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Theory of electron cooling using electron cooling as an intrabeam scattering process

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Abstract

Electron cooling that results when a bunch of electrons overlaps a bunch of ions , with both bunches moving at the same velocity, may be considered to be an intrabeam scattering process. The process is similar to the usual intrabeam scattering, where the ions scatter from each other and usually results in beam growth. An important difference is that in electron cooling the mass of the ion is different from and much larger than the mass of the electron. This difference considerably complicates the intrabeam scattering theory. It introduces a new term in the emittance growth rate, which vanishes when the particles are identical and their masses are equal, and can give rise to emittance cooling of the heavier particles . The term that gives rise to beam growth for the usual intrabeam scattering is also present but is much smaller than the cooling term when one particle is much heavier than the other. This paper derives the results found for the emittance cooling rates due to the scattering of the ions in the ion bunch by the electons in the electron bunch.

1 Introduction

Electron cooling that results when a bunch of electrons overlaps a bunch of ions , with both bunches moving at the same velocity, may be considered to be an intrabeam scattering process. The process is similar to the usual intrabeam scattering, Ref.[1] where the ions scatter from each other and usually results in beam growth. An important difference is that in electron cooling

the mass of the ion is different from and much larger than the mass of the electron. This difference considerably complicates the intrabeam scattering theory. It introduces a new term in the emittance growth rate, which vanishes when the particles are identical and their masses are equal, and can give rise to emittance cooling of the heavier particles . The term that gives rise to beam growth for the usual intrabeam scattering is also present but is much smaller than the cooling term when one particle is much heavier than the other.

This paper derives the results found for the emittance cooling rates due to the scattering of the ions in the ion bunch by the electons in the electron bunch. The derivations given below makes considerable use of the results found in two previous papers, Ref.[2] and Ref.[3]

2 The $f(x,p)$ distribution and the scattering rate δN

The ions are contained within a bunch and their distibution is given by $f_a(x_a, p_a)$ where $N_a f_a(x_a, p_a)$ is the number of ions in $d^3x_a d^3p_a$. N_a is the number of ions in the bunch.

$$\int d^3x_a d^3p_a \; f_a(x_a, p_a) = 1$$

The distribution of the electrons in the electon bunch is given by $f_b(x_b, p_b)$ and N_b is the number of electrons in the electron bunch. Let δN_a be the number of ions with momentum, p_a in d^3p_a and space coordinate x in d^3x which are scattered by the electrons with momentum p_b in d^3p_b which are also in d^3x, in the time interval dt , into the solid angle $d\Omega'$ corresponding to the direction $\hat{p'_a}$. Then δN_a is given by, Ref.[2],

$$\begin{aligned} \delta N_a &= N_a N_b \sigma_{ab} d\Omega' \frac{d^3p_a}{\gamma_a} \frac{d^3p_b}{\gamma_b} f_a(x, p_a) f_b(x, p_b) F(p_a, p_b) d^3x dt \\ F(p_a, p_b) &= \frac{[(p_a p_b)^2 - m_a^2 m_b^2]^{1/2}}{m_a m_b} \end{aligned} \tag{1}$$

σ_{ab} is the scattering cross section for the scattering of the ions from the electrons.In the expression for $F(p_a, p_b)$, we have put $c = 1$. $F(p_a, p_b)$ has the dimensions of a velocity.

For completeness sake this result is given in the form which is valid in any CS. For the electron cooling problem for RHIC, one can do all the calclations in the Rest CS, which is the CS moving along with the two bunches. In the Rest CS, the central particle in either bunch is at rest and the motion of the motion of the particles may be treated non-reletavistically.In the Rest CS , one may put $\gamma_a = \gamma_b = 1$ and

$$F(p_a, p_b) = |\vec{v_a} - \vec{v_b}|$$

3 Growth rates for $< p_{ia}p_{ja} >$

Following Bjorken and Mtingwa, Ref.[4], cooling rates will first be given for $< p_{ia}p_{ja} >$. where the <> indicate an average over all the particles in the bunch. From these one can compute the growth rates for the average emittances of the ions, $< \epsilon_{ia} >$. In a scattering event, where an ion with momentum p_a scatters off an electron with momentum p_b, the momenta will change to p'_a and p'_b. Let δp_{ia} represent the change in p_{ia} in the collision, and similarly for $\delta(p_{ia}p_{ja})$. Then

$$\begin{aligned} \delta p_{ia} &= p'_{ia} - p_{ia} \\ \delta(p_{ia}p_{ja}) &= p'_{ia}p'_{ja} - p_{ia}p_{ja} \end{aligned} \tag{2}$$

Using the scattering rate given by Eq.(1), one can now compute $\delta < p_{ia}p_{ja} >$ in the Rest CS,

$$\begin{aligned} \delta < (p_{ia}p_{ja}) > &= N_b \int \quad d^3x d^3p_a d^3p_b f_a(x,p_a) f_b(x,p_b) |\vec{v_a} - \vec{v_b}| \\ &\qquad \sigma_{ab} d\Omega' \quad \delta(p_{ia}p_{ja}) \quad dt \\ \delta(p_{ia}p_{ja}) &= (p'_{ia}p'_{ja} - p_{ia}p_{ja}) \end{aligned} \tag{3}$$

The 11-dimensional integral in Eq.3 can be reduced to a 3-dimenional integral for gaussian distributions, if one notes that in the Rest CS σ_{ab} depends on $\vec{v_a} - \vec{v_b}$ and one transforms from the momentum variables p_a, p_b to two new variables one of which is $\vec{v_a} - \vec{v_b}$. This can be done by the transformation

$$\begin{aligned} \bar{p}_{ia} &= W_i + \frac{\mu}{m_a}\Delta_i \\ \bar{p}_{ib} &= W_i - \frac{\mu}{m_b}\Delta_i \end{aligned}$$

$$
\begin{aligned}
W_i &= \frac{p_{ia}+p_{ib}}{\gamma_0\beta_0(m_a+m_b)c} \\
\Delta_i &= \bar{p}_{ia}-\bar{p}_{ib}=\frac{v_{ia}-v_{ib}}{\gamma_0\beta_0 c} \\
d^3\bar{p}_a d^3\bar{p}_b &= d^3W d^3\Delta \\
\bar{p}_{ia} &= \frac{p_{ia}}{\gamma_0\beta_0 m_a c} \\
\bar{p}_{ib} &= \frac{p_{ib}}{\gamma_0\beta_0 m_b c} \\
\frac{1}{\mu} &= \frac{1}{m_a}+\frac{1}{m_b} \\
d^3\bar{p}_a d^3\bar{p}_b &= d^3W d^3\Delta
\end{aligned}
\tag{4}
$$

Δ_i is proportional to the relative velocity, $\vec{v_a}-\vec{v_b}$ when the velocities are non-relativistic. A similar transformation is used in Ref.1 and Ref.4 except that for them the particles are identical and the transformation is simpler.

$\delta(p_{ia}p_{ja})$ can be written as

$$
\begin{aligned}
\delta(p_{ia}p_{ja}) &= p_{ia}q_{ja}+p_{ja}q_{ia}+q_{ia}q_{ja} \\
q_{ia} &= p'_{ia}-p_{ia}
\end{aligned}
\tag{5}
$$

This result can written as

$$
\begin{aligned}
\delta(\bar{p}_{ia}\bar{p}_{ja}) &= [(W_i\bar{q}_{ja}+W_j\bar{q}_{ia})\frac{\mu}{m_a}]+[(\frac{\mu}{m_a})^2(\Delta_i\bar{q}_{ja}+\Delta_j\bar{q}_{ia}+\bar{q}_{ia}\bar{q}_{ja})] \\
\bar{q}_{ia} &= q_{ia}/(\gamma_0\beta_0\mu c)
\end{aligned}
\tag{6}
$$

Eq.3 can be rewritten in terms of W,Δ as

$$
\begin{aligned}
<\delta(\bar{p}_{ia}\bar{p}_{ja})> &= N_b\int d^3x d^3W d^3\Delta f_a(x,p_a)f_b(x,p_b)|\vec{v_a}-\vec{v_b}| \\
&\quad \sigma_{ab}d\Omega' \quad \delta(\bar{p}_{ia}\bar{p}_{ja}) \quad dt \\
\delta(\bar{p}_{ia}\bar{p}_{ja}) &= [(W_i\bar{q}_{ja}+W_j\bar{q}_{ia})\frac{\mu}{m_a}]+[(\frac{\mu}{m_a})^2(\Delta_i\bar{q}_{ja}+\Delta_j\bar{q}_{ia}+\bar{q}_{ia}\bar{q}_{ja})]
\end{aligned}
\tag{7}
$$

One may note that σ_{ab} depends only on Δ and not on W. In the expression for $\delta(\bar{p}_{ia}\bar{p}_{ja})$ the second term will be seen to depend only on Δ and gives rise

to the usual intrabeam scattering growth rate, while the first term depends on W and will be seen to vanish for identical particles and gives rise to the cooling rates for ion electron scattering.

The transformation from $\vec{p}_a, \vec{p}_b$ to $\vec{W}, \vec{\Delta}$ allows us to do the integral over $d\Omega'$. Eq.7 holds in any CS where the particle motion is non-relativistic. For each $\vec{p}_a, \vec{p}_b$ one can define a center of mass CS, called the CMS, in which $\vec{p}_a + \vec{p}_b = 0$. In the CMS

$$\Delta_i = \bar{p}_{ia} - \bar{p}_{ib} = p_{ia}/(\gamma_0 \beta_0 \mu c)$$

In the CMS, $\vec{\Delta}$ and $\vec{p}_a$ have the same direction, and $\vec{p}_a$ is scattered by the electrons to $\vec{p}_a{}'$ which is along the direction given by the polar angles θ, ϕ relative to the direction of $\vec{p}_a$ or $\vec{\Delta}$.

In Eq.7, only the $\bar{q}_{ia}$ depend on the scattering angles θ, ϕ . To do the integral over $d\Omega'$ in the Rest CS one has to evaluate the integrals

$$\begin{aligned} d_i &= \int d\Omega' \sigma_{ab} \bar{q}_{ia} \\ c_{ij} &= \int d\Omega' \sigma_{ab} [(\Delta_i \bar{q}_{ja} + \Delta_j \bar{q}_{ia}) + \bar{q}_{ia} \bar{q}_{ja}] \end{aligned} \tag{8}$$

$d\Omega' \sigma_{ab}$ is an invariant and $\vec{\Delta}, \vec{q}_a$ are both the same in the CMS and the Rest CS as they are both the difference of 2 vectors that are proportional to a velocity. d_i, c_{ij} are tensors in 3-space. If these integrals are evaluated in the CMS and the result is written in terms of tensors in 3-space then the result will also hold in the Rest CS.

In the CMS, we introduce a polar coordinate system θ, ϕ where θ is measured relative to the direction of $\vec{p_a}$ or $\vec{\Delta}$ and we assume that $\sigma_{ab}(\theta, \phi)$ is a fumction of θ only. we can then write

$$\begin{aligned} \vec{\Delta} &= (0,0,1)|\vec{\Delta}| \\ \vec{p_a} &= (0,0,1)|\vec{\Delta}|(\gamma_0 \beta_0 \mu c) \\ \vec{p_a}' &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)|\vec{\Delta}|(\gamma_0 \beta_0 \mu c) \\ \vec{q_a} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta - 1)|\vec{\Delta}|(\gamma_0 \beta_0 \mu c) \end{aligned} \tag{9}$$

In the CMS, using Eq.9, one finds

$$d_i = -2\pi \int d\theta sin\theta (1 - cos\theta) \sigma_{ab}(0,0,1)|\vec{\Delta}|$$

$$c_{ij} = \pi \int_0^\pi d\theta \sin^3\theta \sigma_{ab} |\vec{\Delta}|^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{10}$$

In computing c_{ij} one may note that the $\Delta_i \bar{q}_{ja} + \Delta_j \bar{q}_{ia}$ term in Eq.8 only contributes to c_{33} while the $\bar{q}_{ia}\bar{q}_{ja}$ term contributes to to all 3 diagonal elements of c_{ij}.

These results for d_i, c_{ij} in the CMS can be rewritten in terms of tensors in 3-space as

$$\begin{aligned} d_i &= -2\pi \int d\theta sin\theta(1 - cos\theta)\sigma_{ab}\Delta_i \\ c_{ij} &= \pi \int_0^\pi d\theta \sin^3\theta \sigma_{ab} \, (|\vec{\Delta}|^2 \delta_{ij} - 3\Delta_i\Delta_j) \end{aligned} \tag{11}$$

In this form the results will also hold in the Rest CS. Eq. 7 can now be rewritten as

$$\begin{aligned} < \delta(\bar{p}_{ia}\bar{p}_{ja}) > &= N_b \int \quad d^3x d^3W d^3\Delta f_a(x,p_a) f_b(x,p_b) |\vec{v_a} - \vec{v_b}| \\ &\quad ([-2\pi \frac{\mu}{m_a}(W_i\Delta_j + W_j\Delta_i) \int d\theta sin\theta(1 - cos\theta)\sigma_{ab}]_1 \\ &\quad + [\pi(\frac{\mu}{m_a})^2 (|\vec{\Delta}|^2\delta_{ij} - 3\Delta_i\Delta_j) \int d\theta sin^3\theta \sigma_{ab}]_2) \quad dt \end{aligned} \tag{12}$$

Eq.12 can be used to compute either intrabeam scattering for identical particles or electron cooling. If the a and b particles are identical, then the second term indicated by $[\ \ ]_2$ and called the Δ-term gives the growth rates for intrabeam scattering. In this case, the first term, indicated by $[\ \ ]_1$ and called the W-term, will vanish. This is shown below for gaussian distributions and also can be shown to hold for any distribution because of the symmetry of the a and b particles. If the b particle is much lighter than the a particle, the W-term gives rise to cooling of the a particles and the Δ-term is smaller than the W-term by the factor m_b/m_a. This is shown below for gaussian distributions. Eq. 12 holds for any distibutions, $f_a(x,p_a), f_b(x,p_b)$. In the next section, we will specialize to gaussian distributions.

it is often assumed that σ_{ab} is given by the Coulonb cross-section in the CMS CS for the a and b particles. This is given by

$$\begin{aligned} \sigma_{ab} &= (\frac{r_{ab}}{\beta_{ab}^2})^2 \frac{1}{(1-cos\theta)^2} \\ r_{ab} &= \frac{Z_a Z_b e^2}{\mu c^2} \\ \beta_{ab} c &= |\vec{v_a} - \vec{v_b}| \end{aligned} \tag{13}$$

The integrals over θ in Eq.12 can then be written as

$$\begin{aligned} \int d\theta sin\theta (1-cos\theta) \frac{1}{(1-cos\theta)^2} &= ln\left[1 + \left(\frac{\beta_{ab}^2 b_{maxab}}{r_{ab}}\right)^2\right] \\ \int d\theta sin^3\theta \frac{1}{(1-cos\theta)^2} &= 2\left[ln\left[1 + \left(\frac{\beta_{ab}^2 b_{maxab}}{r_{ab}}\right)^2\right] - \frac{1}{1 + (r_{ab}/(\beta_{ab}^2 b_{maxab}))^2}\right] \\ tan(\theta_{min}) &= \frac{r_{ab}}{\beta_{ab}^2 b_{maxab}} \end{aligned} \tag{14}$$

b_{maxab} is the maximun allowed impact parameter in the CMS. θ_{min} is the smallest allowed scattering angle in the CMS.

It will be seen below that to compute the cooling rates for the emittances one will also need the cooling rates for $< x_{ia} p_{ja} >$. When the a and b particles are identical, the $< x_{ia} p_{ja} >$ are zero , but not zero when the particles are different. Using Eq.7, one finds

$$\begin{aligned} < \delta(x_i \bar{p}_{ja}) > &= N_b \int \quad d^3x d^3W d^3\Delta f_a(x,p_a) f_b(x,p_b) |\vec{v_a} - \vec{v_b}| \\ & \qquad \sigma_{ab} d\Omega' \quad \delta(x_{ia}\bar{p}_{ja}) \quad dt \\ \delta(x_i \bar{p}_{ja}) &= x_i \delta\bar{p}_{ja} = x_i \bar{q}_{ja} \frac{\mu}{m_a} \end{aligned}$$

From Eq.11 one has

$$\int d\Omega' \sigma_{ab} \bar{q}_{ja} = -2\pi \int d\theta sin\theta (1-cos\theta) \sigma_{ab} \Delta_j$$

which gives

$$< \delta(x_i \bar{p}_{ja}) > \;=\; N_b \int d^3x d^3W d^3\Delta \quad f_a(x,p_a) f_b(x,p_b) |\vec{v_a} - \vec{v_b}| \\ [-2\pi \int d\theta sin\theta (1-cos\theta)\sigma_{ab} \quad x_i \Delta_j \frac{\mu}{m_a}] \quad dt \tag{15}$$

Eq.15 shows that $< \delta(x_i \bar{p}_{ja}) >$ gives rise to a cooling term which vanishes when the particles are identical , or when $\alpha_i = 0$ for the ion partcle for a gaussian distribution.

4 Cooling rates for $< p_{ia} p_{ja} >$ in the Rest CS for Gaussian distributions

In this section, we will find the cooling rates due to the scattering of the ions by the electrons in the cooling section when the ion and electron bunches have gaussian distributions. In Eq.12 , we will keep only the W-term as the $\Delta-$term , discussed later, is smaller by the factor m_b/m_a In this paper, it will be assumed that the dispersion is zero in the cooling section.

For a gaussian distribution, $f_a(x,p_a)$ is given for the ion bunch for zero dispersion by Ref.[3],

$$\begin{aligned} f_a(x,p_a) &= \frac{1}{\Gamma_a} exp[-S_a(x,p_a)] \\ \Gamma_a &= \int d^3x d^3p \; exp[-S_a(x,p_a)] \\ \Gamma_a &= \pi^3 \bar{\epsilon}_{xa} \bar{\epsilon}_{sa} \bar{\epsilon}_{ya} \end{aligned} \tag{16}$$

$$S_a \;=\; S_{xa} + S_{ya} + S_{sa}$$

$$\begin{aligned} S_{xa} &= \frac{1}{\bar{\epsilon_{xa}}} \epsilon_{xa}(x, x'_a) \quad x'_a = p_{xa}/p_{0a} \\ \epsilon_{xa}(x, x'_a) &= [x^2 + (\beta_x x'_a + \alpha_{xa} x)^2]/\beta_{xa} \end{aligned}$$

$$\begin{aligned} S_{ya} &= \frac{1}{\bar{\epsilon_{ya}}} \epsilon_{ya}(y, y'_a) \quad y'_a = p_{ya}/p_{0a} \\ \epsilon_{ya}(y, y'_a) &= [y^2 + (\beta_y y'_a + \alpha_{ya} y)^2]/\beta_{ya} \end{aligned}$$

$$
\begin{aligned}
S_s &= \frac{1}{\bar{\epsilon_s}}\epsilon_s(s, p_s/p_{0a}) \\
\epsilon_s(s, p_s/p_{0a}) &= \frac{s^2}{2\sigma_s^2} + \frac{(p_s/p_{0a})^2}{2\sigma_p^2} \\
\epsilon_s(s, p_s/p_{0a}) &= \frac{1}{\beta_s}(s)^2 + \beta_s(p_s/p_{0a})^2 \\
\epsilon_s(s, p_s/p_{0a}) &= [(s)^2 + (\beta_s(p_s/p_{0a}))^2]/\beta_s \\
\beta_s &= \sigma_s/\sigma_p \\
\bar{\epsilon}_s &= 2\sigma_s\sigma_p
\end{aligned} \tag{17}
$$

A longitudinal emittance has been introduced so that the longitudinal motion and the transverse motions can be treated in a similar manner. β_s in the Rest CS is larger than β_s in the Laboratory CS by the factor γ_0^2. s, p_s are the paricle longitudinal position and momentum in the Rest CS.

In Eq.12 we will now do the integration over d^3xd^3W using the above gaussian ditributions. Because there is no dispersion in the cooling section the integral over $dxdW_x$ or $dsdW_s$ or $dydW_y$ can each be treated in a similar way. Eq.12 can now be written using the Coulomb cross-section as

$$
\begin{aligned}
\delta < (\bar{p}_{ia}\bar{p}_{ja}) > &= \frac{N_b}{\Gamma_a\Gamma_b}\int \quad d^3xd^3Wd^3\Delta exp[-(S_a + S_b)]|\vec{v_a} - \vec{v_b}| \\
&\quad \frac{\mu}{m_a}\bar{W}_{ij} \quad (\frac{r_{ab}}{\beta_{ab}^2})^2 \quad ln\left[1 + \left(\frac{\beta_{ab}^2 b_{maxab}}{r_{ab}}\right)^2\right] \quad dt \\
\bar{W}_{ij} &= -2\pi(W_i\Delta_j + W_j\Delta_i)
\end{aligned} \tag{18}
$$

We rewrite $S_a + S_b$ as

$$
\begin{aligned}
S_a + S_b &= \Sigma_i(S_{ia} + S_{ib}) \quad i = x, y, s \\
S_{ia} &= \frac{1}{\bar{\epsilon}_{ia}}\left[\frac{x_{ia}^2}{\beta_{ia}} + (\beta_{ia}^{1/2}\bar{p}_{ia} + \frac{\alpha_{ia}x_{ia}}{\beta_{ia}^{1/2}})^2\right] \\
S_{ia} &= \frac{1}{\bar{\epsilon}_{ia}}\left[\frac{x_{ia}^2}{\beta_{ia}} + (\beta_{ia}^{1/2}(W_i + \frac{\mu}{m_a}\Delta_i) + \frac{\alpha_{ia}x_{ia}}{\beta_{ia}^{1/2}})^2\right]
\end{aligned}
$$

$$
S_{ia} + S_{ib} = A_{11i}x_i^2 + A_{22i}W_i^2 + 2A_{12i}x_iW_i + (A_{10i}x_i + A_{01i}W_i)\Delta_i + A_{00i}\Delta_i^2
$$

$$
\begin{aligned}
A_{11i} &= \left[\frac{1+\alpha_i^2}{\beta_i\bar{\epsilon}_i}\right]_+ \qquad A_{22i} = \left[\frac{\beta_i}{\bar{\epsilon}_i}\right]_+ \\
A_{12i} &= \left[\frac{\alpha_i}{\bar{\epsilon}_i}\right]_+ \qquad A_{10i} = \left[2\frac{\mu}{m}\frac{\alpha_i}{\bar{\epsilon}_i}\right]_- \\
A_{01i} &= \left[2\frac{\mu}{m}\frac{\beta_i}{\bar{\epsilon}_i}\right]_- \qquad A_{00i} = \left[(\frac{\mu}{m})^2\frac{\beta_i}{\bar{\epsilon}_i}\right]_+
\end{aligned}
\tag{19}
$$

The symbols $[(\quad)]_+$ and $[(\quad)]_-$ are defined by

$$[(\quad)]_+ = (\quad)_a + (\quad)_b$$

$$[(\quad)]_- = (\quad)_a - (\quad)_b$$

We will now make a transformation to eliminate the $2A_{12i}x_iW_i$ term in $S_{ia} + S_{ib}$. We rewrite $S_{ia} + S_{ib}$ as

$$
\begin{aligned}
S_{ia} + S_{ib} &= A_{11i}x_i^2 + A_{22i}W_i^2 + 2A_{12i}x_iW_i + (A_{10i}x_i + A_{01i}W_i)\Delta_i + A_{00i}\Delta_i^2 \\
&= [A_{11}x^2 + A_{22}W^2 + 2A_{12}xW + (A_{10}x + A_{01}W)\Delta + A_{00}\Delta^2]_i \\
&= [x^2(A_{11} - \frac{A_{12}^2}{A_{22}}) + (A_{22}^{1/2}W + \frac{A_{12}}{A_{22}^{1/2}}x)^2 \\
&\quad +(A_{10}x + A_{01}W)\Delta + A_{00}\Delta^2]_i
\end{aligned}
$$

$$
\begin{aligned}
\eta_i &= \left[\frac{\bar{A}^{1/2}}{A_{22}^{1/2}}x\right]_i \qquad p_{\eta i} = \left[A_{22}^{1/2}W + \frac{A_{12}}{A_{22}^{1/2}}x\right]_i \\
\bar{A}_i &= [A_{11}A_{22} - A_{12}^2]_i \\
x_i &= [x_\eta\eta]_i \qquad W_i = [(W_\eta\eta + W_{p_\eta}p_\eta)]_i \\
dx_i dw_i &= \left[\frac{1}{\bar{A}^{1/2}}d\eta dp_\eta\right]_i \\
x_{\eta i} &= \left[\frac{A_{22}^{1/2}}{\bar{A}^{1/2}}\right]_i \qquad W_{\eta i} = \left[-\frac{A_{12}}{\bar{A}^{1/2}}\right]_i \qquad W_{p_\eta i} = \left[\frac{1}{A_{22}^{1/2}}\right]_i
\end{aligned}
$$

$$
\begin{aligned}
S_{ia} + S_{ib} &= [\eta^2 + p_\eta^2 + (A_{10}x + A_{01}W)\Delta + A_{00}\Delta^2]_i \\
&= [\eta^2 + p_\eta^2 + (B_{10}\eta + B_{01}p_\eta)\Delta + A_{00}\Delta^2]_i \\
B_{10i} &= [A_{10}x_\eta + A_{01}W_\eta]_i \qquad B_{01i} = [A_{01}W_{p_\eta}]_i
\end{aligned}
$$

$$
\begin{aligned}
B_{10i} &= \left[A_{10} \frac{A_{22}^{1/2}}{\bar{A}^{1/2}} - A_{01} \frac{A_{12}}{\bar{A}^{1/2}} \right]_i \qquad B_{01i} = \left[A_{01} \frac{1}{A_{22}^{1/2}} \right]_i \\
\bar{W}_{ij} &= -2\pi [(W_\eta \eta + W_{p_\eta} p_\eta)_i \Delta_j + (W_\eta \eta + W_{p_\eta} p_\eta)_j \Delta_i]
\end{aligned}
\tag{20}
$$

In the expression for $S_{ia} + S_{ib}$ given at the end of Eq.19, the linear terms in η, p_η can be eliminated by the transformation

$$
\begin{aligned}
\bar{\eta}_i &= \left[\eta + \frac{B_{10}}{2} \Delta \right]_i \qquad \bar{p_{\eta}}_i = \left[p_\eta + \frac{B_{01}}{2} \Delta \right]_i \\
S_{ia} + S_{ib} &= [\bar{\eta}^2 + \bar{p_\eta}^2 + (A_{00} - B_{10}^2/4 - B_{01}^2/4) \Delta^2]_i \\
\bar{W}_{ij} &= -2\pi [[W_\eta (\bar{\eta} - \frac{B_{10}}{2} \Delta)]_i \Delta_j + [W_{p_\eta} (\bar{p_\eta} - \frac{B_{01}}{2} \Delta)]_i \Delta_j \\
&\quad + [W_\eta (\bar{\eta} - \frac{B_{10}}{2} \Delta)]_j \Delta_i + [W_{p_\eta} (\bar{p_\eta} - \frac{B_{01}}{2} \Delta)]_j \Delta_i]
\end{aligned}
\tag{21}
$$

Eq.17 can now be rewritten as

$$
\begin{aligned}
< \delta(\bar{p}_{ia} \bar{p}_{ja}) > &= \frac{N_b}{\Gamma_a \Gamma_b} \frac{1}{\bar{A}_p^{1/2}} \int \quad d^3\bar{\eta} d^3\bar{p_\eta} d^3\Delta \quad exp[-(S_a + S_b)] |\vec{v_a} - \vec{v_b}| \\
&\quad \frac{\mu}{m_a} \bar{W}_{ij} (\frac{r_{ab}}{\beta_{ab}^2})^2 \quad ln \left[1 + \left(\frac{\beta_{ab}^2 b_{maxab}}{r_{ab}} \right)^2 \right] \quad dt \\
\bar{W}_{ij} &= -2\pi [[W_\eta (\bar{\eta} - \frac{B_{10}}{2} \Delta)]_i \Delta_j + [W_{p_\eta} (\bar{p_\eta} - \frac{B_{01}}{2} \Delta)]_i \Delta_j \\
&\quad + [W_\eta (\bar{\eta} - \frac{B_{10}}{2} \Delta)]_j \Delta_i + [W_{p_\eta} (\bar{p_\eta} - \frac{B_{01}}{2} \Delta)]_j \Delta_i] \\
\bar{A}_p^{1/2} &= \bar{A}_x^{1/2} \bar{A}_y^{1/2} \bar{A}_s^{1/2}
\end{aligned}
\tag{22}
$$

Using Eq.20 for $S_{ia} + S_{ib}$ and for $\bar{W}_{ij}$, one can do the integral over $d^3\bar{\eta} d^3\bar{p_\eta}$ and get

$$
\begin{aligned}
\delta < (\bar{p}_{ia}\bar{p}_{ja}) > &= \frac{N_b}{\Gamma_a\Gamma_b}\frac{1}{\bar{A}_p^{1/2}}\pi^3 r_{ab}^2 c\frac{\mu}{m_a}\hat{W}_{ij} \\
&\int \quad d^3\Delta \frac{exp[-(\lambda_x\Delta_x^2+\lambda_y\Delta_y^2+\lambda_s\Delta_s^2)]}{\beta_{ab}^3}\Delta_i\Delta_j \\
&ln\left[1+\left(\frac{\beta_{ab}^2 b_{maxab}}{r_{ab}}\right)^2\right] \quad dt \\
\hat{W}_{ij} &= 2\pi\left[(W_\eta\frac{B_{10}}{2}+W_{p_\eta}\frac{B_{01}}{2})_i+(W_\eta\frac{B_{10}}{2}+W_{p_\eta}\frac{B_{01}}{2})_j\right] \\
\beta_{ab} &= \gamma_0\beta_0(\Delta_x^2+\Delta_y^2+\Delta_s^2)^{1/2} \\
\lambda_i &= \left[A_{00}-(\frac{B_{10}}{2})^2-(\frac{B_{01}}{2})^2\right]_i \\
\bar{A}_i &= [A_{11}A_{22}-A_{12}^2]_i \\
\bar{A}_p^{1/2} &= \bar{A}_x^{1/2}\bar{A}_y^{1/2}\bar{A}_s^{1/2} \\
x_{\eta i} &= \left[\frac{A_{22}^{1/2}}{\bar{A}^{1/2}}\right]_i \qquad W_{\eta i}=\left[-\frac{A_{12}}{\bar{A}^{1/2}}\right]_i \qquad W_{p_{\eta i}}=\left[\frac{1}{A_{22}^{1/2}}\right]_i \\
B_{10i} &= [A_{10}x_\eta+A_{01}W_\eta]_i \qquad B_{01i}=[A_{01}W_{p_\eta}]_i \\
B_{10i} &= \left[A_{10}\frac{A_{22}^{1/2}}{\bar{A}^{1/2}}-A_{01}\frac{A_{12}}{\bar{A}^{1/2}}\right]_i \qquad B_{01i}=\left[A_{01}\frac{1}{A_{22}^{1/2}}\right]_i \\
A_{11i} &= \left[\frac{1+\alpha_i^2}{\beta_i\bar{\epsilon}_i}\right]_+ \qquad A_{22i}=\left[\frac{\beta_i}{\bar{\epsilon}_i}\right]_+ \\
A_{12i} &= \left[\frac{\alpha_i}{\bar{\epsilon}_i}\right]_+ \qquad A_{10i}=\left[2\frac{\mu}{m}\frac{\alpha_i}{\bar{\epsilon}_i}\right]_- \\
A_{01i} &= \left[2\frac{\mu}{m}\frac{\beta_i}{\bar{\epsilon}_i}\right]_- \qquad A_{00i}=\left[(\frac{\mu}{m})^2\frac{\beta_i}{\bar{\epsilon}_i}\right]_+
\end{aligned}
\tag{23}
$$

Eq.23 is our final result for the cooling rates for $< p_{ia}p_{ja} >$ in the Rest CS, for two overlapping gaussian bunches , with no dispersion in the cooling section. For this case one gets zero results when $i \neq j$. The remaining 3-

dimensional integral over $d^3\Delta$ is an integral over the relative velocities of the ions and electrons.

It will be seen below that to compute the cooling rates for the emittances one will also need the cooling rates for $< x_{ia}p_{ja} >$. For gaussian distributioins, using the coulomb cross section and Eq.15, Eq.18 is replaced by

$$\begin{aligned}
\delta < (x_i\bar{p}_{ja}) > &= \frac{N_b}{\Gamma_a\Gamma_b}\int \quad d^3x d^3W d^3\Delta exp[-(S_a+S_b)]|\vec{v_a}-\vec{v_b}| \\
&\quad \frac{\mu}{m_a}\bar{x}_{ij} \quad \left[\frac{r_{ab}}{\beta_{ab}^2}\right]^2 \quad ln\left[1+\left(\frac{\beta_{ab}^2 b_{maxab}}{r_{ab}}\right)^2\right] \quad dt \\
\bar{x}_{ij} &= -2\pi x_i\Delta_j
\end{aligned} \tag{24}$$

After going from the x, W coordinates to η, p_η and integrating over η, p_η Eq.23 is replaced by

$$\begin{aligned}
\delta < (x_i\bar{p}_{ja}) > &= \frac{N_b}{\Gamma_a\Gamma_b}\frac{1}{\bar{A}_p^{1/2}}\pi^3 r_{ab}^2 c\frac{\mu}{m_a}\hat{x}_{ij} \\
&\quad \int \quad d^3\Delta \frac{exp[-(\lambda_x\Delta_x^2+\lambda_y\Delta_y^2+\lambda_s\Delta_s^2)]}{\beta_{ab}^3}\Delta_i\Delta_j \\
&\quad ln\left[1+\left(\frac{\beta_{ab}^2 b_{maxab}}{r_{ab}}\right)^2\right] \quad dt \\
\hat{x}_{ij} &= 2\pi\left[x_\eta\frac{B_{10}}{2}\right]_i \\
x_{\eta i} &= \left[\frac{A_{22}^{1/2}}{\bar{A}^{1/2}}\right]_i \qquad W_{\eta i} = \left[-\frac{A_{12}}{\bar{A}^{1/2}}\right]_i \qquad W_{p_{\eta i}} = \left[\frac{1}{A_{22}^{1/2}}\right]_i \\
B_{10i} &= [A_{10}x_\eta + A_{01}W_\eta]_i \qquad B_{01i} = [A_{01}W_{p_\eta}]_i \\
B_{10i} &= \left[A_{10}\frac{A_{22}^{1/2}}{\bar{A}^{1/2}} - A_{01}\frac{A_{12}}{\bar{A}^{1/2}}\right]_i \qquad B_{01i} = \left[A_{01}\frac{1}{A_{22}^{1/2}}\right]_i
\end{aligned} \tag{25}$$

5 Emittance growth rates

One can compute growth rates for the average emittances, $< \epsilon_{ia} >$ in the Laboratory Coordinate System, from the growth rates for $< p_{ia}p_{ja} >$ in the

Rest Coordinate System. In the following , dt is the time interval in the Laboratory System and $d\tilde{t}$ is the time interval in the Rest System. $dt = \gamma d\tilde{t}$. The final results are, for zero dispersion,

$$\frac{d}{dt} < \bar{\epsilon}_{ia} > \;=\; \frac{\beta_{ia}}{\gamma}\frac{d}{d\tilde{t}} < \bar{p}_{ia}^2 > + \frac{2\alpha_{ia}}{\gamma}\frac{d}{d\tilde{t}} < x_i \bar{p}_{ia} > \quad i = x, y, s \tag{26}$$

To derive the above results, the simplest case to treat is that of the vertical emittance. The verical emmitance is given by

$$\begin{aligned} \bar{\epsilon}_{ya}(y, y'_a) &= [y^2 + (\beta_{ya} y'_a + \alpha_{ya} y)^2]/\beta_{ya} \quad y'_a = \bar{p}_{ya} \\ \delta\bar{\epsilon}_{ya} &= \beta_{ya}\delta(\bar{p}_{ya}^2) + \delta(2\alpha_{ya} y(\bar{p}_{ya}) \\ \frac{d}{dt} < \bar{\epsilon}_{ya} > &= \frac{\beta_{ya}}{\gamma}\frac{d}{d\tilde{t}} < \bar{p}_{ya}^2 > + \frac{2\alpha_{ya}}{\gamma}\frac{d}{d\tilde{t}} < y\bar{p}_{ya} > \end{aligned} \tag{27}$$

In Eq.(27), $y'_a = \bar{p}_{ya}$, $\delta\epsilon_{ya}$ is the change in $\bar{\epsilon}_{ya}$ in a scattering event. Similar results will hold for $\bar{\epsilon}_{xa}$ and $\bar{\epsilon}_{sa}$ for zero dispersion.

The Δ term in electron cooling

In the previous section it was assumed that in Eq.12 one could drop the second term or Δ term compared to the first term or W term.This is true when $m_b << m_a$ and $\bar{p}_a \simeq \bar{p}_b$ in the Rest CS. Using Eq.4, one can write

$$\begin{aligned} W_i &= \left[\bar{p}_a \frac{m_a}{m_a + m_b} + \bar{p}_b \frac{m_b}{m_a + m_b}\right]_i \\ \Delta_i &= [\bar{p}_a - \bar{p}_b]_i \\ \\ W_i &\simeq [\bar{p}_a]_i \\ \Delta_i &= [\bar{p}_a - \bar{p}_b]_i \end{aligned} \tag{28}$$

Thus W and Δ are both of the same order as $\bar{p}_a$. If the motion is non-relativistic in the Rest CS, $\bar{q}_a \simeq \Delta \simeq \bar{p}_a$. From this it follows that the Δ term in Eq.12 is smaller than the W term by the factor m_b/m_a.

It has also been assumed that the motion in the Rest CS is non-relativistic. In the Laboratory CS, the rms spread in the relative momentum is given by

$$\sigma_{pi} \quad = \quad \left[\frac{\bar{\epsilon}_i}{2\beta_i}\right]^{1/2} \qquad i = x, y, s \tag{29}$$

For gold ions in RHIC at $\gamma = 100$

$$\bar{\epsilon}_x = \bar{\epsilon}_y = 5e-8, \beta_x = \beta_y = 50 \quad and \quad \sigma_{px} = \sigma_{py} = 2.24e-5$$

$$\bar{\epsilon}_s = 1.8e-4, \beta_s = 300m \quad and \quad \sigma_{ps} = .55e-3$$

In the Rest CS σ_{px}, σ_{py} are unchanged at 2.24e-5 And σ_{ps} is reduced by the factor γ to .55e-5 The spread in each of the momenta in the Rest CS is of the order of $1e-3m_ac$ since $\gamma = 100$ and the ion velocities are of the order of 1e-3c. Similar numbers hold for the electrons in the electron bunch.

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