

Electric Fields of a Uniformly Charged Elliptical Beam

G. Parzen

August 2001

Collider Accelerator Department
Brookhaven National Laboratory

U.S. Department of Energy

USDOE Office of Science (SC)

Notice: This technical note has been authored by employees of Brookhaven Science Associates, LLC under Contract No.DE-AC02-98CH10886 with the U.S. Department of Energy. The publisher by accepting the technical note for publication acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this technical note, or allow others to do so, for United States Government purposes.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or any third party's use or the results of such use of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof or its contractors or subcontractors. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.



Electric fields of a uniformly charged elliptical beam

BNL/SNS TECHNICAL NOTE

NO. 098

George Parzen

August 3, 2001

COLLIDER-ACCELERATOR DEPARTMENT
BROOKHAVEN NATIONAL LABORATORY
UPTON, NEW YORK 11973

Electric fields of a uniformly charged elliptical beam.

George Parzen

August 3, 2001

Introduction

This paper presents results for the electric field due to a uniformly charged elliptical beam outside the beam. Results for the field inside the beam are well known [1, 2] The beam being considered extends indefinitely in the z direction and has an elliptical boundary in x and y given by

$$x^2/a^2 + y^2/b^2 = 1 \quad (1)$$

The charge density, $\rho(x, y, z)$ is uniform within the elliptical boundary, zero outside the elliptical boundary, and does not depend on z . The results given below depend on the observation made by B. Houssais [3], that the result for the electric field of a gaussian charge distribution given by W. Kellog [1] as a one dimensional integral would hold for any elliptical charge distribution as defined below. This may be stated as follows. Let the charge distribution be given as

$$\rho(x, y, z) = \lambda n(x, y) \quad (2)$$

where λ is the charge per unit length and

$$\int dx dy n(x, y) = 1 \quad (3)$$

A charge distribution will be called elliptical if $n(x, y)$ can be written as

$$\begin{aligned} n(x, y) &= \hat{n}(T)/\pi ab \\ T &= x^2/a^2 + y^2/b^2 \end{aligned} \quad (4)$$

For the uniform elliptical beam , $\hat{n}(T)$ is given by

$$\begin{aligned}\hat{n}(T) &= 1, \quad T \leq 1 \\ \hat{n}(T) &= 0, \quad T > 1\end{aligned}\tag{5}$$

For a Gaussian beam, $\hat{n}(T)$ is given by

$$\hat{n}(T) = \exp(-T)\tag{6}$$

One can show, using Eq. 3, that $\hat{n}(T)$ obeys the equation

$$\int_0^\infty dT \hat{n}(T) = 1\tag{7}$$

The generalization of the Kellog result for any elliptical beam is then

$$\begin{aligned}E_x &= 2\lambda \int_0^\infty dt \frac{\hat{n}(\hat{T})}{(a^2 + t)^{3/2}(b^2 + t)^{1/2}} \\ \hat{T} &= x^2/(a^2 + t) + y^2/(b^2 + t)\end{aligned}\tag{8}$$

A similar result, with a, b and x, y interchanged will give E_y

Electric fields for x,y inside the beam

As a first step, the fields inside a uniformly charged elliptical beam will be found using Eq. 8 .In this case, \hat{T} is always ≤ 1 since for $t = 0$, $\hat{T} = x^2/a^2 + y^2/b^2$, which is ≤ 1 for x,y inside the beam, and decreases further for larger t. Eq. 8 then becomes

$$\begin{aligned}E_x &= 2\lambda x \int_0^\infty dt \frac{1}{(a^2 + t)^{3/2}(b^2 + t)^{1/2}} \\ \hat{T} &= x^2/(a^2 + t) + y^2/(b^2 + t)\end{aligned}\tag{9}$$

The integral in Eq. 9 can be done using the result

$$\int_{t_1}^\infty dt \frac{1}{(a^2 + t)^{3/2}(b^2 + t)^{1/2}} = 2 \frac{1}{(a^2 + t_1)^{1/2}} \frac{1}{(a^2 + t_1)^{1/2} + (b^2 + t_1)^{1/2}}\tag{10}$$

This gives

$$E_x = 4\lambda x \frac{1}{a(a+b)}\tag{11}$$

and a similar result for E_y with a and b interchanged and x replaced by y.

Electric fields outside the beam when $y = 0$

As the next step, the fields outside a uniformly charged elliptical beam will be found using Eq. 8 for the case when $y = 0$. The results in this case are simpler and the mathematics is easier to comprehend. In this case, \hat{T} is > 1 for $t = 0$ since for $t = 0$, $\hat{T} = x^2/a^2 + y^2/b^2$, which is > 1 for x, y outside the beam. For larger t , \hat{T} decreases and reaches the value of 1 at $t = t_1$, and at still larger t , \hat{T} decreases further always remaining smaller than 1. The integral in Eq. 8 then goes from $t = t_1$, to $t = \infty$. Eq. 8 then becomes

$$\begin{aligned} E_x &= 2\lambda x \int_{t_1}^{\infty} dt \frac{1}{(a^2 + t)^{3/2}(b^2 + t)^{1/2}} \\ \hat{T} &= x^2/(a^2 + t) \end{aligned} \quad (12)$$

$$\begin{aligned} t_1 &= x^2 - a^2 \\ y &= 0 \end{aligned} \quad (13)$$

Using Eq. 10. one finds

$$\begin{aligned} E_x &= 4\lambda \frac{1}{x + (x^2 + b^2 - a^2)^{1/2}} \\ E_y &= 0 \\ y &= 0 \end{aligned} \quad (14)$$

$E_{xx} = \partial E_x / \partial x$ is given by

$$E_{xx} = -\frac{E_x}{(x^2 + b^2 - a^2)^{1/2}} \quad (15)$$

Electric fields outside the beam when $y \neq 0$

As the final step, the fields outside a uniformly charged elliptical beam will be found using Eq. 8 for the general case. In this case, \hat{T} is > 1 for $t = 0$ since for $t = 0$, $\hat{T} = x^2/a^2 + y^2/b^2$, which is > 1 for x, y outside the beam. For larger t , \hat{T} decreases and reaches the value of 1 at $t = t_1$, and at still larger t , \hat{T} decreases further always remaining smaller than 1. The integral in Eq. 8 then goes from $t = t_1$, to $t = \infty$. Eq. 8 then becomes

$$\begin{aligned} E_x &= 2\lambda x \int_{t_1}^{\infty} dt \frac{1}{(a^2 + t)^{3/2}(b^2 + t)^{1/2}} \\ x^2/(a^2 + t_1) + y^2/(b^2 + t_1) &= 1 \end{aligned} \quad (16)$$

t_1 is the positive root of the equation

$$x^2/(a^2 + t_1) + y^2/(b^2 + t_1) = 1 \quad (17)$$

The quadratic equation for t_1 , Eq. 17, can be solved to give

$$\begin{aligned} t_1 &= (B^2/4 + C)^{1/2} + B/2 \\ B &= x^2 + y^2 - a^2 - b^2 \\ C &= x^2b^2 + y^2a^2 - a^2b^2 \end{aligned} \quad (18)$$

Eq. 16 gives the result for E_x

$$E_x = 4\lambda x \frac{1}{(a^2 + t_1)^{1/2}} \frac{1}{(a^2 + t_1)^{1/2} + (b^2 + t_1)^{1/2}} \quad (19)$$

and a similar result for E_y with a and b interchanged and x replaced by y .

It may be usefull to also have results for the derivatives of the fields, $E_{xx}, E_{yy}, E_{xy} = E_{yx}$, where $E_{xx} = \partial E_x / \partial x$, $E_{yy} = \partial E_y / \partial y$ and $E_{xy} = \partial E_x / \partial y$. E_{xx} is found using Eq. 16 for E_x

$$E_{xx} = \frac{E_x}{x} - 2\lambda x \frac{1}{(a^2 + t_1)^{3/2} (b^2 + t_1)^{1/2}} \frac{dt_1}{dx} \quad (20)$$

dt_1/dx can be found from Eq. 17 for t_1 as

$$\frac{dt_1}{dx} = 2x \frac{(a^2 + t_1)(b^2 + t_1)^2}{x^2(b^2 + t_1)^2 + y^2(a^2 + t_1)^2} \quad (21)$$

This gives for E_{xx}

$$E_{xx} = \frac{E_x}{x} - 4\lambda x^2 \frac{(a^2 + t_1)^{-1/2} (b^2 + t_1)^{3/2}}{x^2(b^2 + t_1)^2 + y^2(a^2 + t_1)^2} \quad (22)$$

E_{yy} and dt_1/dy can be found by interchanging x and y, and a and b. E_{xy} can be found in the same way as

$$E_{xy} = -4\lambda xy \frac{(a^2 + t_1)^{1/2} (b^2 + t_1)^{1/2}}{x^2(b^2 + t_1)^2 + y^2(a^2 + t_1)^2} \quad (23)$$

References

- [1] W.Kellog, Foundations of Potential Theory, (Dover Publications,New York,1953), p. 192.
- [2] L. Teng, Report ANLAD-59 (1963)
- [3] F.J. Sacherer, PAC71, p.1105, (1971)