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Electric Fields of a Uniformly Charged Elliptical Beam

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Introduction

This paper presents results for the electric field due to a uniformly charged elliptical beam outside the beam. Results for the field inside inside the beam are well known [1, 2] The beam being considered extends indefinitly in the z direction and has an elliptical boundary in x and y given by

$$x^2/a^2 + y^2/b^2 = 1 \tag{1}$$

The charge density, $\rho(x, y, z)$ is uniform within the elliptical boundary, zero outside the elliptical boundary, and does not depend on z. The results given below depend on the observation made by B. Houssais [3], that the result for the electric field of a gaussian charge distribution given by W. Kellog [1] as a one dimensional integral would hold for any elliptical charge distribution as defined below. This may be stated as follows. Let the charge distribution be given as

$$\rho(x, y, z) = \lambda n(x, y) \tag{2}$$

where λ is the charge per unit length and

$$\int dxdy \ n(x,y) = 1 \tag{3}$$

A charge distribution will be called elliptical if n(x, y) can be written as

$$n(x,y) = \hat{n}(T)/\pi ab$$

$$T = x^{2}/a^{2} + y^{2}/b^{2}$$
(4)

For the uniform elliptical beam , $\hat{n}(T)$ is given by

$$\hat{n}(T) = 1, \ T \le 1$$

 $\hat{n}(T) = 0, \ T > 1$
(5)

For a Gaussian beam, $\hat{n}(T)$ is given by

$$\hat{n}(T) = exp(-T) \tag{6}$$

One can show, using Eq. 3, that $\hat{n}(T)$ obeys the equation

$$\int_0^\infty dT \ \hat{n}(T) = 1 \tag{7}$$

The generalization of the Kellog result for any elliptical beam is then

$$E_x = 2\lambda \int_0^\infty dt \frac{\hat{n}(\hat{T})}{(a^2 + t)^{3/2} (b^2 + t)^{1/2}}$$
$$\hat{T} = x^2/(a^2 + t) + y^2/(b^2 + t)$$
(8)

A similar result, with a, b and x, y interchanged will give E_y

Electric fields for x,y inside the beam

As a first step, the fields inside a uniformly charged elliptical beam will be found using Eq. 8. In this case, \hat{T} is always ≤ 1 since for t = 0, $\hat{T} = x^2/a^2 + y^2/b^2$, which is ≤ 1 for x,y inside the beam, and decreases further for larger t. Eq. 8 then becomes

$$E_x = 2\lambda x \int_0^\infty dt \frac{1}{(a^2 + t)^{3/2} (b^2 + t)^{1/2}}$$
$$\hat{T} = x^2/(a^2 + t) + y^2/(b^2 + t)$$
(9)

The integral in Eq. 9 can be done using the result

$$\int_{t_1}^{\infty} dt \frac{1}{(a^2+t)^{3/2}(b^2+t)^{1/2}} = 2\frac{1}{(a^2+t_1)^{1/2}} \frac{1}{(a^2+t_1)^{1/2}+(b^2+t_1)^{1/2}} (10)$$

This gives

$$E_x = 4\lambda x \frac{1}{a(a+b)} \tag{11}$$

and a similar result for E_y with a and b interchanged and x replaced by y.

Electric fields outside the beam when y = 0

As the next step, the fields outside a uniformly charged elliptical beam will be found using Eq. 8 for the case when y = 0. The results in this case are simpler and the mathematics is easier to comprehend. In this case, \hat{T} is > 1 for t = 0 since for t = 0, $\hat{T} = x^2/a^2 + y^2/b^2$, which is > 1 for x,y outside the beam. For larger t, \hat{T} decreases and reaches the vaue of 1 at $t = t_1$, and at still larger t, \hat{T} decreases further always remaining smaller than 1. The integral in Eq. 8 then goes from $t = t_1$, to $t = \infty$. Eq. 8 then becomes

$$E_x = 2\lambda x \int_{t_1}^{\infty} dt \frac{1}{(a^2 + t)^{3/2} (b^2 + t)^{1/2}}$$
$$\hat{T} = x^2 / (a^2 + t)$$
$$t_1 = x^2 - a^2$$
(12)

$$y = 0 \tag{13}$$

Using Eq. 10. one finds

$$E_x = 4\lambda \frac{1}{x + (x^2 + b^2 - a^2)^{1/2}}$$

$$E_y = 0$$

$$y = 0$$
(14)

 $E_{xx} = \partial E_x / \partial x$ is given by

$$E_{xx} = -\frac{E_x}{(x^2 + b^2 - a^2)^{1/2}}$$
(15)

Electric fields outside the beam when $y \neq 0$

As the final step, the fields outside a uniformly charged elliptical beam will be found using Eq. 8 for the general case. In this case, \hat{T} is > 1 for t = 0since for t = 0, $\hat{T} = x^2/a^2 + y^2/b^2$, which is > 1 for x,y outside the beam. For larger t, \hat{T} decreases and reaches the value of 1 at $t = t_1$, and at still larger t, \hat{T} decreases further always remaining smaller than 1. The integral in Eq. 8 then goes from $t = t_1$, to $t = \infty$. Eq. 8 then becomes

$$E_x = 2\lambda x \int_{t_1}^{\infty} dt \frac{1}{(a^2 + t)^{3/2} (b^2 + t)^{1/2}}$$
$$x^2/(a^2 + t_1) + y^2/(b^2 + t_1) = 1$$
(16)

 t_1 is the positive root of the equation

$$x^{2}/(a^{2}+t_{1})+y^{2}/(b^{2}+t_{1}) = 1$$
(17)

The quadratic equation for t_1 , Eq. 17, can be solved to give

$$t_{1} = (B^{2}/4 + C)^{1/2} + B/2$$

$$B = x^{2} + y^{2} - a^{2} - b^{2}$$

$$C = x^{2}b^{2} + y^{2}a^{2} - a^{2}b^{2}$$
(18)

Eq. 16 gives the result for E_x

$$E_x = 4\lambda x \frac{1}{(a^2 + t_1)^{1/2}} \frac{1}{(a^2 + t_1)^{1/2} + (b^2 + t_1)^{1/2}}$$
(19)

and a similar result for E_y with a and b interchanged and x replaced by y.

It may be usefull to also have results for the derivatives of the fields, $E_{xx}, E_{yy}, E_{xy} = E_{yx}$, where $E_{xx} = \partial E_x / \partial x$, $E_{yy} = \partial E_y / \partial y$ and $E_{xy} = \partial E_x / \partial y$. E_{xx} is found using Eq. 16 for E_x

$$E_{xx} = \frac{E_x}{x} - 2\lambda x \frac{1}{(a^2 + t_1)^{3/2} (b^2 + t_1)^{1/2}} \frac{dt_1}{dx}$$
(20)

 dt_1/dx can be found from Eq. 17 for t_1 as

$$\frac{dt_1}{dx} = 2x \frac{(a^2 + t_1)(b^2 + t_1)^2}{x^2(b^2 + t_1)^2 + y^2(a^2 + t_1)^2}$$
(21)

This gives for E_{xx}

$$E_{xx} = \frac{E_x}{x} - 4\lambda x^2 \frac{(a^2 + t_1)^{-1/2} (b^2 + t_1)^{3/2}}{x^2 (b^2 + t_1)^2 + y^2 (a^2 + t_1)^2}$$
(22)

 E_{yy} and dt_1/dy can be found by interchanging x and y, and a and b. E_{xy} can be found in the same way as

$$E_{xy} = -4\lambda xy \frac{(a^2 + t_1)^{1/2} (b^2 + t_1)^{1/2}}{x^2 (b^2 + t_1)^2 + y^2 (a^2 + t_1)^2}$$
(23)

References

- W.Kellog, Foundations of Potential Theory, (Dover Publications, New York, 1953), p. 192.
- [2] L. Teng, Report ANLAD-59 (1963)
- [3] F.J. Sacherer, PAC71, p.1105, (1971)