$

where

$$\omega_c = \sqrt{5}\,\frac{\pi c}{w} \tag{8}$$

is the cut-off frequency which depends only on the dimension of the waveguide.

It is convenient to introduce the form factor

$$\begin{aligned} q &= \sqrt{5}\,\frac{\pi c}{w\omega} \\ &= \omega_c/\omega \end{aligned} \tag{9}$$

and the phase velocity, normalized to the speed of light,

$$\beta_w = \omega/kc \tag{10}$$

It is then seen from Eq. (6) that

$$\beta_w = \frac{1}{\sqrt{1-q^2}} \tag{11}$$

The range of values of the form factor q fulfilling the condition of propagation is

$$0 < q < 1 \tag{12}$$

It is also seen that $\beta_w > 1$, that is the wave phase velocity is always larger than the speed of light.*

* In principle, it is also possible to use material for the walls of the waveguide which is not perfectly conductive; if the material properties are chosen conveniently, it is possible to obtain $\beta_w < 1$, if required.

An inspection of the dispersion relation, Eq. (6) shows that below the cut-off, $\omega < \omega_c$ there is no propagation, and k assumes no real values. For large values of ω, k increases about linearly. It is also convenient to introduce the wavelength $\lambda = 2\pi/k$; it is then seen that the wavelength is very long in proximity of the cut- off and it gets very short as ω increases.

An interesting plot, shown in Fig. 2, is the display of the wave phase velocity β_w versus the form factor q as given by Eq. (11). There are two regimes:

(i) $q << 1$. In this case

$$\beta_w \approx 1 + \frac{1}{2}\left(\frac{\omega_c}{\omega}\right)^2 \tag{13}$$

Since $\omega >> \omega_c$, that is well above the cut-off, the phase velocity is very close to the speed of light and the wavelength λ is very short.

(ii) $q \sim 1$, in proximity of the cut-off and $\omega \sim \omega_c$. In this case the wavelength λ is very long.*

According to the conventional waveguide terminology, a TM-mode is defined as that traveling wave with vanishing magnetic field in the main direction of propagation, that is the z-axis of the waveguide. This mode is associated to solutions of the vector potential **A** which actually is completely directed along the z-axis and has vanishing components in the directions (x and y) perpendicular to the direction of the wave propagation, that is

$$\mathbf{A} \equiv (0, 0, A) \tag{14}$$

where A satisfies an equation similar to Eq. (4). Moreover, to satisfy the Lorentz condition represented by Eq. (3),

$$A = \beta_w\ V \tag{15}$$

Thus the vector potential **A** is completely determined from the knowledge of the scalar potential V. The only parameter left to be defined is the amplitude V_0 of the scalar potential distribution.

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* In proximity of the cut-off with $\omega \sim \omega_c$, the properties of the wall material do not affect critically the value of the phase velocity. This is not true for the other case where $q << 1$.

3. The Field Distribution

The electric **E** and magnetic **B** fields can be determined from the usual relations

$$\mathbf{E} = -\text{grad } V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{16}$$

$$\mathbf{B} = \text{rot } \mathbf{A} \tag{17}$$

We obtain after inserting Eqs. (5, 14 and 15)

$$E_x = -2\pi \frac{V_0}{w} \cos\left(2\pi \frac{x}{w}\right) \cos\left(\pi \frac{y}{w}\right) \cos\phi \tag{18}$$

$$E_y = \pi \frac{V_0}{w} \sin\left(2\pi \frac{x}{w}\right) \sin\left(\pi \frac{y}{w}\right) \cos\phi \tag{19}$$

$$E_z = -k\left(\beta_w^2 - 1\right) V_0 \sin\left(2\pi \frac{x}{w}\right) \cos\left(\pi \frac{y}{w}\right) \sin\phi \tag{20}$$

and

$$B_x = -\beta_w E_y \tag{21}$$

$$B_y = \beta_w E_x \tag{22}$$

$$B_z = 0 \tag{23}$$

where

$$\phi(z, t) = kz - \omega t \tag{24}$$

is the wave phase function.

A property of a perfectly conductive wall is that the components of the electric field tangent to the wall and those of the magnetic field perpendicular to the wall are to vanish identically. It is easily verified that these conditions are satisfied by the solution shown by Eqs. (18-23).

From the point of view of an electron moving along the longitudinal direction z, the field distribution given by Eqs. (18-24) is different from that the same electron would experience in the case of a wiggler magnet or a photon beam. To see this we can operate a transformation of the field distribution in the frame where the electron is at rest. Let β and γ be the relativistic factors respectively for velocity and energy of the particle. Let us first consider the case of a planar electromagnetic wave moving in the z-direction. The electric and magnetic fields are perpendicular to each other and perpendicular to the direction of motion. In the coordinate frame we have adopted, the components of the fields of the plane wave in the laboratory frame are

$$E_y = E_z = B_x = B_z = 0 \tag{25}$$

$$E_x = B_y = E_0 \; e^{i(kz-\omega t)} \tag{26}$$

where E_0 is a constant amplitude, ω the angular frequency and k the propagation constant of the wave. After performing the relativistic transformation, the components of field distribution of the same wave, in the system where the electron is at rest, are

$$E'_y = E'_z = B'_x = B'_z = 0 \tag{27}$$

$$E'_x = B'_y = \gamma\,(1+\beta)\,E_0 \; e^{i\left(k'z'-\omega' t'\right)} \tag{28}$$

where we have denoted with prime the variables in the frame at rest. We have

$$k' = \gamma\left(k - \omega\frac{\beta}{c}\right) \tag{29}$$

$$\omega' = \gamma\,(\omega - \beta kc) \tag{30}$$

In the vacuum $k = \omega/c$ and the previous relations become

$$\omega' = \gamma\,(1-\beta)\,\omega \tag{31}$$

$$k' = \omega'/c \tag{32}$$

Now let us consider the case of a wiggler magnet, that is a sequence of dipole magnets where the magnetic field is in the y-direction and changes periodically sign from one magnet to the next. Let L be the periodicity of the wiggler. In the laboratory frame the field components are

$$B_x = B_z = 0 \tag{33}$$

$$E_x = E_y = E_z = 0 \tag{34}$$

$$B_y = \pm B_w \tag{35}$$

where B_w is the wiggler field. There is no electric field in a wiggler. After performing the Lorentz transformation where the electron is at rest, one obtains a field distribution similar to that given by Eqs. (27-28) except that

$$E'_x = \beta\gamma B_w \tag{36}$$

$$B'_y = \gamma\ B_w \tag{37}$$

To a relativistic electron with $\beta \sim 1$, the wiggler is equivalent to a plane electromagnetic wave with amplitude $E_0 \simeq B_w/2$ and

$$\omega' = c\gamma k_w \tag{38}$$

where $k_w = 2\pi/L$. Thus the interaction of an electron with a planar electromagnetic wave or with a wiggler magnet is the same; if the electron moves against the wave, the consequence is a Compton scattering by which the electron loses energy and the photon field intensity increases.

Let us turn now our attention to the field distribution in the waveguide given by Eqs. (18-24). After the Lorentz transformation to the frame where the electron is at rest is performed, the new field distribution is given by

$$E'_z = E_z \tag{39}$$

$$E'_x = \gamma\,(1 - \beta\beta_w)\,E_x \tag{40}$$

$$E'_y = \gamma\,(1 - \beta\beta_w)\,E_y \tag{41}$$

$$B'_z = 0 \tag{42}$$

$$B'_x = -\gamma\,(\beta_w - \beta)\,E_y \tag{43}$$

$$B'_y = \gamma\,(\beta_w - \beta)\,E_x \tag{44}$$

where E_x and E_y are given by Eqs. (18-19) with ϕ replaced by $\phi' = k'z' - \omega' t'$ where k' and ω' are also given by Eqs. (29-30). Since ω and k are related to each other by the phase velocity given by Eq. (10), it is

$$k' = \gamma k\,(1 - \beta\beta_w) \tag{45}$$

$$\omega' = \gamma k c\,(\beta_w - \beta) \tag{46}$$

from which the phase velocity in the rest frame is

$$\beta'_w = \frac{\omega'}{k'c} = \frac{\beta_w - \beta}{1 - \beta\beta_w} \tag{47}$$

If we neglect, for a moment, the longitudinal component E'_z, it is seen that the two vectors $\mathbf{E}'$ and $\mathbf{B}'$ are perpendicular to the main direction of motion and to each other. Also, for a relativistic electron, $\beta \sim 1$ and the two vectors have about the same magnitude. This represents also an electromagnetic wave, but having nonplanar properties; for instance, the amplitude of the wave vanishes at the origin $x = y = 0$ and increases for small displacements from the waveguide axis to vanish again at the four corners. Since $\beta_w > 1$, it is seen from Eq. (47) that in order for the wave to propagate in the positive direction of the z-axis also in the frame at rest, the electron should move in the opposite direction ($\beta < 0$) to start with. This will create an enhancement of the equivalent field as seen by the electron; otherwise, if the wave and the particle would move in the same direction there would be a cancellation, which is the most effective when $\beta_w \sim 1$ that is in the short wavelength regime, with frequencies ω well above the cut-off value ω_c. In the other regime with $\beta_w >> 1$, in proximity of the cut-off, which corresponds to long wavelengths λ, the cancellation does not apply; a part from a sign, the amplitude of the wave is then proportional to $\gamma\beta_w$ independently of the direction of motion of the electron with respect to the wave.

Thus there are substantial differences between the field distribution generated by an electromagnetic wave in a waveguide, and those due to either a wiggler magnet or a pure plane wave in the vacuum. We shall rely on these differences to work out the new method of interaction between the electron and the surrounding.

A case of interest is the following. The electron and the e.m. wave, as seen in the laboratory frame, are moving in the same direction, which is the positive direction of the z-axis of the waveguide, as shown in Fig. 1. Assume $\beta \sim 1$ and $\beta_w >> 1$; then, from Eq. (47), $\beta'_w \sim -1$ and, in its own frame at rest, the electron sees a wave moving against its position; the consequence is again a forward Compton scattering. Let us also suppose that initially $x = y = 0$, then the only nonvanishing field components are

$$E'_x = 2\pi \frac{V_0}{w} \gamma \left(\beta\beta_w - 1\right) \cos\phi \tag{48}$$

$$B'_y = -2\pi \frac{V_0}{w} \gamma \left(\beta_w - \beta\right) \cos\phi \tag{49}$$

which have an amplitude

$$\gamma E_w = 2\pi \frac{V_0}{w} \gamma \beta_w \tag{50}$$

that can be compared to the value γB_w corresponding to a magnetic wiggler. Inspection of Eq. (50) shows that the field amplitude can reach very large values when driving the waveguide in proximity of the cut off.

4. Power Flow in the Waveguide

The constant V_0 in Eq. (5) determines the amplitude of the field potential and is related to the power flux in the waveguide. The flow of energy is described by the Poynting vector

$$\mathbf{S} = \frac{c}{8\pi}\,\mathbf{E} \times \mathbf{B} \tag{51}$$

To evaluate the total power flow P we integrate the axial component of $\mathbf{S}$ over the cross-section of the waveguide, that is:

$$P = \frac{c}{8\pi} \int_{-\frac{w}{2}}^{+\frac{w}{2}} \int_{-\frac{w}{2}}^{+\frac{w}{2}} (E_x B_y - B_x E_y)\, dx\ dy \tag{52}$$

Insertion of Eqs. (18,19 and 21,22) in Eq. (51) gives

$$P = \frac{5\pi c}{64}\,\beta_w\,V_0^2 \tag{53}$$

Inserting this equation at the right hand side of Eq. (50) gives

$$E_w \simeq \sqrt{\frac{256\pi\beta_w P}{5cw^2}} \tag{54}$$

For a constant input rf power P, the equivalent amplitude of the field has still large values of driving frequencies close to the cut-off.

5. The Equations of Motion

Consider now an electron with mass at rest m and electric charge e moving down the waveguide. The motion is relativistic and mainly along the axis of the waveguide but it could have either direction, that is along or against the direction of propagation of the electromagnetic wave.* The components of the equations of motion can be derived from

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + e\frac{\mathbf{v}}{c} \times \mathbf{B} \tag{55}$$

where $\mathbf{p}$ is the electron vector momentum and $\mathbf{v} \equiv (\dot{x}, \dot{y}, \dot{z})$ the velocity vector. By inserting Eqs. (21-23) in Eq. (55) we obtain

$$\frac{dp_x}{dt} = -e\left(\frac{\dot{z}}{c}\beta_w - 1\right) E_x \tag{56}$$

$$\frac{dp_y}{dt} = -e\left(\frac{\dot{z}}{c}\beta_w - 1\right) E_y \tag{57}$$

$$\frac{dp_z}{dt} = e\left(\frac{\dot{x}}{c}\beta_w E_x + \frac{\dot{y}}{c}\beta_w E_y + E_z\right) \tag{58}$$

The equations of motion simplify considerably if we assume that the motion of the electron is confined to the $y = 0$ plane. In this case $y = \dot{y} = 0$ is a solution of the equations of motion since $E_y = 0$. We have then, with Eqs. (18-20) taken into account

$$\frac{dp_x}{dt} = 2\pi e\left(\frac{\dot{z}}{c}\beta_w - 1\right) \frac{V_0}{w} \cos\left(2\pi\frac{x}{w}\right) \cos\phi \tag{59}$$

$$\frac{dp_y}{dt} = 0 \tag{60}$$

$$\frac{dp_z}{dt} = -2\pi e\frac{\dot{x}}{c}\beta_w \frac{V_0}{w} \cos\left(2\pi\frac{x}{w}\right) \cos\phi + \\ - ek\left(\beta_w^2 - 1\right) V_0 \sin\left(2\pi\frac{x}{w}\right) \sin\phi \tag{61}$$

where the phase ϕ is given by Eq. (24). These equations are exact as long as energy losses to radiation are neglected.

* Throughout the paper, the electron velocity β can take positive and negative values.

6. Hamiltonian Formalism

If again the radiation effects are neglected, the motion of the electron satisfies energy conservation principles and can be described by the following Hamiltonian

$$H = eV + c\sqrt{m^2c^2 + \left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right)^2} \tag{62}$$

where $\mathbf{P}$ is the generalize momentum vector related to the actual momentum $\mathbf{p}$ by the relation

$$\mathbf{P} = \mathbf{p} + \frac{e}{c}\mathbf{A} \tag{63}$$

and canonical conjugated to the vector $\mathbf{r}$ made of the position coordinates x, y and z of the particle.

We shall continue to consider only the case of planar motion with $y = P_y = 0$. For this case, since $\mathbf{A}$ is given by Eqs. (14-15),

$$H = eV + c\sqrt{m^2c^2 + P_x^2 + \left(P_z - \frac{e}{c}\,\beta_w V\right)^2} \tag{64}$$

where from Eqs. (5 and 24)

$$V = V_0 \ \sin\ u \quad \cos\ \phi \tag{65}$$

and we have set

$$u = 2\pi\frac{x}{w}\ . \tag{66}$$

It is possible to replace the canonically conjugated variables x, P_x with u, P_y by adopting the following new Hamiltonian

$$H = eV_0 \sin u \cos\phi + c\sqrt{m^2c^2 + 4\frac{\pi^2}{w^2}P_u^2 + \left(P_z - \frac{e}{c}\beta_w V_0 \sin u \cos\phi\right)^2} \tag{67}$$

and

$$P_u = \frac{w}{2\pi}P_x \tag{68}$$

We found also convenient to perform a canonical transformation which replaces the longitudinal variable z with the phase angle ϕ as given by Eq. (24). For this purpose we introduce the canonical generating function of the third kind

$$\begin{aligned} F &= -\frac{P_z}{k}(\phi + \omega t) \\ &= F(\phi, P_z, t) \end{aligned} \tag{69}$$

which relates the new variables ϕ, P_ϕ to the old pair z, P_z with the relations

$$z = -\frac{\partial F}{\partial P_z} = \frac{\phi + \omega t}{k} \tag{70}$$

$$P_\phi = -\frac{\partial F}{\partial \phi} = \frac{P_z}{k} \tag{71}$$

The new Hamiltonian, expressed in the new variables, is obtained by adding to the right hand side of Eq. (67) the partial derivative of F with respect to time, that is

$$\begin{aligned} H = {} & e\ V_0 \sin u\ \ \cos\phi - \omega\ P_\phi + \\ & + c\sqrt{m^2c^2 + 4\frac{\pi^2}{w^2}P_u^2 + \left(kP_\phi - \frac{e}{c}\beta_w V_0 \sin u\ \ \cos\phi\right)^2} \end{aligned} \tag{72}$$

Some of the parameters appearing can be eliminated with proper scaling. We chose to replace the independent variable t (time) with

$$s = ct \tag{73}$$

that is the distance traveled by an e.m. wave moving at the speed of light. Also, we introduce the reduced momenta

$$Q_{u,\phi} = \frac{P_{u,\phi}}{mc}, \tag{74}$$

the normalized parameter for the voltage amplitude

$$\Gamma = \frac{eV_0}{mc^2} \tag{75}$$

and the new Hamiltonian is

$$\begin{aligned} K &= \frac{H}{mc^2} \\ &= \Gamma \sin u\ \ \cos\phi - \frac{\omega}{c}Q_\phi + \\ &+ \sqrt{1 + 4\frac{\pi^2}{w^2}Q_u^2 + \left(kQ_\phi - \beta_w \Gamma \sin u\ \ \cos\phi\right)^2} \end{aligned} \tag{76}$$

As usual the equations of motion are given by the following system of differential equations of the first order

$$\frac{du}{ds} = \frac{\partial K}{\partial Q_u} \tag{77}$$

$$\frac{d\phi}{ds} = \frac{\partial K}{\partial Q_\phi} \tag{78}$$

$$\frac{dQ_u}{ds} = -\frac{\partial K}{\partial u} \tag{79}$$

$$\frac{dQ_\phi}{ds} = -\frac{\partial K}{\partial \phi} \tag{80}$$

Though there are five parameters entering the Hamiltonian given by Eq. (76), only three are really required since the others can be derived from relations established in Section 2. We prefer to assign the amplitude parameter Γ, the angular frequency ω of the e.m. wave, and the form factor q. From Eq. (9) we then derive the cut-off frequency ω_c and the required width w of the waveguide from Eq. (8); finally Eq. (11) is used to evaluate β_w and Eq. (10) the propagation constant k.

To determine the motion of the electron, the system of differential Eqs. (77-80) is solved by specifying the initial values of the four variables u, ϕ, P_u, P_ϕ at the starting location that can be taken to be $s = 0$.

7. The Longitudinal Motion

It is important to understand first of all the motion of the electron along the axis of the waveguide. It is easily seen that if $u = P_u = 0$ then the trajectory is a straight line and coincides exactly with the axis of the waveguide. For this case the Hamiltonian becomes

$$K = -\frac{\omega}{c} Q_\phi + \sqrt{1 + k^2 Q_\phi^2} \tag{81}$$

which involves only one of the two canonically conjugated variables ϕ, P_ϕ and the two of the three previously defined parameters: Γ, ω and q. The equations of motion are then

$$\frac{d\phi}{ds} = -\frac{\omega}{c} + \frac{k^2 Q_\phi}{\sqrt{1 + k^2 Q_\phi^2}} \tag{82}$$

$$\frac{dQ_\phi}{ds} = 0 \tag{83}$$

It is seen that, since the phase velocity $\beta_w > 1$, the second term at the right hand side of Eq. (82) is always less than unit, so that $d\phi/ds$ can never vanish and actual takes always negative values. We can refer to this as the *phase slippage*. This fact could also have been seen easily from the definition of the phase given by Eq. (24); taking the time derivative of both sides

$$\begin{aligned} \dot{\phi} &= k\dot{z} - \omega \\ &\approx kv - \omega \\ &\approx ck\,(\beta - \beta_w) = -\Omega_0 \end{aligned} \tag{84}$$

Since $\beta_w > \beta$ this quantity is indeed always negative. A slow phase slippage occurs when the particle and the electromagnetic wave are traveling in the same direction and for frequencies ω well above the cut-off value ω_0, that is $\beta_w \sim 1$. In proximity of the cut off where $\beta_w >> 1$ the phase slippage $\Omega_0 \approx \omega$.

It is reasonable to assume that during the interaction with the electromagnetic wave, the velocity β of the electron does not change considerably and it remains close to unit. In line with this approximation, Eqs. (82-83) can then be replaced by

$$\frac{d\phi}{ds} \simeq k\,(\beta - \beta_w) = -\Omega_0/c \tag{85}$$

$$\frac{dQ_\phi}{ds} \simeq 0 \tag{86}$$

In particular the integration of Eq. (86) leads to a constant value of the particle momentum.

8. The Transverse Motion

We can solve Eq. (59) in the approximation that the particle energy and velocity do not change appreciably. Thus we replace $\dot{z}$ with the total velocity v and neglect the $\dot{\gamma}$ term at the left hand side of Eq. (59) itself. Moreover we shall consider only small displacement $x << w$, so that the cos function at the right hand side can be replaced by unit. We obtain

$$\ddot{x} = \Omega^2 w \cos\phi \tag{87}$$

where

$$\Omega^2 = 2e\frac{\pi^2}{w^2}V_0\frac{\beta\beta_w - 1}{m_0\gamma} \tag{88}$$

Eq. (87) can also be written as

$$\frac{d^2x}{d\phi^2} = \nu^2 w \cos\phi \tag{89}$$

with

$$\nu = \Omega/\Omega_0 \tag{90}$$

The solution of Eq. (89) can be easily derived to be

$$x = -a \ \cos \ \phi \tag{91}$$

that is an oscillation at the frequency equal to the phase slippage Ω_0 given by Eq. (83) and amplitude

$$a = w \ \nu^2 \tag{92}$$

The solution given by Eq. (91) is correct only as long as the amplitude a of the oscillation is small compared to the width w of the waveguide, that is $\nu^2 << 1$, which gives the condition

$$\frac{\Gamma}{\gamma} << \frac{5}{2}\pi\frac{(\beta - \beta_w)^2}{q^2\beta_w^2(\beta\beta_w - 1)} \tag{93}$$

setting a limit on the value of the voltage amplitude V_0.

9. Energy Loss by Radiation

Consider an electron which is moving at relativistic velocity along the z-axis and at the same time is performing small amplitude oscillations on the $y = 0$ plane at the angular frequency Ω_0. It is well known that the electron will lose energy by radiating electromagnetic waves moving forward in the same direction of the motion of the particle, within an angular aperture of about $1/\gamma$. In the approximation that the oscillatory motion has been occurring for an infinitely long period of time, the spectrum of the radiation is made of only one line at the angular frequency

$$\omega_{\rm rad} = 2\gamma^2\Omega_0 \tag{94}$$

The radiated power at that frequency by one electron can also be calculated

$$P_0 = \frac{1}{3}\frac{e^2}{c^3}\gamma^4 a^2 \Omega_0^4 \tag{95}$$

where a is the amplitude of the oscillation which in our case may be given by Eq. (89) combined to Eq. (90) and Eqs. (84,88). In the case the electron has been performing only a limited number of oscillations, say n, then the frequency spectrum of the radiation is more complicated and similar to that usually observed in a wiggler or undulator, with more lines appearing beside the main frequency given by Eq. (94), spaced in wavelengths by about $1/n$. We shall assume that $n >> 1$ and still consider only the frequency line given by Eq. (94).

The spatial distribution of the electrons in a short beam bunch all performing the same oscillatory motion is also important. In the extreme case where the beam bunch is much longer than the wavelength of the radiation $2\pi c/\omega_{\rm rad}$, each electron will radiate independently from the others and the total power radiated is $P_{\rm rad} = NP_0$ where P_0 is the power from a single electron, given by Eq. (95), and N the total number of electrons in the bunch. On the other hand, when the bunch length ℓ in considerably smaller than the radiated wavelength, that is

$$\ell << 2\pi c/\omega_{\rm rad} \tag{96}$$

it is conceivable that all the electrons are radiating *coherently* and in this case the total power is

$$P_{\rm rad} = N^2 P_0 \tag{97}$$

At the same time, though, in order to take advantage of this effect, it is also important that the transverse dimensions of the electron beam are made as small as possible. Indeed, they should not exceed the amplitude of the oscillations given by Eq. (96) and should be smaller than the bunch length itself.

10. Applications

It is convenient to define two parameters that best summarize the interaction between the electron motion and the field in the waveguide. One is the *frequency transformer ratio* $r = \omega_{\rm rad}/\omega$, that is the ratio of the *radiated* frequency to the *input* frequency to the waveguide. From Eqs. (84 and 94) we derive

$$r = 2\gamma^2 \frac{\beta_w - \beta}{\beta_w} \tag{98}$$

Since the phase velocity $\beta_w > 1$, it is easily seen that the frequency transformer ratio can never be less than one; that is, the radiated frequency is always larger than the frequency in input to the waveguide.

The second parameter is the *power amplification factor* $\eta = P_{\rm rad}/P$, that is the ratio of the power radiated by the beam bunch to the *input* power to the waveguide. Assuming that the condition of short bunches expressed by Eq. (96) is satisfied, we derive from Eqs. (53, 88, 90, 92, 95 and 97)

$$\eta = \frac{256}{15}\pi \frac{r_0^2}{w^2} N^2 \gamma^2 \frac{(\beta\beta_w - 1)^2}{\beta_w} \tag{99}$$

which is valid of course in the case of small amplitude oscillations ($\nu < 1$), that is when the condition given by Eq. (93) is satisfied. As it could have been expected, the power amplification factor does not depend on the voltage amplitude V_0 in the waveguide.

We suggest two possible modes of application to take advantage of the interaction between the motion of the electrons and the field traveling in the waveguide. They are expressed by the following conditions.

A. Power Amplifier

$$r \sim 1 \qquad \text{and} \qquad \eta >> 1 \tag{100}$$

In this model the electrons are radiating power at a frequency very close to the one in input to the waveguide. It is known that there are no power amplifiers for frequencies larger than 3 GHz which can deliver power in significant amount, for instance few MWatt. The waveguide can be driven nevertheless by a low-input available power source at a given frequency ω and stimulate a short electron bunch to radiate to a frequency very close to ω. In this mode, then one requires a large power amplification, that is $\eta >> 1$. Ideally, one would require exactly $r = 1$. Inverting Eq. (98), this is obtained only if

$$\beta_w = \frac{2\beta}{1+\beta^2} \tag{101}$$

Since $\beta < 1$, this condition can be satisfied only allowing $\beta_w < 1$. For a ultra-relativistic electron, $\gamma >> 1$, actually one derives

$$\beta_w \approx 1 - \frac{1}{8\gamma^4} \tag{102}$$

The only way to obtain a wave traveling with a phase velocity slower than the speed of light is to modify the boundary conditions at the wall of the waveguide. This can be done by making use of a non-perfectly conducting material as it will be shown in a subsequent paper. We shall continue here with the model of perfectly conductive wall of a waveguide in the vacuum. For the operation of a *power amplifier* the input frequency is well above the cut-off value, that is $\beta_w \sim 1$, in which case as a good approximation

$$r \sim 1 + (\gamma q)^2 \tag{103}$$

We found some difficulties to find optimum beam and waveguide parameters for this mode of operation.

B. Frequency Transformer

$$r >> 1 \qquad \text{and} \qquad \eta \sim 1 \tag{104}$$

In this mode of operation the power radiated by a short electron bunch is at a frequency larger than that used in input to the waveguide. In this case it is sufficient that the power gain is close to unit. Inversion of Eq. (98) gives

$$\beta_w = \frac{2\gamma^2\beta}{2\gamma^2 - r} \tag{105}$$

Very likely $r << 2\gamma^2$ and with a good approximation

$$\beta_w \approx 1 + \frac{r-1}{2\gamma^2} \tag{106}$$

Thus again it seems that one should drive the waveguide at frequencies well above the cut-off. Nevertheless another interesting situation corresponds to values of r very close to but not exceeding $2\gamma^2$; this makes the denominator at the right hand side of Eq. (105) positive and very small. The phase velocity now is $\beta_w >> 1$, which requires driving the waveguide at a frequency dose to the cut-off value. An example of this case is shown in Table 1.

11. Conclusions

A practical demonstration of the method exposed in this paper would be very valuable. It would be sufficient to start with a demonstration of the principles involved, for instance with lower initial power amplification factor and more modest electron beam intensity. There are nevertheless still a number of technical and design issues that ought to be solved. For instance

(i) How to drive efficiently the waveguide with the 1.3 GHz power source?

(ii) How to excite the wanted traveling mode and exclude all the others?

(iii) How close to the cut-off can one drive the waveguide, without loss of stability in the rf power source?

(iv) The electron beam energy is low and the required intensity large. How does one cope with space charge effects? How small can one make the bunch length and the transverse emittance?

(v) Experimental evidence of *coherent* radiation is crucial in our model.

(vi) The electron energy may be too low, so that one should re-evaluate the radiation mechanism for non ultra-relativistic particles.

(vii) To produce a clean narrow-band spectrum of the radiation, it is important that electrons travel a distance long enough to include several transverse oscillations. If the waveguide is 2.6m long, about ten complete oscillations would be executed; what are the consequences on the radiation spectrum? In the meantime each particle would have lost about 0.1 MeV; what are the consequences from the energy loss?

This list of unsolved issues can be made very long indeed; but we are just at the beginning of our research. We plan to continue by examining closely the interaction of a short electron bunch with an electromagnetic wave of the most general (TM or TE) propagating mode, traveling in a waveguide. We would like to consider also more realistic, less conductive wall material which could have the effect of slowing down the phase velocity. We plan also to determine numerically the general equations of motion outlined in Sections 5 and 6. For instance we would like to find ways to take advantage of the longitudinal electric field to restore the energy lost by the electrons to the radiation. As more results will be available, other possibilities will surface for improved optimization of the design. We are just at the beginning of our program...

Table 1: An Example of Frequency Transformer

Kinetic Energy of Electrons	4 MeV
Number of Electrons	4×10^{11}
Bunch Length	1 mm
Input Frequency	1.3 GHz
Radiated Frequency	190 GHz
Frequency Transform Ratio	146
Power Amplification Factor	1.06
Phase Velocity, β_w	15.5
Cutoff Frequency	1.2973 GHz
Waveguide Dimension, w	25.8 cm
Period of Oscillations	24.5 cm

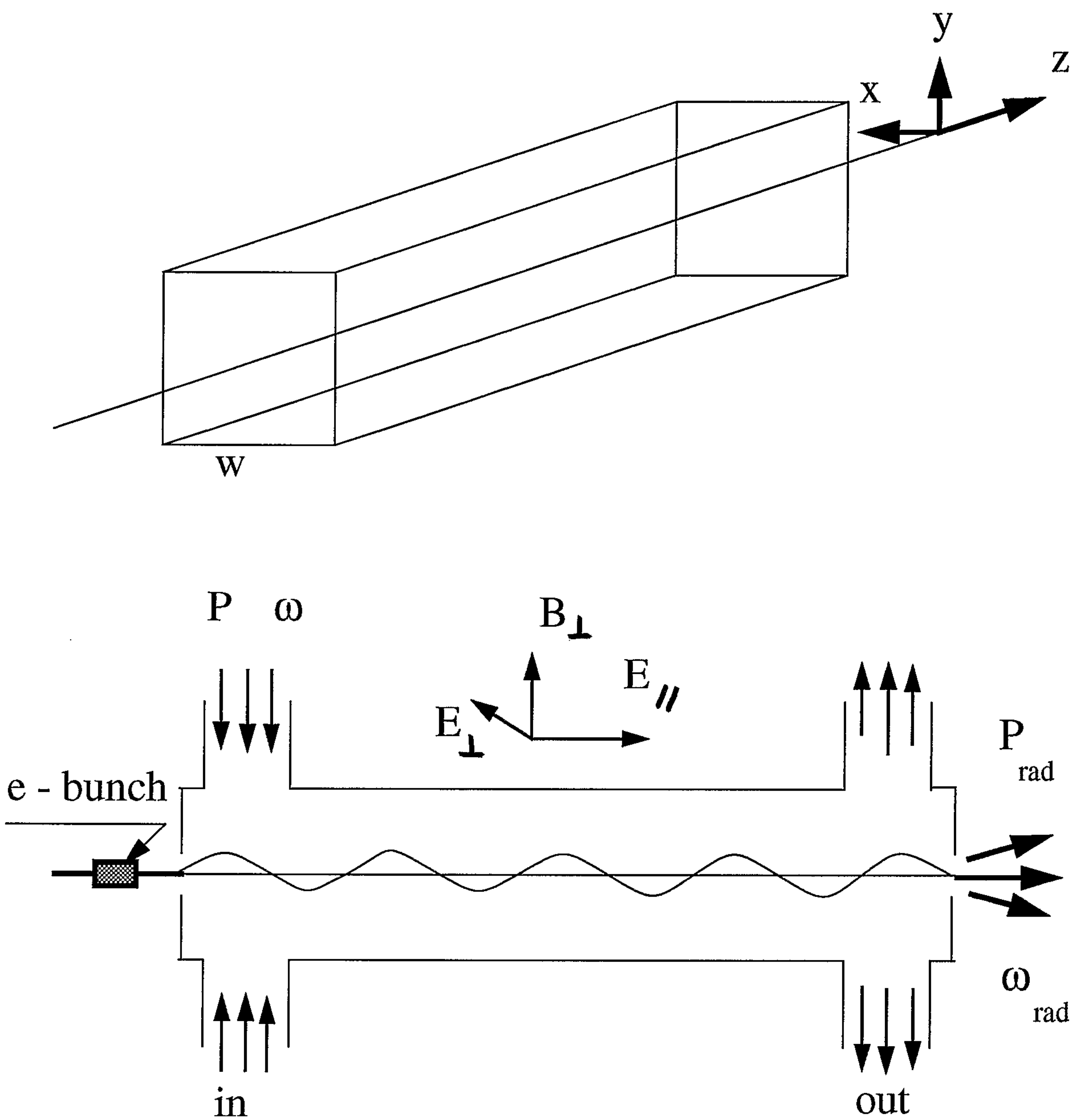

Fig. 1 - The rf Wiggler Concept

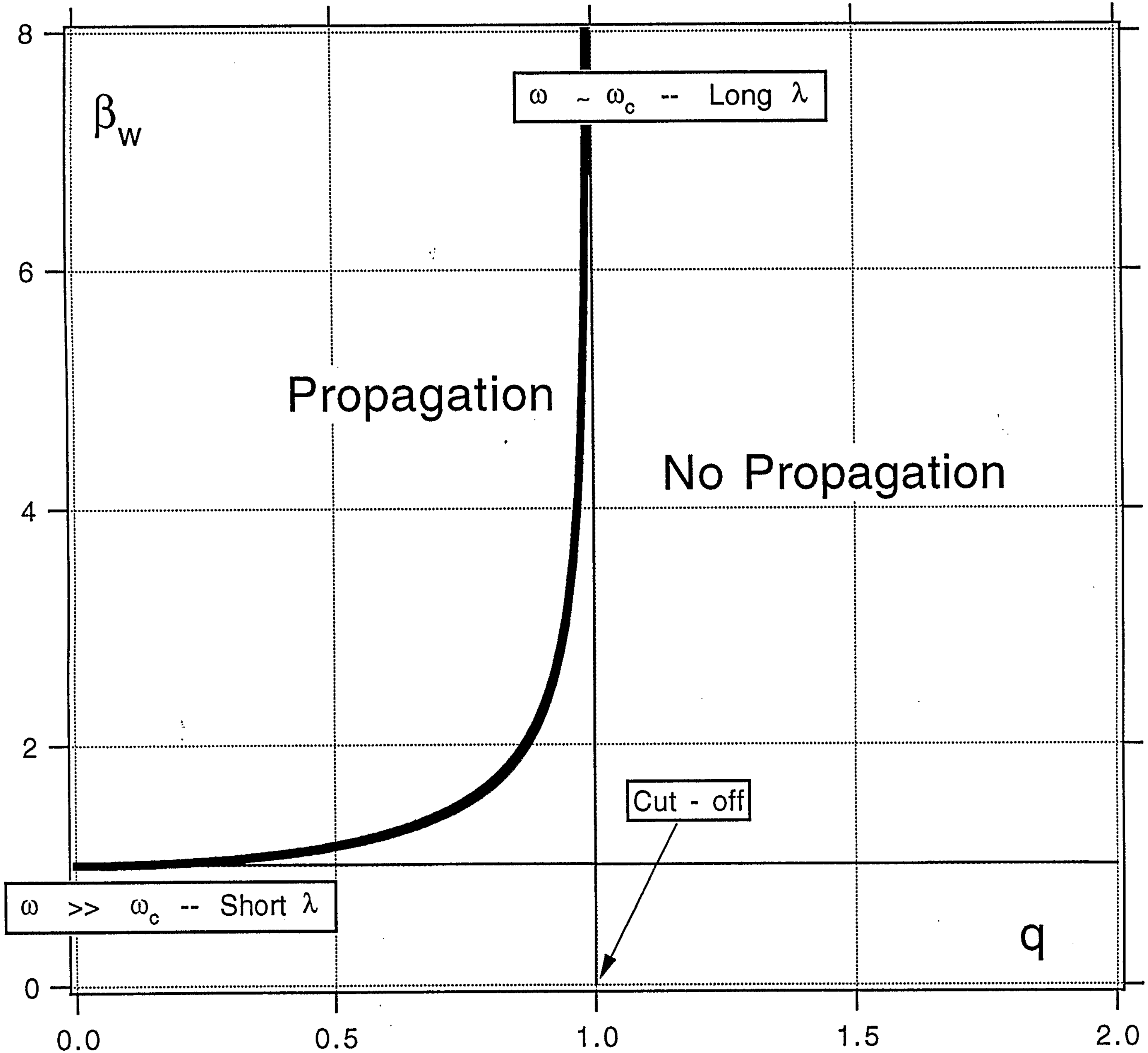

Fig. 2 Phase Velocity vs. Form Factor