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# Revised Spin Motion Equations Spin Motion and Resonances in Accelerators and Storage Rings

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# Revised Spin Motion Equations

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## Abstract

In the spin dynamics of accelerators and storage rings, the components of spin parallel and perpendicular to the particle orbit behave differently. The conventional treatment employs a coordinate system derived from a reference orbit. However, it has recently been pointed out that in this treatment the distinction between longitudinal and transverse components is not treated accurately. We present revised equations for the strengths of depolarizing resonances and for calculating the spin rotations effected by "Siberian snakes" and other spin rotators, showing that the strengths of resonances usually contain a factor G$\gamma$ rather than 1+ G$\gamma$.

# 1. Fundamentals

The Froissart-Stora [1] formulation of the Thomas-BMT equation [2] may be written

$$\frac{d\vec{S}}{dt} = \frac{q}{m\gamma}\vec{S}\times[\vec{B} + G(\gamma\vec{B}_{\perp} + \vec{B}_{\parallel})] \tag{1.1}$$

where $\vec{B}_{\parallel} = (\hat{v}\cdot\vec{B})\hat{v}$ and $\vec{B}_{\perp} = \vec{B} - \vec{B}_{\parallel} = (\hat{v}\times\vec{B})\times\hat{v}$ are the longitudinal and transverse parts of $\vec{B}$, $\hat{v}$ being the unit vector in the direction of the particle velocity. Similarly the longitudinal and transverse parts of the spin are $\vec{S}_{\parallel} = (\hat{v}\cdot\vec{S})\hat{v}$ and $\vec{S}_{\perp} = \vec{S} - \vec{S}_{\parallel} = (\hat{v}\times\vec{S})\times\hat{v}$.

The Lorentz force equation is

$$\frac{d\hat{v}}{dt} = \frac{q}{m\gamma}\hat{v}\times\vec{B} \tag{1.2}$$

Combining (1.1) and (1.2) we obtain

$$\begin{aligned}\frac{d\vec{S}_{\parallel}}{dt} &= \left(\frac{d\vec{S}}{dt}\cdot\hat{v} + \vec{S}\cdot\frac{d\hat{v}}{dt}\right)\hat{v} + (\vec{S}\cdot\hat{v})\frac{d\hat{v}}{dt} \\ &= \frac{q}{m\gamma}[\vec{S}_{\parallel}\times\vec{B}_{\perp} + G\gamma(\vec{S}\times\vec{B}_{\perp}\cdot\hat{v})\hat{v}]\end{aligned} \tag{1.3}$$

and

$$\frac{d\vec{S}_{\perp}}{dt} = \frac{q}{m\gamma}[\vec{S}_{\perp}\times\vec{B} + G(\gamma\vec{B}_{\perp} + \vec{B}_{\parallel}) - G\gamma(\vec{S}\times\vec{B}_{\perp}\cdot\hat{v})\hat{v}] \tag{1.4}$$

The Froissart-Stora equation (1) is independent of the coordinate system. But, since the particle moves in the vicinity of a closed orbit, it is convenient to use a coordinate system based on a closed *reference orbit* as we consider particles whose motion takes place near (though not exactly on) that orbit. We assume the reference orbit is plane and has a circumference we denote by $2\pi R$. We transform to a coordinate system (a Frenet-Serret system) based on this reference orbit. The position of a particle is characterized by the vector $\vec{\xi}$ from the point on the reference orbit closest to the particle, and we define the coordinates to be:

$s$ = the distance along the reference orbit from an origin point (arbitrarily chosen) on the reference orbit to the point on the reference orbit closest to the particle.

$z$ = the vertical component of $\vec{\xi}$, i.e. the distance from the plane of the reference orbit to the particle.

$x$ = the horizontal component of $\vec{\xi}$, which is the length of the projection of $\vec{\xi}$ on the orbit plane.

We also define $\rho(s)$ to be the radius of curvature of the reference orbit at $s$.; in a straight section the curvature $1/\rho(s)$ is zero, and the coordinates are locally Cartesian.

It is convenient to change to $s$ instead of the time $t$ as the independent variable, with

$$ds = \frac{v}{1 + x/\rho} dt \tag{1.5}$$

(note that $s$ is the distance along the reference orbit, not exactly the distance traversed by the particle).

In what follows we shall use the prime for differentiation by $s$; i.e.

$$X' \equiv \frac{dX}{ds}$$

for any variable $X$.

We define a frame with the unit vectors in the $x$, $s$, $z$ directions as basis vectors. These basis vectors form a right-handed system. They rotate with s:

$$\frac{d\hat{e}_1}{ds} = \frac{\hat{e}_2}{\rho}; \quad \frac{d\hat{e}_2}{ds} = -\frac{\hat{e}_1}{\rho}; \quad \frac{d\hat{e}_3}{ds} = 0 \tag{1.6}$$

# 2. Trajectory-based Frame

The particles do not necessarily travel on the reference orbit, therefore the spin components in the directions $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are not exactly in the directions transverse and longitudinal to the motion of a particle. Since the dynamical equation (1.1) shows that the transverse and longitudinal spin components behave differently, it is desirable to formulate equations of motion that maintain this distinction.

Following Kondratenko and Sivers [3] we introduce a "natural" or "local" reference frame based on the actual trajectory of the particle. The basis vector $\hat{u}_2$ is taken to be exactly the unit vector $\hat{v} = (\hat{e}_2 + x'\hat{e}_1 + z'\hat{e}_3)/\sqrt{1+x'^2+z'^2}$ in the direction of the instantaneous particle velocity, and the other two are in the local radial and vertical direction orthogonal to $\hat{v}$:

$$\begin{aligned} \hat{u}_1 &= [\hat{v}\times\hat{e}_3]_N \approx \hat{e}_1 - x'\hat{e}_2 \\ \hat{u}_2 &= \hat{v} \approx \hat{e}_2 + x'\hat{e}_1 + z'\hat{e}_3 \\ \hat{u}_3 &= \hat{u}_1\times\hat{u}_2 \approx \hat{e}_3 - z'\hat{e}_2 \end{aligned} \tag{2.1}$$

where the subscript *N* denotes normalization to unit length. The $\approx$ relations are correct to first order in the excursions *x* and *z* from the reference orbit.

The new basis vectors, of course, also rotate; using (1.6) and (2.1) we obtain, to first order in *x* and *z*,

$$\begin{aligned} \hat{u}_1' &= \frac{\hat{e}_2}{\rho} - x''\hat{e}_2 + x'\frac{\hat{e}_1}{\rho} = \left(\frac{1}{\rho} - x''\right)\hat{u}_2 - \frac{z'}{\rho}\hat{u}_3 \\ \hat{u}_2' &= -\frac{\hat{e}_1}{\rho} + x'\frac{\hat{e}_2}{\rho} + x''\hat{e}_1 + z''\hat{e}_3 = -\left(\frac{1}{\rho} - x''\right)\hat{u}_1 + z''\hat{u}_3 \\ \hat{u}_3' &= \frac{z'}{\rho}\hat{e}_1 - z''\hat{e}_2 = \frac{z'}{\rho}\hat{u}_1 - z''\hat{u}_2 \end{aligned} \tag{2.2}$$

Here the independent variable *s* and the excursions *x* and *z* are still defined with respect to the reference orbit, while the basis vectors are derived from the actual trajectory.

With *s* as the independent variable the F-S equation is

$$\vec{S}' \equiv \frac{d\vec{S}}{ds} = \vec{S}\times\vec{F}; \quad \vec{F} = \frac{1}{B\rho}[\vec{B} + G(\gamma\vec{B}_\perp + \vec{B}_\parallel)] \tag{2.3}$$

where $B\rho = \dfrac{m\gamma v}{q}$ is the magnetic rigidity of the particle,

so that

$$\begin{pmatrix} S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} \vec{S}\times\vec{F}\cdot\hat{u}_1 + \vec{S}\cdot\hat{u}_1' \\ \vec{S}\times\vec{F}\cdot\hat{u}_2 + \vec{S}\cdot\hat{u}_2' \\ \vec{S}\times\vec{F}\cdot\hat{u}_3 + \vec{S}\cdot\hat{u}_3' \end{pmatrix} = \vec{S}\times\vec{W} \tag{2.4}$$

with

$$\vec{W} = \vec{F} + z''\hat{u}_1 + \frac{z'}{\rho}\hat{u}_2 + \left(\frac{1}{\rho} - x''\right)\hat{u}_3 \tag{2.5}$$

It is convenient to express this in terms of the excursions of the particle. Courant and Ruth [4] and Lee [5] express $\vec{B}_\perp$, $\vec{B}_\parallel$ and $\vec{F}$ in terms of the particle excursions, governed by the Lorentz force equation (1.2). In terms of the fixed vectors $\hat{e}$

$$\begin{aligned} \vec{B}_\perp &= B\rho\left[\left(x'' - \frac{1}{\rho}\right)\hat{e}_3 + \frac{z'}{\rho}\hat{e}_2 - z''\hat{e}_1\right] \\ \vec{B}_\parallel &= \left[B_{sol} - B\rho\left(\frac{z}{\rho}\right)'\right]\hat{e}_2 \end{aligned} \tag{2.6}$$

where $B_{sol}$ is the solenoidal field on the reference orbit, which was not included in [4] and [5].

In the trajectory-based coordinate system (2.2) this becomes

$$\begin{aligned} \vec{B}_\perp &= B\rho\left[\left(x'' - \frac{1}{\rho}\right)\hat{u}_3 - z''\hat{u}_1\right] \\ \vec{B}_\parallel &= \left[B_{sol} - B\rho\left(\frac{z}{\rho}\right)'\right]\hat{u}_2 \end{aligned} \tag{2.7}$$

Note that $\vec{B}_\perp$ has no component in the direction $\hat{u}_2$.

We thus have (to first order in the displacements *x* and *z*)

$$\begin{aligned} \vec{F} &= -(1+G\gamma)z''\hat{u}_1 + (1+G)\left[\frac{B_{sol}}{B\rho} - \left(\frac{z}{\rho}\right)'\right]\hat{u}_2 - (1+G\gamma)\left(\frac{1}{\rho} - x''\right)\hat{u}_3 \\ \vec{W} &= -G\gamma z''\hat{u}_1 + \left\{(1+G)\left[\frac{B_{sol}}{B\rho} - z\left(\frac{1}{\rho}\right)'\right] - G\frac{z'}{\rho}\right\}\hat{u}_2 - G\gamma\left(\frac{1}{\rho} - x''\right)\hat{u}_3 \end{aligned} \tag{2.8}$$

and

$$
\begin{aligned}
S_1' &= -S_2 G\gamma\left(\frac{1}{\rho} - x''\right) - S_3\left\{(1+G)\left[\frac{B_{sol}}{B\rho} - z\left(\frac{1}{\rho}\right)'\right] - G\frac{z'}{\rho}\right\} \\
S_2' &= G\gamma\left[-S_3 z'' + S_1\left(\frac{1}{\rho} - x''\right)\right] \qquad (2.9) \\
S_3' &= \left\{(1+G)\left[\frac{B_{sol}}{B\rho} - z\left(\frac{1}{\rho}\right)'\right] - G\frac{z'}{\rho}\right\} + S_2 G\gamma z''
\end{aligned}
$$

Note that if $G$=0, i.e. if there is no anomalous magnetic moment, the longitudinal spin component $S_2$ is constant: helicity is conserved.

The dominant terms in the equations for $S_1'$ and $S_2'$ are $\mp G\gamma / \rho$ , leading to the precession frequency (spin tune) $G\gamma$.

The dominant depolarizing term (contribution to $S_3'$) is the last term in the equation for $S_3'$ and is proportional to $G\gamma$, not to (1+ $G\gamma$), in agreement with Kondratenko, Sivers and others [3]. In the calculation of depolarization due to transverse field perturbations (including magnet errors and rf excitation dipoles) and/or vertical betatron oscillations, appearing in much of the literature on spin dynamics including [4], [5] and [6] the basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are used; the relations corresponding to (2.9) in S Y Lee's book [5], are his equations (2.44), rewritten in our notation as

$$
\begin{aligned}
S_x' &= -S_s\left[\frac{G\gamma}{\rho} - (1+G\gamma)x''\right] - S_z\left[(\gamma-1)G\frac{z'}{\rho} - (1+G)z\left(\frac{1}{\rho}\right)'\right] \\
S_s' &= -S_z z''(1+G\gamma) + S_x\left[\frac{G\gamma}{\rho} - (1+G\gamma)x''\right] \qquad (2.10) \\
S_z' &= S_x\left[(\gamma-1)G\frac{z'}{\rho} - (1+G)z\left(\frac{1}{\rho}\right)'\right] + S_s z''(1+G\gamma)
\end{aligned}
$$

Both formulations (2.9) and (2.10) are correct. But the components $S_1$, $S_2$, $S_3$ addressed here in (2.9) have a direct physical significance, $S_2$ being the helicity (spin component along the velocity direction) which is strictly longitudinal, while $S_1$ and $S_3$ are strictly transverse components. The components $S_x$, $S_s$, $S_z$ in (2.10), along the axes of the coordinate system defined by the reference orbit, all contain a mixture of the longitudinal and the transverse, and therefore have much less physical significance. *Therefore (2.9) and not (2.10) is the relation that should be used in calculations of polarization, including resonance strength and strengths of (full or partial) Siberian snakes.* Fortunately this makes very little practical difference because we almost always deal with large values of $G\gamma$. But in the case of deuterons $G$ is small and

negative, and indeed analysis of some recent experimental COSY data, by Leonova and others, also point to the factor $G\gamma$ and not 1+ $G\gamma$.

# 3. Depolarization Resonances

We recall how resonance strengths are calculated in the literature [4], [5], [6]:

Equation (2.4), in the form

$$\begin{pmatrix} S_1' \\ S_2' \\ S_3' \end{pmatrix} = \vec{S} \times \vec{W} \tag{3.1}$$

is equivalent to the spinor equation

$$\psi' = \frac{i}{2}(\vec{\sigma} \cdot \vec{W})\psi = \frac{i}{2}\begin{pmatrix} W_3 & W_1 - iW_2 \\ W_1 + iW_2 & -W_3 \end{pmatrix}\psi \tag{3.2}$$

where the relation between the 2-component spinor $\psi$ and the spin $\vec{S}$ is

$$\vec{S} = \psi^\dagger \vec{\sigma} \psi,$$

$$\text{i.e.} \tag{3.3}$$

$$S_1 = \psi^\dagger \sigma_1 \psi,\ S_2 = \psi^\dagger \sigma_2 \psi,\ S_3 = \psi^\dagger \sigma_3 \psi$$

Here $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and $W$ is given by (2.8). We simplify $W_3$ by averaging it over a revolution and noting that x" averages to zero: $W_3 \approx G\gamma / R$ . Then (3.2) becomes

$$\psi' = -\frac{i}{2}\begin{pmatrix} \frac{G\gamma}{R} & -\zeta \\ -\zeta^* & -\frac{G\gamma}{R} \end{pmatrix}\psi$$

$$\text{with } \zeta = W_1 - iW_2 = G\gamma z'' - i\left\{(1+G)\left[\frac{B_{sol}}{B\rho} - z\left(\frac{1}{\rho}\right)'\right] - G\frac{z'}{\rho}\right\} \qquad (3.4)$$

The diagonal elements of the matrix in (3.4) lead to the precession frequency $\nu_s = G\gamma$. The off-diagonal terms $\zeta$ and $\zeta^*$ will, therefore, produce resonant behavior if they contain this frequency. Since $\zeta$ arises from $z$ oscillations this will occur when these oscillations have a component of frequency $G\gamma$. Therefore we have the well-known resonance criteria: Resonance occurs at those energies where the spin precession $\nu_s = G\gamma$ equals a frequency present in the spectrum of of $z$, specifically of $\zeta$.

We expand $\zeta$ as a Fourier series

$$\zeta = W_1 - iW_2 = \sum_i \varepsilon_{K_i} e^{-iK_i\theta} \qquad (3.5)$$

where $K_i$ , the $i$-th resonance value of $G\gamma$, may be

Imperfection resonances: $K_i$ = an integer $k$, for imperfection resonances.

"Intrinsic" resonances (due to vertical betatron oscillations): $K_i = kP \pm \nu_z$, where $\nu_z$ is the vertical betatron tune, $P$ is the periodicity of the magnet structure, k is any integer.

Broken periodicity resonances $K_i = k \pm \nu_z$ occur when the structure periodicity $P$ is inexact.

RF resonances $K_i = \omega_{rf}/\omega_{orbit}$ induced by rf dipoles and/or solenoids placed somewhere on the orbit.

In addition there are higher order resonances and/or resonances due to horizontal oscillations, which we shall not consider here.

The resonance strengths are calculated by Fourier analysis of $\zeta$ (Eq. 3.5):

$$\varepsilon_K = \frac{1}{2\pi R}\int_0^{2\pi R} \zeta e^{iK\theta} ds \qquad (3.6)$$

The computer program DEPOL, introduced and described in [4], computes this quantity for intrinsic, imperfection and broken periodicity resonances. But it is employs the reference-orbit based basis vectors and the dynamical equations (2.10); therefore it is now being rewritten on the basis of the trajectory-based formulation (2.9).

Since the major difference between (2.9) and (2.10) is the replacement of the factor $(1+G\gamma)$ by $G\gamma$, calculations by the old method are approximately right if the results are multiplied by a factor $G\gamma/(1+G\gamma)$. In most practical cases $G\gamma$ is large, therefore the difference is minor.

The same applies to calculations of the strength of Siberian snakes and rotators.

In the case of deuterons, where $G = -.138$ is small and negative, the distinction is important. A. Luccio [7] has taken the case of COSY with low energy deuterons, and shows that experimental results disagree widely with his tracking program SPINK when the factor $1+G\gamma$ is used, but agree very well if the program is modified to use the factor $G\gamma$ instead.


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