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Notes on the Injection of EBIS Ions into Booster

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Ions from the proposed EBIS are to be injected into Booster by means of the existing electrostatic inflector in the C3 straight section and the four existing programmable injection dipoles in the C1, C3, C7, and D1 straights. These devices have been in use for many years for the injection of ions from Tandem as described in References [1] and [2].

All ions from EBIS have the same velocity $c\beta$ where c is the velocity of light and

$$\beta = 0.0655. \tag{1}$$

The kinetic energy of the ions is 2 MeV per nucleon and the magnetic rigidities range from 0.2054 Tm for protons to 0.5703 Tm for Fe^{20+} ions and 1.255 Tm for Au^{32+} ions. The revolution period of the ions in Booster at injection is 10.3 μs.

The rigidities of ions from Tandem range from 0.6073 Tm for protons to 0.6669 Tm for Fe^{20+} ions and 0.8538 Tm for Au^{32+} ions. The revolution periods range from 3.5 μs for protons to 8.8 μs for Fe^{20+} ions and 15 μs for Au^{32+} ions.

Unlike Tandem, which delivers a long pulse of ions over many turns around Booster, EBIS delivers a short pulse over one or a few turns. The typical pulse width from Tandem is 500 μs during which time the currents in the four injection dipoles are decreased to move the closed orbit away from the inflector septum. As discussed in Reference [2], the rate at which the orbit has to be moved depends on the size of the incoming beam and the horizontal tune in Booster. For EBIS beams the transverse emittance (un-normalized) in both planes is 11π mm milliradians; for Tandem beams it is approximately 1π mm milliradians.

In these notes we explore the conditions under which EBIS ions can be injected into Booster using the C3 inflector and the four injection dipoles.

1 Beam and Machine Ellipses at Inflector Exit

We assume that the EBIS beam is contained within an ellipse called the beam ellipse which evolves linearly as the beam is transported from EBIS through the inflector and into Booster. The beam ellipse at the inflector exit is given by

$$\gamma_0(x-x_0)^2 + 2\alpha_0(x-x_0)(x'-x'_0) + \beta_0(x'-x'_0)^2 = \epsilon_0 \tag{2}$$

where x and x' are the position and angle of a beam particle with respect to the Booster closed orbit, x_0 and x'_0 are the position and angle of the center of the ellipse with respect to the closed orbit, and $\pi\epsilon_0$ is the beam emittance. The ellipse parameters α_0, β_0, γ_0 satisfy

$$\beta_0\gamma_0 - \alpha_0^2 = 1, \quad \beta_0 > 0. \tag{3}$$

Writing

$$\mathbf{X} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad \mathbf{X}_0 = \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \tag{4}$$

and

$$\mathbf{E}_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}, \quad \mathbf{E}_0^{-1} = \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \tag{5}$$

the equation for the beam ellipse becomes

$$(\mathbf{X}^T - \mathbf{X}_0^T)\mathbf{E}_0^{-1}(\mathbf{X} - \mathbf{X}_0) = \epsilon_0 \tag{6}$$

where the superscript T denotes the transpose of a vector or matrix.

Once the beam ellipse has entered Booster it is contained within an ellipse that is matched to the Booster lattice. This is called the machine ellipse. The machine ellipse at the inflector exit is given by

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon \tag{7}$$

where x and x' are the position and angle of a beam particle with respect to the Booster closed orbit and α, β, and γ are the Courant-Snyder parameters of the lattice at the inflector exit. Writing

$$\mathbf{E} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}, \quad \mathbf{E}^{-1} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \tag{8}$$

the equation for the machine ellipse becomes

$$\mathbf{X}^T\mathbf{E}^{-1}\mathbf{X} = \epsilon. \tag{9}$$

The parameter ϵ is adjusted so that the machine ellipse contains the beam ellipse as shown in **Figure 1**. Here the black ellipse is the machine ellipse and the red ellipse is the beam ellipse. The machine ellipse is centered on the closed orbit at the inflector exit. (The ellipse centers are marked with plus signs.) The vertical blue line shows the position of the inflector septum. In **Figure 2** the parameter ϵ has been adjusted to give the smallest machine ellipse that contains the beam ellipse. We call this the minimum machine ellipse. The size of the minimum machine ellipse depends on the size and shape of the beam ellipse. On subsequent turns around the machine the beam ellipse remains inside the machine ellipse; eventually it will hit the septum unless the closed orbit is moved away. The amount the orbit has to be moved depends on the size of the minimum machine ellipse.

2 Transformation to Circular Coordinates

In order to find the minimum machine ellipse, it is helpful to transform to coordinates

$$y = \frac{x}{\sqrt{\beta}}, \quad y' = \frac{1}{\sqrt{\beta}}\left\{\alpha x + \beta x'\right\} \tag{10}$$

and

$$y_0 = \frac{x_0}{\sqrt{\beta}}, \quad y_0' = \frac{1}{\sqrt{\beta}}\left\{\alpha x_0 + \beta x_0'\right\}. \tag{11}$$

In terms of these coordinates the machine ellipse becomes the circle

$$y^2 + {y'}^2 = \epsilon \tag{12}$$

and the beam ellipse becomes

$$G(y-y_0)^2 + 2A(y-y_0)(y'-y_0') + B(y'-y_0')^2 = \epsilon_0 \tag{13}$$

where

$$B = \frac{\beta_0}{\beta}, \quad A = \alpha_0 - \alpha B \tag{14}$$

$$BG - A^2 = 1, \quad B + G = \gamma\beta_0 - 2\alpha\alpha_0 + \beta\gamma_0. \tag{15}$$

Writing

$$\mathbf{Y} = \begin{pmatrix} y \\ y' \end{pmatrix}, \quad \mathbf{Y}_0 = \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}, \quad \mathbf{N} = \frac{1}{\sqrt{\beta}} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \tag{16}$$

we have

$$\mathbf{Y} = \mathbf{N}\mathbf{X}, \quad \mathbf{Y}_0 = \mathbf{N}\mathbf{X}_0, \quad \mathbf{E}^{-1} = \mathbf{N}^T\mathbf{N}. \tag{17}$$

The machine ellipse is then

$$\mathbf{Y}^T\mathbf{Y} = \epsilon \tag{18}$$

and the beam ellipse is

$$(\mathbf{Y}^T - \mathbf{Y}_0^T)\mathbf{F}^{-1}(\mathbf{Y} - \mathbf{Y}_0) = \epsilon_0 \tag{19}$$

where

$$\mathbf{F} = \mathbf{N}\mathbf{E}_0\mathbf{N}^T = \begin{pmatrix} B & -A \\ -A & G \end{pmatrix}, \quad \mathbf{F}^{-1} = \begin{pmatrix} G & A \\ A & B \end{pmatrix}. \tag{20}$$

3 Transformation to Dimensionless Coordinates

Going one step further we define dimensionless coordinates

$$Y = \frac{y}{\sqrt{\epsilon_0}} = \frac{x}{\sqrt{\epsilon_0\beta}}, \quad Y' = \frac{y'}{\sqrt{\epsilon_0}} = \frac{1}{\sqrt{\epsilon_0\beta}}\left\{\alpha x + \beta x'\right\} \tag{21}$$

$$Y_0 = \frac{y_0}{\sqrt{\epsilon_0}} = \frac{x_0}{\sqrt{\epsilon_0\beta}}, \quad Y'_0 = \frac{y'_0}{\sqrt{\epsilon_0}} = \frac{1}{\sqrt{\epsilon_0\beta}}\left\{\alpha x_0 + \beta x'_0\right\} \tag{22}$$

and emittance

$$E = \epsilon/\epsilon_0. \tag{23}$$

The machine ellipse then becomes

$$Y^2 + Y'^2 = E \tag{24}$$

and the beam ellipse becomes

$$G(Y - Y_0)^2 + 2A(Y - Y_0)(Y' - Y'_0) + B(Y' - Y'_0)^2 = 1 \tag{25}$$

where

$$B = \frac{\beta_0}{\beta}, \quad A = \alpha_0 - \alpha B, \quad BG = 1 + A^2 \tag{26}$$

and

$$B + G = \gamma\beta_0 - 2\alpha\alpha_0 + \beta\gamma_0. \tag{27}$$

Thus the machine ellipse becomes a circle with area πE. The area of the beam ellipse becomes π and its half-width becomes $\sqrt{B}$. **Figure 3** shows the machine and beam ellipses of **Figure 2** in terms of the dimensionless coordinates.

4 Position and Orientation of the Beam Ellipse

The size of the minimum machine ellipse will depend on the position and orientation of the beam ellipse. It is clear from **Figure 3** that in order to make the minimum machine ellipse as small as possible, we must have a beam ellipse that is upright (in terms of coordinates Y and Y') and centered on $Y' = 0$. Thus we must have

$$Y_0' = 0 \tag{28}$$

and

$$A = 0. \tag{29}$$

The beam ellipse then becomes

$$G(Y - Y_0)^2 + BY'^2 = 1 \tag{30}$$

where

$$G = (1 + A^2)/B = 1/B. \tag{31}$$

Using the second of equations (22) and the second of equations (14) we find that in terms of the original coordinates, the condition $Y_0' = 0$ implies

$$x_0' = -\alpha x_0/\beta \tag{32}$$

and the condition $A = 0$ implies

$$\alpha_0 = \alpha B = \alpha\beta_0/\beta. \tag{33}$$

5 Angle and Radius Parameters

Let us now introduce angle and radius parameters, ϕ and R, such that

$$Y - Y_0 = R\cos\phi, \quad Y' = R\sin\phi. \tag{34}$$

The equation for the beam ellipse then becomes

$$GR^2 \cos^2 \phi + BR^2 \sin^2 \phi = 1 \tag{35}$$

where

$$G = 1/B. \tag{36}$$

Using

$$1 + \cos 2\phi = 2\cos^2 \phi, \quad 1 - \cos 2\phi = 2\sin^2 \phi \tag{37}$$

we have

$$GR^2 \{1 + \cos 2\phi\} + BR^2 \{1 - \cos 2\phi\} = 2 \tag{38}$$

$$(G + B)R^2 + (G - B)R^2 \cos 2\phi = 2 \tag{39}$$

and therefore

$$R^2 F(\phi) = 2 \tag{40}$$

where

$$F(\phi) = (G + B) + (G - B)\cos 2\phi, \quad G = 1/B. \tag{41}$$

Thus the points on the beam ellipse are given by

$$Y(\phi) = Y_0 + R(\phi)\cos\phi, \quad Y'(\phi) = R(\phi)\sin\phi. \tag{42}$$

where

$$R(\phi) = \left\{ \frac{2}{F(\phi)} \right\}^{1/2}. \tag{43}$$

6 The Minimum Machine Ellipse

Now for each point on the beam ellipse we define

$$E(\phi) = Y^2 + Y'^2 = (Y_0 + R\cos\phi)^2 + R^2 \sin^2 \phi \tag{44}$$

which gives

$$E(\phi) = Y_0^2 + 2Y_0 R(\phi)\cos\phi + R^2(\phi). \tag{45}$$

The minimum machine ellipse is then given by the *maximum* value attained by $E(\phi)$ as ϕ varies between 0 and 2π. We assume that

$$Y_0 > 0. \tag{46}$$

One then finds (as shown in the Appendixes) that for

$$B^{3/2} \left\{ Y_0 + \sqrt{B} \right\} > 1 \tag{47}$$

the minimum machine ellipse is given by

$$Y^2 + Y'^2 = E = \left\{Y_0 + \sqrt{B}\right\}^2 \tag{48}$$

and for

$$B^{3/2}\left\{Y_0 + \sqrt{B}\right\} < 1 \tag{49}$$

the minimum machine ellipse is given by

$$Y^2 + Y'^2 = E = \frac{1 - B^2 + BY_0^2}{B(1 - B^2)}. \tag{50}$$

Figure 3 shows a red beam ellipse with parameter $B = 0.314077$ and center $Y_0 = 2 + \sqrt{B}$. In this case we have

$$B^{3/2}\left\{Y_0 + \sqrt{B}\right\} = 0.5493 < 1 \tag{51}$$

and the minimum machine ellipse is given by

$$Y^2 + Y'^2 = E = \frac{1 - B^2 + BY_0^2}{B(1 - B^2)} = 10.4572. \tag{52}$$

This is the black circle in the Figure.

7 Septum Position Parameter

We now define dimensionless parameter

$$Y_s = \frac{x_s}{\sqrt{\epsilon_0 \beta}} \tag{53}$$

where x_s is the position of the outer side of the septum with respect to the closed orbit. x_s and Y_s are indicated by the vertical blue lines in **Figures 1**, **2** and **3**. We assume that the incoming beam ellipse just touches the outer side of the septum as shown in the Figures. Then we have

$$x_0 = x_s + \sqrt{\epsilon_0 \beta_0} \tag{54}$$

and

$$Y_0 = \frac{x_0}{\sqrt{\epsilon_0 \beta}} = Y_s + \sqrt{B}. \tag{55}$$

The equations for the minimum machine ellipse then become

$$Y^2 + Y'^2 = E = \left\{Y_s + 2\sqrt{B}\right\}^2 \tag{56}$$

for

$$B^{3/2}\left\{Y_s+2\sqrt{B}\right\}>1 \tag{57}$$

and

$$Y^2+Y'^2=E=\frac{1+BY_s^2+2Y_sB^{3/2}}{B(1-B^2)} \tag{58}$$

for

$$B^{3/2}\left\{Y_s+2\sqrt{B}\right\}<1. \tag{59}$$

Note that for

$$Y_s>0 \tag{60}$$

we have

$$Y_s+2\sqrt{B}>2\sqrt{B} \tag{61}$$

and

$$B^{3/2}\left\{Y_s+2\sqrt{B}\right\}>2B^2. \tag{62}$$

Thus we see that if

$$Y_s>0, \quad B\geq 1/\sqrt{2} \tag{63}$$

then

$$B^{3/2}\left\{Y_s+2\sqrt{B}\right\}>1. \tag{64}$$

8 Beam Ellipses with Different Shapes

As already noted, the size of the minimum machine ellipse depends on the shape of the beam ellipse. This is illustrated in **Figures 4**, **5**, **6**, and **7**.

Figure 4 shows Red and Blue beam ellipses and the corresponding minimum machine ellipses for the case in which $Y_s=2$. The two beam ellipses have the same area $\pi\epsilon_0$. The Blue ellipse is matched to the machine ellipse; the red one is not. The minimum machine ellipse containing the unmatched red ellipse is significantly smaller than the minimum machine ellipse containing the matched blue ellipse.

Figure 5 shows the machine and beam ellipses of **Figure 4** in terms of dimensionless coordinates. In these coordinates the minimum machine ellipses are circles, the area of each beam ellipse is π, and the matched blue ellipse is a circle. The half-widths of the beam ellipses are $\sqrt{B}$ where $B=1$ for the matched ellipse and $B=0.314077$ for the red ellipse.

Figure 6 shows beam ellipses and the corresponding minimum machine ellipses for the case in which $Y_s=2$. Here the values of B for the Blue, Green, and Red beam ellipses are 1, 0.65, and 0.314077 respectively. The minimum machine ellipse is smallest for the beam ellipse with $B = 0.314077$.

In **Figure 7**, the values of B for the Red, Green, and Violet beam ellipses are 0.314077, 0.14, and 0.09405 respectively. Here again the minimum machine ellipse is smallest for the beam ellipse with $B = 0.314077$.

9 The Optimum Beam Ellipse

As noted in Section 1, the amount the closed orbit has to be moved away from the septum depends on the size of the minimum machine ellipse, and this in turn depends on the size and shape of the beam ellipse. For given closed orbit position and incoming beam emittance $\pi\epsilon_0$, we define the optimum beam ellipse to be the one for which the minimum machine ellipse is smallest. This will give the smallest required motion of the closed orbit away from the septum. Since the size of the minimum beam ellipse is given by the parameter E, and since the shape of the beam ellipse is governed by the parameter B, the problem of finding the optimum beam ellipse becomes that of finding the value of B for which E is smallest. We call this the optimum value of B.

We assume that

$$Y_s \geq 0 \tag{65}$$

and define B_1 such that

$$B_1^{3/2}\left\{Y_s + 2\sqrt{B_1}\right\} = 1. \tag{66}$$

Then

$$B_1^{3/2}Y_s + 2B_1^2 = 1, \quad \frac{1-2B_1^2}{B_1^{3/2}} = Y_s \geq 0 \tag{67}$$

and we see that

$$B_1^2 \leq 1/2. \tag{68}$$

For $B > B_1$ we have

$$B^{3/2}\left\{Y_s + 2\sqrt{B}\right\} > 1 \tag{69}$$

and

$$E(B) = \left\{Y_s + 2\sqrt{B}\right\}^2. \tag{70}$$

In this case the smallest value of E occurs when $B = B_1$. Using (66) we have

$$E(B_1) = \left\{Y_s + 2\sqrt{B_1}\right\}^2 = 1/B_1^3. \tag{71}$$

For $B < B_1$ we have

$$B^{3/2}\left\{Y_s + 2\sqrt{B}\right\} < 1 \tag{72}$$

and

$$E(B) = \frac{1 + BY_s^2 + 2Y_s B^{3/2}}{B(1 - B^2)}. \tag{73}$$

Using (67) we have

$$E(B_1) = \frac{1 + B_1 Y_s^2 + 2Y_s B_1^{3/2}}{B_1(1 - B_1^2)} = 1/B_1^3 \tag{74}$$

which shows that (70) and (73) give the same value of E at $B = B_1$.

Now, differentiating (73) with respect to B we obtain

$$\frac{dE}{dB} = \frac{f(B)\,g(B)}{B^2(1 - B^2)^2} \tag{75}$$

where

$$f(B) = 2Y_s B^{3/2} + 3B^2 - 1, \quad g(B) = Y_s B^{3/2} + 1. \tag{76}$$

Let B_2 be such that

$$f(B_2) = 2Y_s B_2^{3/2} + 3B_2^2 - 1 = 0. \tag{77}$$

Then for $B = B_2$ we have

$$\frac{dE}{dB} = 0. \tag{78}$$

Using (67) we have

$$f(B_1) = 1 - B_1^2 > 0 \tag{79}$$

and therefore

$$f(B_2) < f(B_1), \quad B_2 < B_1. \tag{80}$$

Thus for $B_2 < B < B_1$ we have

$$\frac{dE}{dB} > 0 \tag{81}$$

and for $0 < B < B_2$

$$\frac{dE}{dB} < 0. \tag{82}$$

It follows that

$$E(B_2) < E(B_1) \tag{83}$$

and $E(B)$ reaches its minimum when $B = B_2$. This is the optimum value of B and the minimum E is

$$E(B_2) = \frac{1 + B_2 Y_s^2 + 2Y_s B_2^{3/2}}{B_2(1 - B_2^2)}. \tag{84}$$

Using (77) we have

$$2Y_s B_2^{3/2} = 1 - 3B_2^2, \quad \frac{1 - 3B_2^2}{2B_2^{3/2}} = Y_s \tag{85}$$

and it follows that

$$E(B_2) = \frac{1 + 3B_2^2}{4B_2^3}. \tag{86}$$

Note that for

$$Y_s \geq 0 \tag{87}$$

we have

$$B_2^2 \leq 1/3. \tag{88}$$

The optimum values of B obtained from (85) for various values of Y_s are listed in **Table 1**. The third column of the Table gives the corresponding values of E obtained from (86).

Figure 8 and shows a plot of the optimum B versus Y_s obtained from (85). In the next section we will see that Y_s is limited to a maximum of 3 by the emittance of the EBIS beam and the Booster acceptance. **Figure 9** shows a plot of optimum B versus Y_s for the range $0 \leq Y_s \leq 3$. For this range we have $0.26 < B < 0.58$.

10 Change in Closed Orbit Required to Avoid Hitting the Septum

In order to prevent injected beam from hitting the septum, the closed orbit must be moved away by amount $\Delta x + d$, where

$$\Delta x = \sqrt{\epsilon\beta} - x_s, \quad \epsilon = \epsilon_0 E, \quad x_s = Y_s\sqrt{\epsilon_0\beta} \tag{89}$$

Table 1: Optimum B and Other Parameters for Various Values of Y_s

Y_s	B	E	ΔY	x_s	ϵ	Δx	Q
0.0	0.57735	2.59808	1.61185	0.0	28.6	17.7	0.5000
0.2	0.53082	3.08440	1.55625	2.2	33.9	17.1	0.4065
0.4	0.49140	3.63310	1.50607	4.4	40.0	16.5	0.3691
0.6	0.45769	4.24622	1.46063	6.6	46.7	16.0	0.3415
0.8	0.42859	4.92545	1.41934	8.8	54.2	15.6	0.3192
1.0	0.40326	5.67226	1.38165	11.0	62.4	15.2	0.3004
1.2	0.38102	6.48786	1.34713	13.2	71.4	14.8	0.2842
1.4	0.36137	7.37331	1.31538	15.4	81.1	14.4	0.2701
1.6	0.34387	8.32950	1.28609	17.6	91.6	14.1	0.2575
1.8	0.32820	9.35723	1.25896	19.8	102.9	13.8	0.2463
2.0	0.31408	10.45718	1.23376	22.0	115.0	13.5	0.2362
2.2	0.30129	11.62993	1.21027	24.2	127.9	13.3	0.2270
2.4	0.28966	12.87602	1.18832	26.4	141.6	13.0	0.2187
2.6	0.27903	14.19592	1.16775	28.5	156.2	12.8	0.2110
2.8	0.26927	15.59003	1.14842	30.7	171.5	12.6	0.2039
3.0	0.26028	17.05873	1.13022	32.9	187.6	12.4	0.1974

and d is the septum thickness. Dividing Δx and d by $\sqrt{\epsilon_0\beta}$ we obtain dimensionless parameters

$$\Delta Y = \frac{\Delta x}{\sqrt{\epsilon_0\beta}} = \sqrt{E} - Y_s, \quad D = \frac{d}{\sqrt{\epsilon_0\beta}}. \tag{90}$$

The fourth column of **Table 1** gives the values of ΔY obtained from the first of equations (90) using the tabulated values of Y_s and E. Note that ΔY decreases as Y_s increases. This means that the amount the closed orbit has to be moved away from the septum decreases as the distance of the orbit from the septum increases.

Multiplying equations (90) by $\sqrt{\epsilon_0\beta}$ we have

$$\Delta x = \sqrt{\epsilon_0\beta}\,\Delta Y, \quad \Delta x + d = \sqrt{\epsilon_0\beta}\,\{\Delta Y + D\}. \tag{91}$$

Here we see that for a given ΔY, the amount the orbit has to be moved is proportional to $\sqrt{\epsilon_0}$ where $\pi\epsilon_0$ is the horizontal emittance (un-normalized) of the incoming beam. For EBIS beams we have $\pi\epsilon_0 = 11\pi$ mm milliradians and for Tandem beams $\pi\epsilon_0 = 1\pi$ mm milliradians. Thus, for a given ΔY, the amount the orbit has to be moved is a factor of $\sqrt{11} = 3.3$ larger for EBIS beams than it is for Tandem beams. Using $\pi\epsilon_0 = 11\pi$ mm milliradians in the second of equations (89) we obtain the values of ϵ (in mm milliradians) listed in **Table 1**.

Now the Courant-Snyder parameter β of the Booster lattice at the inflector exit is 10.96 meters (as determined by the MAD code). Thus we have (for EBIS beams)

$$\sqrt{\epsilon_0 \beta} = 10.98 \text{ mm.} \tag{92}$$

Using this in (89) and (91) we obtain the values of x_s and Δx listed in the table. The units of x_s and Δx are mm. Note that for a septum with thickness $d = 1$ mm, we have

$$D = \frac{d}{\sqrt{\epsilon_0 \beta}} = 0.0911. \tag{93}$$

Since the horizontal acceptance of the Booster is 185π mm milliradians we see from the table that Y_s can be no larger than 3 which gives a machine ellipse area of $\pi\epsilon = 187.6\pi$ mm milliradians.

Note that for Tandem beams with $\pi\epsilon_0 = 1\pi$ mm milliradians we have

$$\sqrt{\epsilon_0 \beta} = 3.31 \text{ mm} \tag{94}$$

and for $Y_s = 0$ we have $\Delta x = 5.34$ mm compared to $\Delta x = 17.7$ mm for EBIS beams.

11 Time Needed to Move Closed Orbit

In Reference [2] we find that a maximum angular kick of 6.71 milliradians is required in the injection dipoles to move the closed orbit by 47.5 mm at the inflector exit. The integrated strength of each dipole is

$$K = 1.33 \times 10^{-5} \text{ Tm/Amp} \tag{95}$$

and the angular kick (per Amp) delivered by the dipoles is

$$\psi = \frac{K}{B\rho} \tag{96}$$

where $B\rho$ is the magnetic rigidity of whatever ion is coming from EBIS. For Au^{32+} ions, the magnetic rigidity is

$$B\rho = 1.255 \text{ Tm} \tag{97}$$

which gives

$$\psi = \frac{K}{B\rho} = 0.010598 \text{ milliradians/Amp.} \tag{98}$$

The change in current required to move the orbit by 47.5 mm is then $6.71/\psi = 633$ Amps; this amounts to 13.3 Amps per mm. Now, the maximum rate at which the dipole current can be decreased is 12 Amps per microsecond. (This comes from a 1992 Memo from S.Y. Zhang which states that the change in current over 100 μs should be less than 1200 Amps.) Thus for Au^{32+} ions with a rigidity of 1.255 Tm, the maximum rate at which the closed orbit can be moved away from the inflector septum is

$$v = \frac{12}{13.3} = 0.902 \text{ mm}/\mu\text{s}. \tag{99}$$

The time needed to move the orbit by $\Delta x + d$ is then

$$\tau = \frac{\Delta x + d}{v}. \tag{100}$$

Taking the septum thickness to be $d = 1$ mm and using the values of Δx from **Table 1**, we find that $\Delta x + d$ ranges from 13.4 to 18.7 mm and τ ranges from 15 to 21 μs. Since the revolution period at injection is 10.3 μs, this amounts to 1.5 to 2 turns around the Booster. In the next section we will see that the horizontal tune can be adjusted so that the injected beam can go two to four turns around Booster before hitting the septum. This means that we should be able to move the orbit away from the septum fast enough for the injection of Au^{32+} ions and other ions from EBIS with rigidities less than or equal to 1.255 Tm.

Note that for protons from Tandem the rigidity is 0.6073 Tm. This gives $v = 1.86$ mm/μs for the maximum rate at which the orbit can be moved away from the septum. For $\Delta x = 5.34$ mm (obtained at the end of the previous section) and $d = 1$ mm, this gives $\tau = (\Delta x + d)/v = 3.4$ μs. Since the revolution period of the Tandem protons is 3.5 μs, this amounts to one turn around Booster compared to two turns for Au^{32+} ions from EBIS.

12 Optimum Horizontal Tune

As discussed in Reference [2], the number of turns that injected beam can go around the machine before hitting the septum depends on the horizontal tune. By adjusting the tune to maximize this number of turns, the closed orbit may be moved away from the septum more slowly. This gives more time for the currents in the injection dipoles to be reduced. Let

$$T_{11} = C + \alpha S, \quad T_{12} = \beta S \tag{101}$$

where

$$C = \cos\mu, \quad S = \sin\mu, \quad \mu = 2\pi Q \tag{102}$$

and Q is the horizontal tune. Then the horizontal position of the beam ellipse center (with respect to the closed orbit) after one turn around the machine is

$$x_1 = T_{11}x_0 + T_{12}x_0'. \tag{103}$$

The beam ellipse beta parameter after one turn is

$$\beta_1 = T_{11}^2\beta_0 - 2T_{11}T_{12}\alpha_0 + T_{12}^2\gamma_0 \tag{104}$$

which gives

$$\beta_0\beta_1 = \{T_{11}\beta_0 - T_{12}\alpha_0\}^2 + T_{12}^2. \tag{105}$$

Using

$$x_0' = -\alpha x_0/\beta, \quad \alpha_0 = \alpha B, \quad \beta_0 = \beta B \tag{106}$$

we then have

$$x_1 = (C + \alpha S)x_0 - \beta S(\alpha x_0/\beta) = Cx_0 \tag{107}$$

$$T_{11}\beta_0 - T_{12}\alpha_0 = (C + \alpha S)\beta B - \beta S\alpha B = C\beta B \tag{108}$$

and

$$\beta_1 = \frac{1}{\beta B}\left\{C^2\beta^2B^2 + \beta^2S^2\right\} = \beta\left\{C^2B + S^2/B\right\}. \tag{109}$$

Now, in order for the beam ellipse to just clear the septum after one turn around the machine we must have

$$x_1 + \sqrt{\epsilon_0\beta_1} = x_s - d. \tag{110}$$

This is the condition that allows the injected beam to go the most number of turns before hitting the septum. Using (107) and (109) in (110) we have

$$Cx_0 + \sqrt{\epsilon_0\beta}\left\{C^2B + S^2/B\right\}^{1/2} = x_s - d \tag{111}$$

and dividing by $\sqrt{\epsilon_0\beta}$ gives

$$CY_0 + \left\{C^2B + S^2/B\right\}^{1/2} = Y_s - D \tag{112}$$

where

$$D = \frac{d}{\sqrt{\epsilon_0 \beta}}. \tag{113}$$

Using $Y_0 = Y_s + \sqrt{B}$ we have finally

$$C\left\{Y_s + \sqrt{B}\right\} + \left\{C^2 B + S^2/B\right\}^{1/2} = Y_s - D. \tag{114}$$

For given values of Y_s, B, and D, this equation can be solved numerically to obtain $\mu = 2\pi Q$. The values of Q obtained for various values of Y_s and B are listed in **Table 1**. Here it has been assumed that $D = 0$. The number of turns N that injected beam can go around the machine before hitting the septum is given by

$$N < 1/Q < N + 1 \tag{115}$$

where

$$0 < Q < 0.5. \tag{116}$$

For the tunes listed in the table, N ranges from 2 to 5. For the case in which $Y_S = 2.0$, $B = 0.314077$, and $D = 0.0911$ we obtain $Q = 0.2421$ which gives $N = 4$. The positions of the beam ellipse on the first, second, and third turns around the machine in this case are shown in **Figure 10**.

13 Horizontal Acceptance of the Inflector

The Inflector is described in Reference [1]. The nominal path length is

$$L = 2.37274 \text{ meters} \tag{117}$$

and the radius of curvature is

$$\rho = 8.74123 \text{ meters}. \tag{118}$$

This gives turning angle

$$\theta = L/\rho = 15.5525^\circ. \tag{119}$$

The horizontal half-aperture is

$$a = 8.5 \text{ mm}. \tag{120}$$

The beam ellipse parameters at the inflector entrance (i.e. the upstream end) are given by

$$\beta_1 = M_{11}^2 \beta_0 - 2M_{11}M_{12}\alpha_0 + M_{12}^2 \gamma_0 \tag{121}$$

$$\alpha_1 = -M_{11}M_{21}\beta_0 + (1 + 2M_{12}M_{21})\alpha_0 - M_{12}M_{22}\gamma_0 \tag{122}$$

$$\gamma_1 = M_{21}^2 \beta_0 - 2M_{21}M_{22}\alpha_0 + M_{22}^2 \gamma_0 \tag{123}$$

where α_0, β_0, γ_0 are the beam ellipse parameters at the inflector exit, and

$$M_{11} = \cos\theta, \quad M_{12} = -\rho \sin\theta \tag{124}$$

$$M_{21} = \frac{1}{\rho} \sin\theta, \quad M_{22} = \cos\theta. \tag{125}$$

Here M_{11}, M_{12}, M_{21}, M_{22} are the elements of the transfer matrix from the inflector exit *backwards* to the entrance. Note that for the given values of L and ρ, the transfer matrix is approximately that of a drift of length L. For a given B, the beam ellipse parameters at the inflector exit are given by

$$\alpha_0 = \alpha B, \quad \beta_0 = \beta B, \quad \gamma_0 = (1 + \alpha_0^2)/\beta_0 \tag{126}$$

where α and β are the Courant-Snyder parameters of the Booster lattice at the inflector exit. The MAD code gives $\alpha = -1.736$ and $\beta = 10.96$ meters.

In order for the incoming beam to fit through the inflector channel we must have

$$\sqrt{\epsilon_0 \beta_0} < a, \quad \sqrt{\epsilon_0 \beta_1} < a \tag{127}$$

where $\pi\epsilon_0$ is the area of incoming beam ellipse.

Let

$$e_0 = a^2/\beta_0, \quad e_1 = a^2/\beta_1. \tag{128}$$

These are shown as functions of B in **Figure 11**; the red curve is e_0 and the blue curve is e_1. The two curves intersect at $B = 0.265$ which gives the maximum acceptance

$$\pi e_0 = \pi e_1 = 24.8\pi \tag{129}$$

mm milliradians. For $B < 0.265$, the acceptance is given by the blue curve. For $B > 0.265$, the acceptance is given by the red curve. Note that for $B = 1$, the beam ellipse is matched to the machine ellipse at the inflector exit. In this case the acceptance of the inflector channel is only 6.59π mm milliradians.

In Sections 9 and 10 we found that the usable values of B range from around 0.26 to 0.58 as shown in **Figure 9**. The red and blue acceptance curves for this range are shown in **Figure 12**. Here we see that for this range, a beam ellipse with emittance $\pi\epsilon_0 = 11\pi$ mm milliradians will fit through the inflector channel.

14 Vertical Acceptance of the Inflector

Let us assume that the incoming beam ellipse is vertically matched to the machine lattice at the inflector exit. The vertical Courant-Snyder parameters at this point (as obtained by the MAD code) are

$$\beta = 4.761 \text{ meters}, \quad \alpha = 0.823, \quad \gamma = \frac{1+\alpha^2}{\beta}. \tag{130}$$

The beam ellipse parameter beta at the inflector entrance is then

$$\beta_1 = \beta + 2\alpha L + L^2\gamma \tag{131}$$

where $L = 2.37274$ meters is the drift length from entrance to exit. (In the vertical plane the inflector is simply a drift.) This gives

$$\beta_1 = 10.650 \text{ meters}. \tag{132}$$

The vertical half-aperture a of the inflector channel is set by the height of the uniform field region between the cathode and septum. AGS Department Drawings D36-M-2256 and D36-M-2009 show that the height of the cathode is 3 inches (76.2 mm). A conservative estimate of the height of the uniform field region is 42 mm. The vertical half-aperture is then $a = 21$ mm which gives vertical acceptance

$$\pi a^2/\beta_1 = 41\pi \text{ mm milliradians}. \tag{133}$$

This is more than adequate to accept the nominal vertical emittance $\pi\epsilon_0 = 11\pi$ mm milliradians of EBIS beams.

Note that, as shown in Reference [3], the maximum acceptance of a drift of length L and half-aperture a is

$$\pi e = \pi a^2/L. \tag{134}$$

Putting in numbers for the inflector gives a maximum vertical acceptance of 186π mm milliradians which is more than twice the vertical acceptance

(87π mm milliradians) of the Booster. The maximum acceptance occurs when the beam ellipse parameter beta has a local minimum of $L/2$ halfway through the inflector. The beam ellipse parameters at the inflector entrance are then

$$\alpha_1 = 1, \quad \beta_1 = L, \quad \gamma_1 = 2/L \tag{135}$$

and those at the exit are

$$\alpha_0 = -1, \quad \beta_0 = L, \quad \gamma_0 = 2/L. \tag{136}$$

The mismatch factor [4, 5] between the beam ellipse and lattice in this case is

$$F = H + \sqrt{H^2 - 1} \tag{137}$$

where

$$H = \frac{1}{2}\left\{\gamma\beta_0 - 2\alpha\alpha_0 + \beta\gamma_0\right\}. \tag{138}$$

Putting in numbers gives

$$H = 3.2475, \quad F = 6.3372. \tag{139}$$

15 Dispersion in the Inflector

Let d_1 and d_1' be the dispersion and its derivative at the upstream end of the inflector. The dispersion $d(s)$ and its derivative $d'(s)$ a distance s along the central trajectory of the inflector are then

$$d(s) = M_{11}d_1 + M_{12}d_1' + M_{13} \tag{140}$$

$$d'(s) = M_{21}d_1 + M_{22}d_1' + M_{23} \tag{141}$$

where [6, 7]

$$M_{11} = \cos\theta, \quad M_{12} = \rho\sin\theta, \quad M_{13} = \rho(1-\cos\theta) \tag{142}$$

$$M_{21} = -\frac{1}{\rho}\sin\theta, \quad M_{22} = \cos\theta, \quad M_{23} = \sin\theta \tag{143}$$

and

$$\theta = s/\rho. \tag{144}$$

Here $\rho = 8.74123$ (meters) and s ranges from 0 at the upstream end of the inflector to $L = 2.37274$ (meters) at the downstream end. Setting $\theta = L/\rho$

in (142) and (143), gives the dispersion and its derivative at the inflector exit,

$$d_0 = M_{11}d_1 + M_{12}d_1' + M_{13} \tag{145}$$

$$d_0' = M_{21}d_1 + M_{22}d_1' + M_{23}. \tag{146}$$

Let us first consider the case in which $d_0 = d_0' = 0$. The positions and angles of incoming beam particles at the inflector exit are then independent of momentum and we have

$$M_{11}d_1 + M_{12}d_1' + M_{13} = 0 \tag{147}$$

$$M_{21}d_1 + M_{22}d_1' + M_{23} = 0. \tag{148}$$

Solving for d_1 and d_1' we obtain

$$d_1 = M_{12}M_{23} - M_{22}M_{13} \tag{149}$$

$$d_1' = M_{21}M_{13} - M_{11}M_{23} \tag{150}$$

and using

$$M_{12}M_{23} - M_{22}M_{13} = \rho(1 - \cos\theta) = M_{13} \tag{151}$$

$$M_{21}M_{13} - M_{11}M_{23} = -\sin\theta = -M_{23} \tag{152}$$

we have

$$d_1 = M_{13} = 0.3201 \text{ m}, \quad d_1' = -M_{23} = -0.2681. \tag{153}$$

These are the maximum values attained by $d(s)$ and $d'(s)$. Since the fractional momentum spread $\Delta p/p$ of EBIS beams can be as large as ± 0.001, we see that $d(s)\Delta p/p$ can be at most ± 0.3201 mm. This is a small fraction of the inflector horizontal half-aperture $a = 8.5$ mm.

Now consider the case in which the dispersion at the inflector exit is matched to the periodic dispersion of the lattice. We then have

$$d_0 = M_{11}d_1 + M_{12}d_1' + M_{13} = -D \tag{154}$$

$$d_0' = M_{21}d_1 + M_{22}d_1' + M_{23} = -D' \tag{155}$$

where D and D' are the periodic dispersion and its derivative. Here we have put the negative of D and D' in (154) and (155) because the direction of positive dispersion (radially outward) in Booster is opposite to the direction of positive dispersion in the inflector. Solving these equations for d_1 and d_1' we obtain

$$d_1 = (M_{12}M_{23} - M_{22}M_{13}) - M_{22}D + M_{12}D' \tag{156}$$

$$d_1' = (M_{21}M_{13} - M_{11}M_{23}) + M_{21}D - M_{11}D'. \tag{157}$$

Using (151) and (152) we then have

$$d_1 = M_{13} - M_{22}D + M_{12}D' \tag{158}$$

$$d_1' = -M_{23} + M_{21}D - M_{11}D' \tag{159}$$

and using (142) and (143) these become

$$d_1 = \rho(1 - \cos\theta) - D\cos\theta + \rho D' \sin\theta \tag{160}$$

$$d_1' = -\sin\theta - (D/\rho)\sin\theta - D'\cos\theta. \tag{161}$$

Then using the values

$$D = 2.521 \text{ m}, \quad D' = 0.408 \tag{162}$$

obtained by the MAD code we have

$$d_1 = -1.152 \text{ m}, \quad d_1' = -0.738. \tag{163}$$

Thus we see that $d(s)$ ranges from -1.152 m at the inflector entrance to -2.521 m at the exit. For EBIS beams with $\Delta p/p = \pm 0.001$ the maximum $d(s)\Delta p/p$ is ± 2.521 mm at the inflector exit, which is a significant fraction of the inflector horizontal half-aperture $a = 8.5$ mm.

16 Appendix I. The Critical Points of $E(\phi)$

We wish to find the maximum value attained by $E(\phi)$ as ϕ varies between 0 and 2π. In order to do so we need to examine the critical points given by

$$\frac{dE}{d\phi} = 0. \tag{164}$$

We have

$$E(\phi) = Y_0^2 + 2Y_0 R\cos\phi + R^2 \tag{165}$$

where

$$R^2 F(\phi) = 2 \tag{166}$$

and

$$F(\phi) = (G + B) + (G - B)\cos 2\phi, \quad G = 1/B. \tag{167}$$

Differentiating E with respect to ϕ we have

$$\frac{dE}{d\phi} = 2R\,\frac{dR}{d\phi} + 2Y_0\left\{\frac{dR}{d\phi}\cos\phi - R\sin\phi\right\} \tag{168}$$

$$\frac{dE}{d\phi} = 2\left\{R + Y_0\cos\phi\right\}\frac{dR}{d\phi} - 2Y_0 R\sin\phi \tag{169}$$

and

$$\frac{1}{R}\frac{dE}{d\phi} = 2\left\{R + Y_0\cos\phi\right\}\frac{1}{R}\frac{dR}{d\phi} - 2Y_0\sin\phi. \tag{170}$$

Differentiating $R^2F(\phi) = 2$ we have

$$2R\frac{dR}{d\phi}F + R^2\frac{dF}{d\phi} = 0 \tag{171}$$

which gives

$$\frac{1}{R}\frac{dR}{d\phi} = -\frac{1}{2}\,\frac{1}{F}\frac{dF}{d\phi}. \tag{172}$$

Thus we have

$$\frac{1}{R}\frac{dE}{d\phi} = -\left\{R + Y_0\cos\phi\right\}\frac{1}{F}\frac{dF}{d\phi} - 2Y_0\sin\phi. \tag{173}$$

The critical points of $E(\phi)$ are then given by

$$\left\{R + Y_0\cos\phi\right\}\frac{dF}{d\phi} + 2FY_0\sin\phi = 0 \tag{174}$$

where

$$\frac{dF}{d\phi} = 2(B-G)\sin 2\phi, \quad F = (G+B) + (G-B)\cos 2\phi. \tag{175}$$

Using

$$\sin 2\phi\cos\phi - \cos 2\phi\sin\phi = \sin\phi \tag{176}$$

we then have

$$R\,\frac{dF}{d\phi} + 4Y_0 B\sin\phi = 0 \tag{177}$$

and

$$R(B-G)\sin 2\phi + 2Y_0 B\sin\phi = 0. \tag{178}$$

Using

$$\sin 2\phi = 2\cos\phi\sin\phi \tag{179}$$

this becomes

$$\{R(B-G)\cos\phi + Y_0 B\}\sin\phi = 0. \tag{180}$$

The critical points are then given by

$$\sin\phi = 0 \tag{181}$$

and by

$$R(B-G)\cos\phi + Y_0 B = 0. \tag{182}$$

The first equation implies

$$\cos\phi = \pm 1, \quad \cos 2\phi = 1. \tag{183}$$

In the second equation we have

$$R^2 = \frac{2}{F(\phi)}, \quad F(\phi) = (G+B) + (G-B)\cos 2\phi \tag{184}$$

and therefore

$$\frac{2(G-B)^2\cos^2\phi}{(G+B)+(G-B)\cos 2\phi} = Y_0^2 B^2. \tag{185}$$

Using

$$2\cos^2\phi = 1 + \cos 2\phi \tag{186}$$

we have

$$\frac{(G-B)^2 + (G-B)^2\cos 2\phi}{(G+B)+(G-B)\cos 2\phi} = Y_0^2 B^2 \tag{187}$$

which gives

$$\cos 2\phi = \frac{Y_0^2 B^2 (G+B) - (G-B)^2}{(G-B)^2 - Y_0^2 B^2 (G-B)} \tag{188}$$

and

$$1 + \cos 2\phi = \left(\frac{2B}{G-B}\right)\left\{\frac{Y_0^2 B^2}{(G-B) - Y_0^2 B^2}\right\}. \tag{189}$$

Thus we must have

$$0 \leq \left(\frac{B}{G-B}\right)\left\{\frac{Y_0^2 B^2}{(G-B) - Y_0^2 B^2}\right\} \leq 1. \tag{190}$$

If this is true then the critical points are

$$\phi = \pm\frac{1}{2}\arccos\left\{\frac{Y_0^2 B^2 (G+B) - (G-B)^2}{(G-B)^2 - Y_0^2 B^2 (G-B)}\right\}. \tag{191}$$

We also have

$$\cos^2\phi = \left(\frac{B}{G-B}\right)\left\{\frac{Y_0^2 B^2}{(G-B) - Y_0^2 B^2}\right\}. \tag{192}$$

17 Appendix II. $E(\phi)$ at the Critical Points

For the critical points given by

$$\sin\phi = 0 \tag{193}$$

we have $\cos 2\phi = 1$ and the equation

$$F(\phi) = (G+B) + (G-B)\cos 2\phi \tag{194}$$

becomes

$$F = 2G = 2/B. \tag{195}$$

This gives

$$R^2 = 2/F = B, \quad R = \sqrt{B} \tag{196}$$

and the equation

$$E(\phi) = Y_0^2 + 2Y_0 R\cos\phi + R^2 \tag{197}$$

becomes

$$E = Y_0^2 \pm 2Y_0\sqrt{B} + B = \left\{Y_0 \pm \sqrt{B}\right\}^2 \tag{198}$$

where the upper and lower signs are for $\cos\phi = +1$ and $\cos\phi = -1$ respectively.

For the critical points

$$\phi = \pm\frac{1}{2}\arccos\left\{\frac{Y_0^2B^2(G+B) - (G-B)^2}{(G-B)^2 - Y_0^2B^2(G-B)}\right\} \tag{199}$$

we have

$$\cos 2\phi = \frac{Y_0^2B^2(G+B) - (G-B)^2}{(G-B)^2 - Y_0^2B^2(G-B)} \tag{200}$$

and the equation

$$F(\phi) = (G+B) + (G-B)\cos 2\phi \tag{201}$$

becomes

$$F = G + B - \frac{(G-B)^2 - Y_0^2B^2(G+B)}{(G-B) - Y_0^2B^2} \tag{202}$$

$$F = \frac{2B(G-B)}{(G-B) - Y_0^2B^2}. \tag{203}$$

Thus

$$R^2 = \frac{2}{F} = \frac{(G-B) - Y_0^2B^2}{B(G-B)}. \tag{204}$$

We also have

$$R(B-G)\cos\phi + Y_0 B = 0. \tag{205}$$

Thus the equation

$$E(\phi) = Y_0^2 + 2Y_0 R\cos\phi + R^2 \tag{206}$$

becomes

$$E = Y_0^2 + \frac{2Y_0^2 B^2}{B(G-B)} + \frac{(G-B) - Y_0^2 B^2}{B(G-B)} \tag{207}$$

$$E = \frac{BGY_0^2 + G - B}{B(G-B)} = \frac{1}{B} + \frac{GY_0^2}{G-B}. \tag{208}$$

Using $BG = 1$ this becomes

$$E = \frac{1}{B} + \frac{Y_0^2}{1-B^2} = \frac{1 - B^2 + BY_0^2}{B(1-B^2)}. \tag{209}$$

18 Appendix III. Analysis of the Critical Points

To determine whether $E(\phi)$ has a local minimum or maximum at the critical points we must evaluate the second derivative. We have

$$\frac{dE}{d\phi} = 2\left\{R + Y_0\cos\phi\right\}\frac{dR}{d\phi} - 2Y_0 R\sin\phi \tag{210}$$

and

$$\frac{d^2E}{d\phi^2} = D_1 + D_2 \tag{211}$$

where

$$D_1 = 2\left\{\frac{dR}{d\phi} - Y_0\sin\phi\right\}\frac{dR}{d\phi} + 2\left\{R + Y_0\cos\phi\right\}\frac{d^2R}{d\phi^2} \tag{212}$$

$$D_2 = -2Y_0\frac{dR}{d\phi}\sin\phi - 2Y_0 R\cos\phi. \tag{213}$$

Here

$$\frac{dR}{d\phi} = -\frac{1}{2}\frac{R}{F}\frac{dF}{d\phi} \tag{214}$$

$$\frac{d^2R}{d\phi^2} = -\frac{1}{2}\left\{\frac{1}{F}\frac{dR}{d\phi} - \frac{R}{F^2}\frac{dF}{d\phi}\right\}\frac{dF}{d\phi} - \frac{1}{2}\frac{R}{F}\frac{d^2F}{d\phi^2} \tag{215}$$

$$R^2 = 2/F, \quad F = (G+B) + (G-B)\cos 2\phi \tag{216}$$

and

$$\frac{dF}{d\phi} = 2(B-G)\sin 2\phi, \quad \frac{d^2F}{d\phi^2} = 4(B-G)\cos 2\phi. \tag{217}$$

Thus, at the critical point given by

$$\sin\phi = 0 \tag{218}$$

we have $\cos 2\phi = 1$ and

$$\frac{dF}{d\phi} = 0, \quad \frac{dR}{d\phi} = 0, \quad \frac{d^2F}{d\phi^2} = 4(B-G) \tag{219}$$

$$F = 2G = \frac{2}{B}, \quad R^2 = \frac{2}{F} = B, \quad \frac{R}{F} = \frac{1}{2}B\sqrt{B} \tag{220}$$

$$\frac{d^2R}{d\phi^2} = -\frac{1}{2}\frac{R}{F}\frac{d^2F}{d\phi^2} = B\sqrt{B}\left\{\frac{1}{B} - B\right\} = \sqrt{B}(1-B^2). \tag{221}$$

This gives

$$\frac{d^2E}{d\phi^2} = 2(R \pm Y_0)\frac{d^2R}{d\phi^2} \mp 2Y_0R \tag{222}$$

$$\frac{d^2E}{d\phi^2} = 2\sqrt{B}\left\{\sqrt{B} \pm Y_0\right\}\left\{1 - B^2\right\} \mp 2Y_0\sqrt{B} \tag{223}$$

$$\frac{d^2E}{d\phi^2} = 2\sqrt{B}\left\{\sqrt{B} \pm Y_0 - B^2\sqrt{B} \mp B^2Y_0 \mp Y_0\right\} \tag{224}$$

$$\frac{d^2E}{d\phi^2} = 2\sqrt{B}\left\{\sqrt{B} - B^2\sqrt{B} \mp B^2Y_0\right\} \tag{225}$$

$$\frac{d^2E}{d\phi^2} = 2B\left\{1 - B^2 \mp B\sqrt{B}Y_0\right\} \tag{226}$$

and

$$\frac{d^2E}{d\phi^2} = 2B\left\{1 - B^{3/2}\left(\sqrt{B} \pm Y_0\right)\right\} \tag{227}$$

where the upper and lower signs are for $\cos\phi = +1$ and $\cos\phi = -1$ respectively.

We assume that

$$Y_0 > 0. \tag{228}$$

Then for $\cos\phi = -1$ and $B \le 1$ we have

$$\frac{d^2E}{d\phi^2} = 2B\left\{1 - B^{3/2}\left(\sqrt{B} - Y_0\right)\right\} > 0 \tag{229}$$

and it follows that $E(\phi)$ has a minimum at the critical point $\phi = \pi$.

For $\cos\phi = 1$ and $B^{3/2}\left(\sqrt{B} + Y_0\right) > 1$ we have

$$\frac{d^2E}{d\phi^2} = 2B\left\{1 - B^{3/2}\left(\sqrt{B} + Y_0\right)\right\} < 0 \tag{230}$$

and $E(\phi)$ has a maximum at the critical point $\phi = 0$. In this case the maximum is

$$E = \left\{Y_0 + \sqrt{B}\right\}^2. \tag{231}$$

For $\cos\phi = 1$ and $B^{3/2}\left(Y_0 + \sqrt{B}\right) < 1$ we have

$$\frac{d^2E}{d\phi^2} = 2B\left\{1 - B^{3/2}\left(\sqrt{B} + Y_0\right)\right\} > 0 \tag{232}$$

and $E(\phi)$ has a minimum at the critical point $\phi = 0$. Since $E(\phi)$ also has a minimum at $\phi = \pi$ it must then reach a maximum for some ϕ between 0 and π. This is the critical point given by

$$\cos 2\phi = \frac{Y_0^2 B^2(G+B) - (G-B)^2}{(G-B)^2 - Y_0^2 B^2(G-B)}. \tag{233}$$

In this case the maximum is

$$E = \frac{1 - B^2 + BY_0^2}{B(1-B^2)}. \tag{234}$$

Note that the condition $B^{3/2}\left(Y_0 + \sqrt{B}\right) < 1$ implies

$$Y_0 B < \frac{1}{\sqrt{B}}(1-B^2), \quad Y_0^2 B^2 < \frac{1}{B}(1-B^2)^2 \tag{235}$$

and it follows that

$$0 \le \left(\frac{B}{G-B}\right)\left\{\frac{Y_0^2 B^2}{(G-B) - Y_0^2 B^2}\right\} \le 1 \tag{236}$$

as required.

References

[1] C.J. Gardner, “Booster Inflector Specifications”, Booster Technical Note 159, February 28, 1990.

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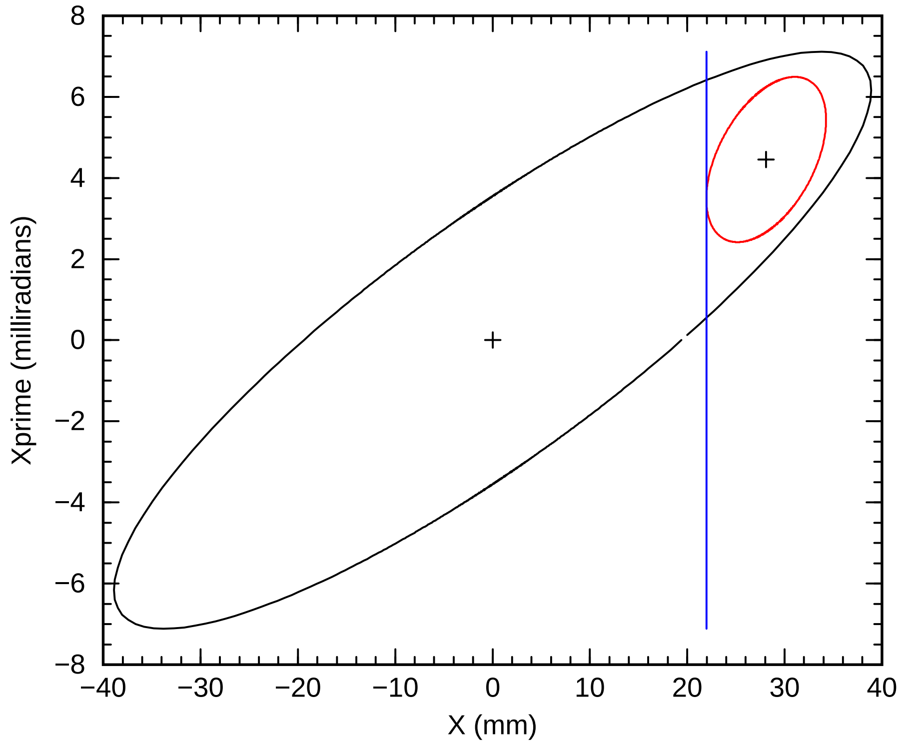

Figure 1: Beam and Machine Ellipses at Inflector Exit. Here the black ellipse is the machine ellipse and the red ellipse is the beam ellipse. The machine ellipse is matched to the machine lattice at the inflector exit and is centered on the closed orbit. (The ellipse centers are marked with plus signs.) The vertical blue line shows the position of the inflector septum.

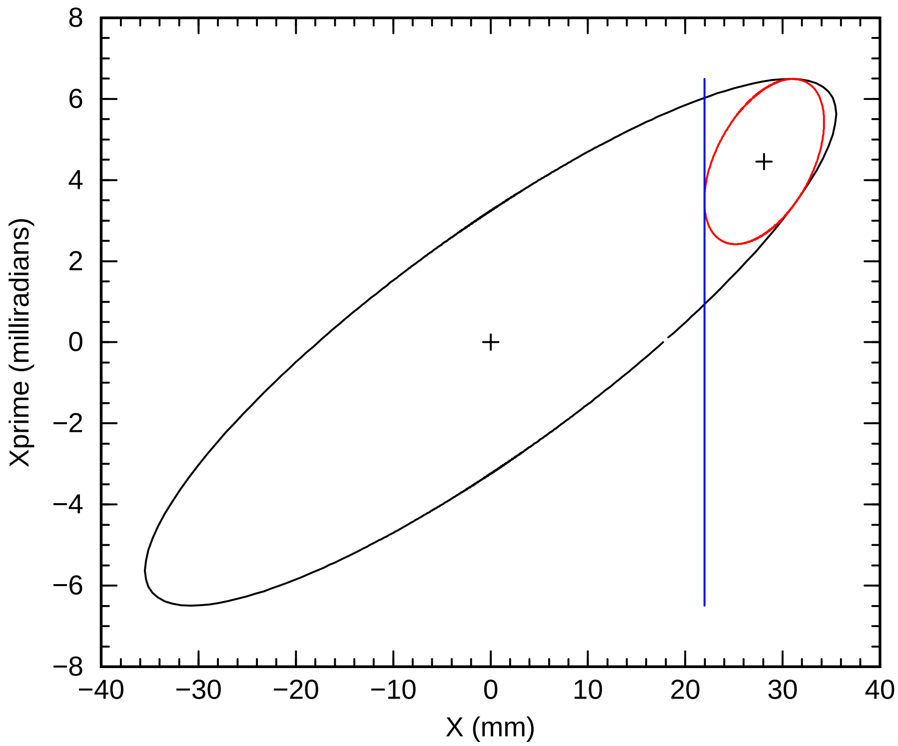

Figure 2: Beam and Machine Ellipses at Inflector Exit. Here the machine ellipse area $\pi\epsilon$ has been adjusted to give the smallest machine ellipse that contains the beam ellipse. We call this the minimum machine ellipse. The size of the minimum machine ellipse depends on the size and shape of the beam ellipse. On subsequent turns around the machine the beam ellipse remains inside the machine ellipse; eventually it will hit the septum unless the closed orbit is moved away. The amount the orbit has to be moved depends on the size of the minimum machine ellipse.

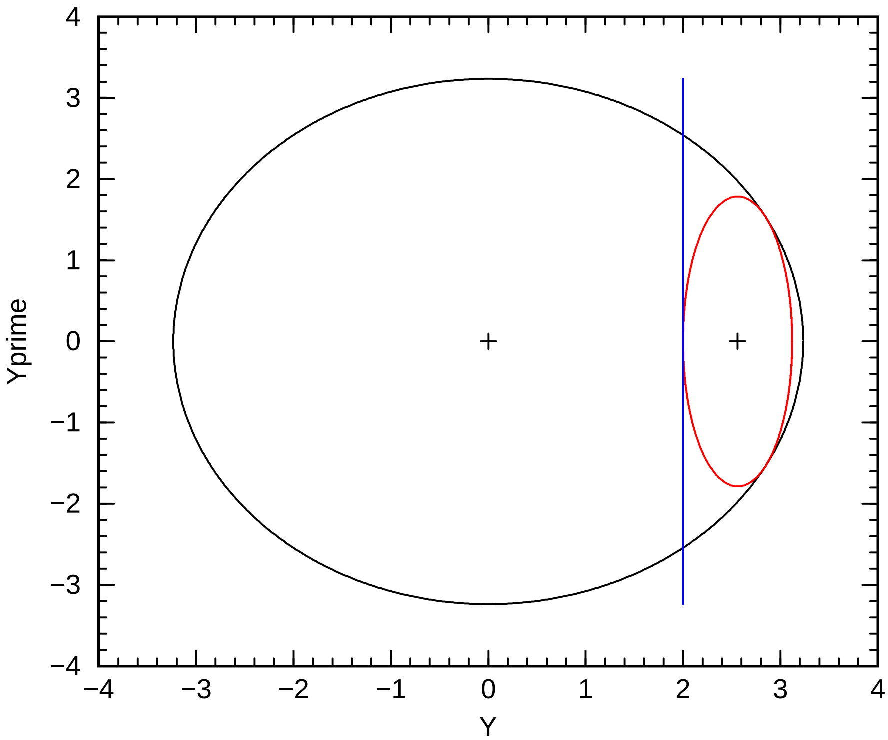

Figure 3: These are the ellipses of **Figure 2** in terms of the dimensionless coordinates introduced in Section 3. In these coordinates the minimum machine ellipse becomes a circle with area πE. The area of the beam ellipse becomes π and its half-width becomes $\sqrt{B}$. Here the parameter $B = 0.314077$ and the center of the beam ellipse is $Y_0 = 2 + \sqrt{B}$. In this case we have $B^{3/2}\left\{Y_0 + \sqrt{B}\right\} = 0.5493 < 1$ and the minimum machine ellipse is given by the circle $E = Y^2 + Y'^2 = \left\{1 - B^2 + BY_0^2\right\} / \left\{B(1 - B^2)\right\} = 10.4572$.

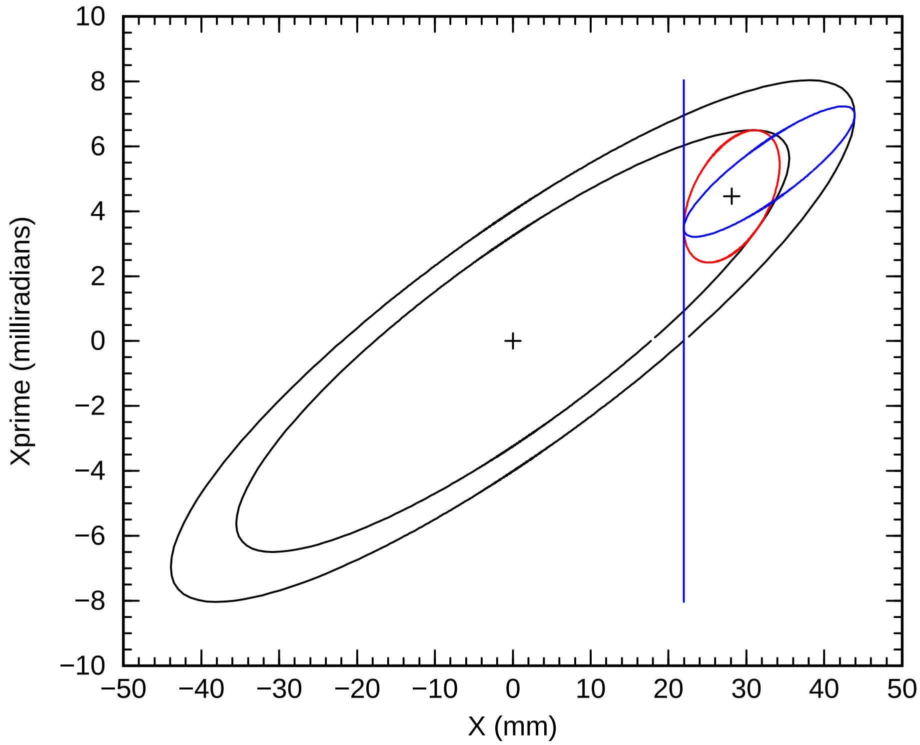

Figure 4: Beam and Machine Ellipses at Inflector Exit. Here the two beam ellipses have the same area $\pi\epsilon_0$. The blue ellipse is matched to the lattice; the red one is not. The minimum machine ellipse containing the unmatched red ellipse is significantly smaller than the minimum machine ellipse containing the matched blue ellipse.

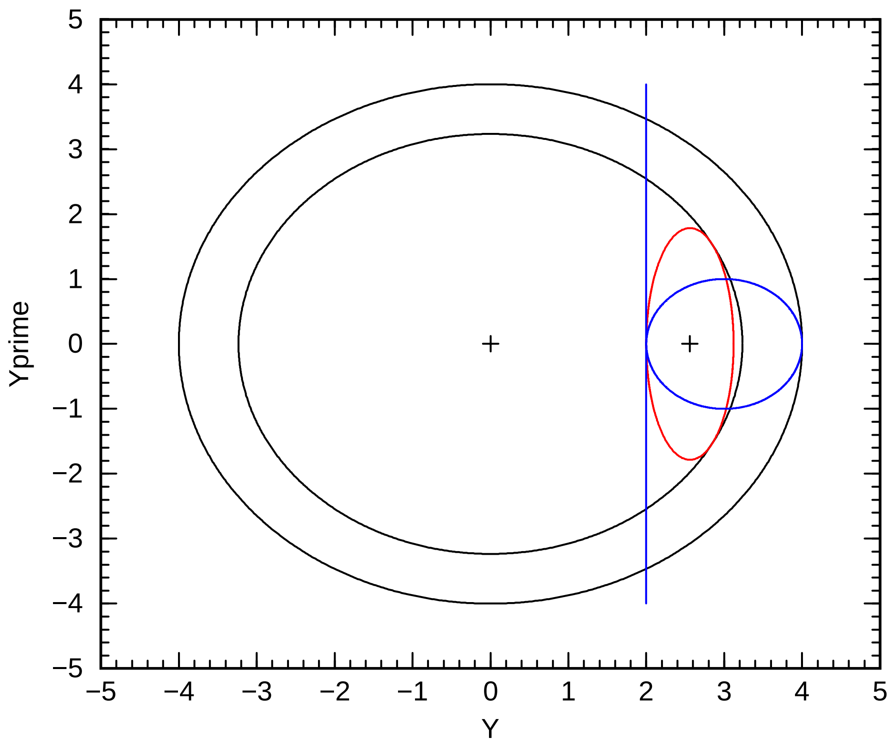

Figure 5: These are the ellipses of **Figure 4** in terms of the dimensionless coordinates introduced in Section 3. In these coordinates the minimum machine ellipses are circles, the area of each beam ellipse is π, and the matched blue ellipse is a circle. The half-widths of the beam ellipses are $\sqrt{B}$ where $B = 1$ for the matched ellipse and $B = 0.314077$ for the red ellipse.

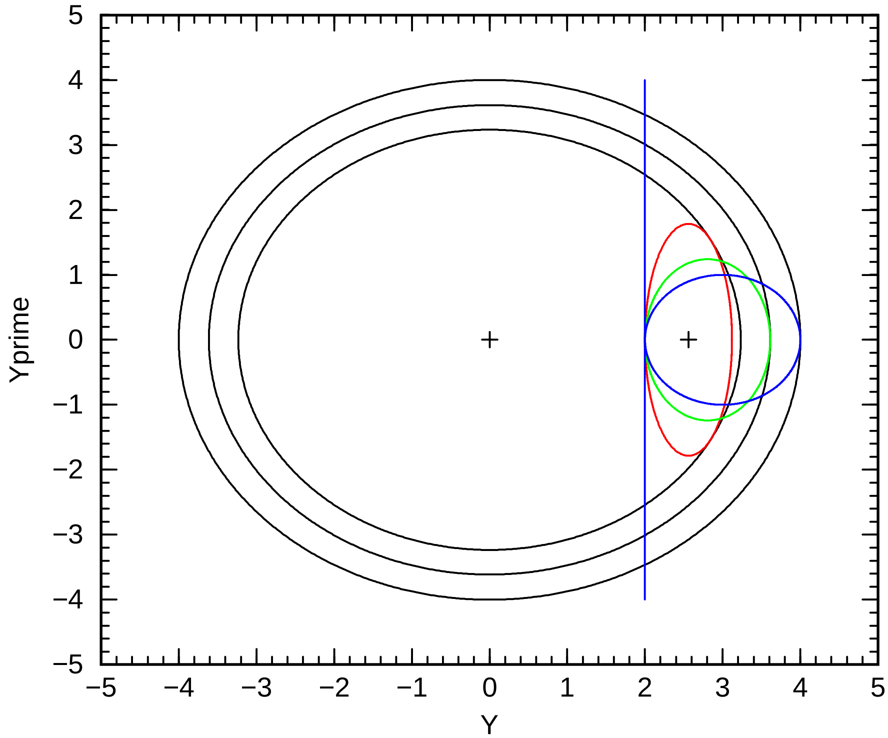

Figure 6: Beam ellipses and the corresponding minimum machine ellipses for the case in which $Y_s = 2$. Here the values of B for the Blue, Green, and Red beam ellipses are 1, 0.65, and 0.314077 respectively. The minimum machine ellipse is smallest for the beam ellipse with $B = 0.314077$.

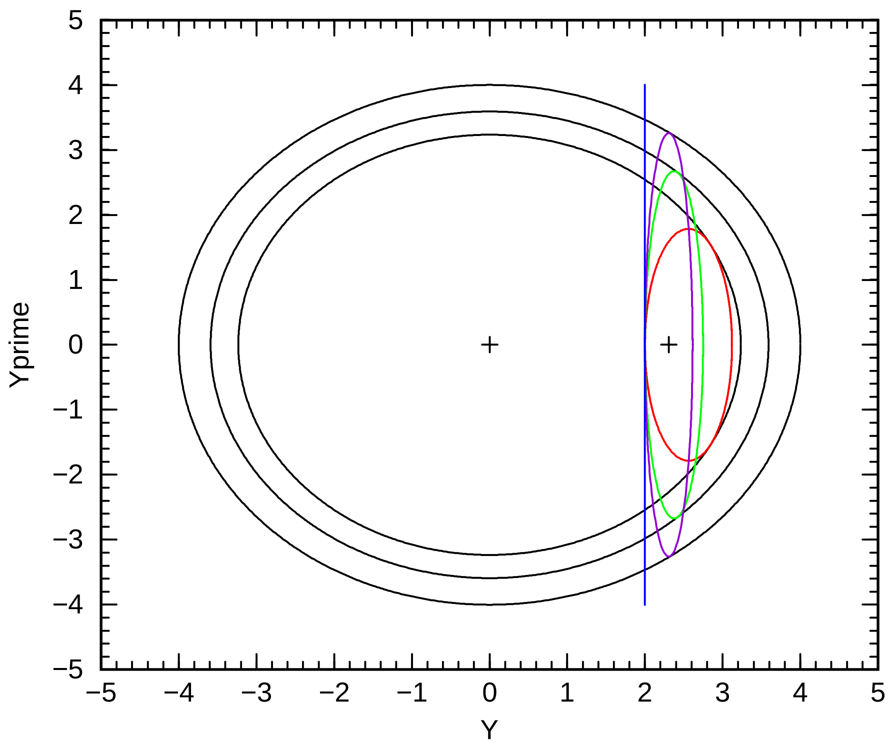

Figure 7: Beam ellipses and the corresponding minimum machine ellipses for the case in which $Y_s = 2$. The values of B for the Red, Green, and Violet beam ellipses are 0.314077, 0.14, and 0.09405 respectively. Here again the minimum machine ellipse is smallest for the beam ellipse with $B = 0.314077$.

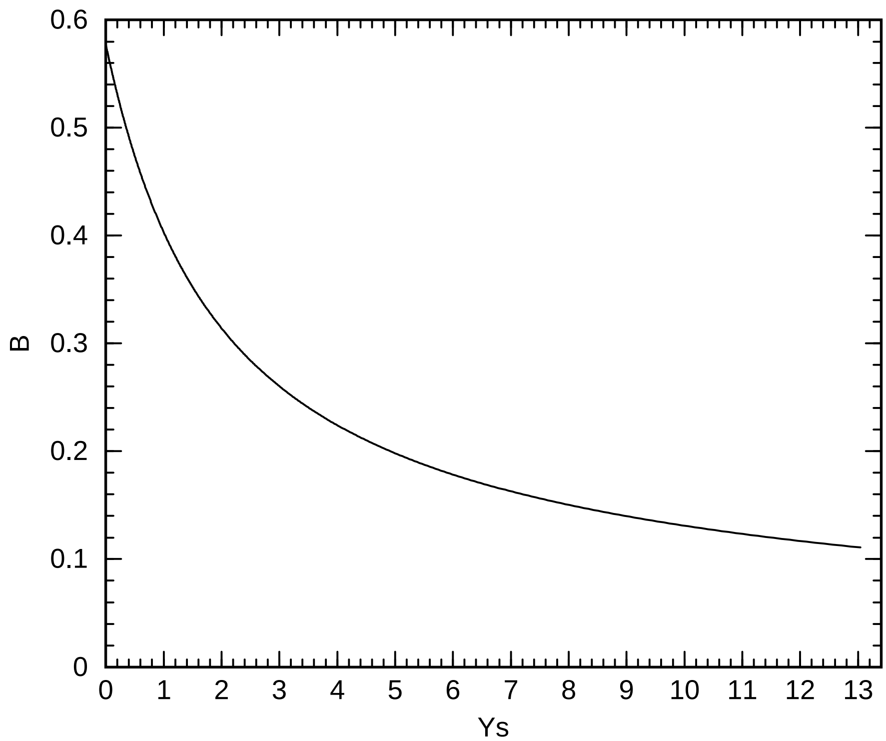

Figure 8: Optimum B versus Y_s obtained from the equation $2Y_sB^{3/2}+3B^2 = 1$. For $Y_s = 0$ we have $B = 1/\sqrt{3}$.

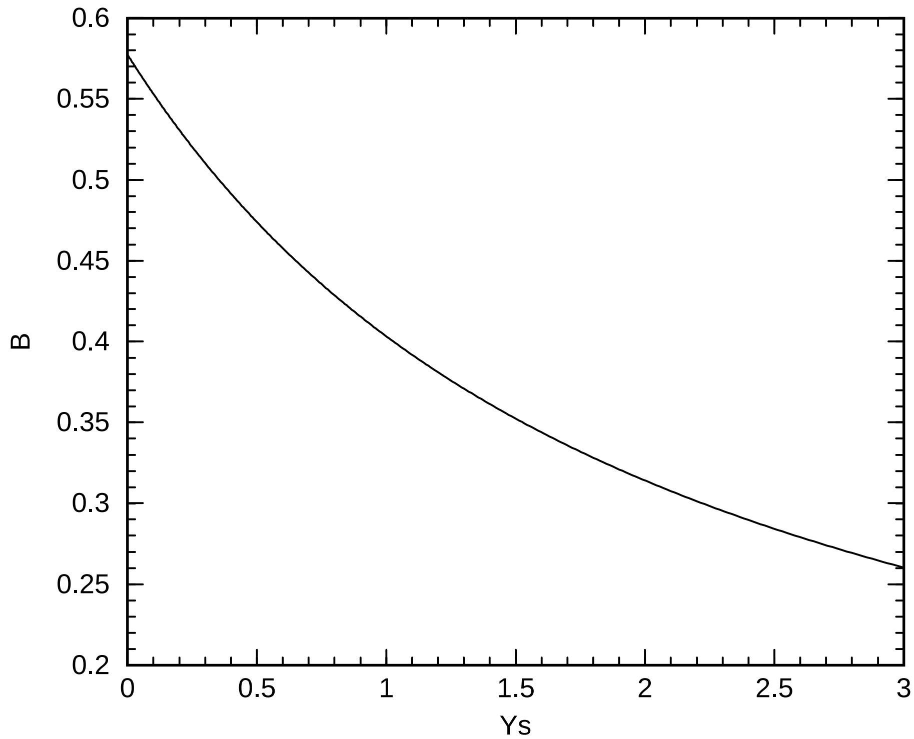

Figure 9: Optimum B versus Y_s for $0 \leq Y_s \leq 3$.

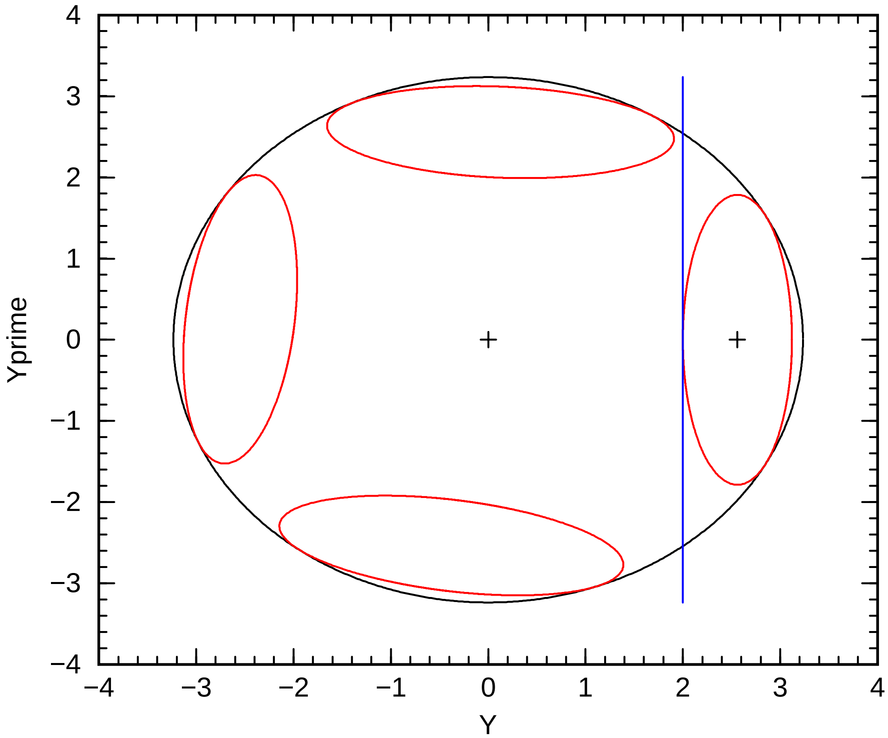

Figure 10: Positions of the beam ellipse on the first, second, and third turns around the machine. Here $Y_S = 2.0$, $B = 0.314077$, and $D = 0.0911$ which give $Q = 0.2421$. In this case one has $N = 4$ turns to move the closed orbit away from the septum.

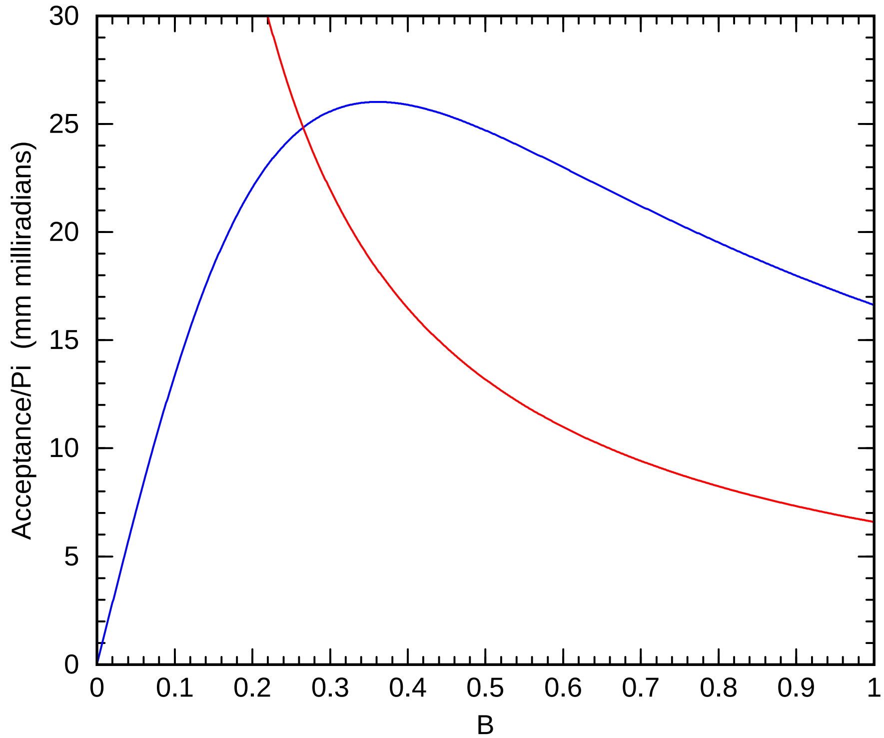

Figure 11: Inflector Acceptance versus B. Here the red curve is $e_0 = a^2/\beta_0$ and the blue curve is $e_1 = a^2/\beta_1$. The two curves intersect at $B = 0.265$ which gives the maximum acceptance $\pi e_0 = \pi e_1 = 24.8\pi$ mm milliradians. For $B < 0.265$, the acceptance is given by the blue curve. For $B > 0.265$, the acceptance is given by the red curve. Note that for $B = 1$, the beam ellipse is matched to the machine ellipse at the inflector exit. In this case the acceptance of the inflector channel is only 6.59π mm milliradians.

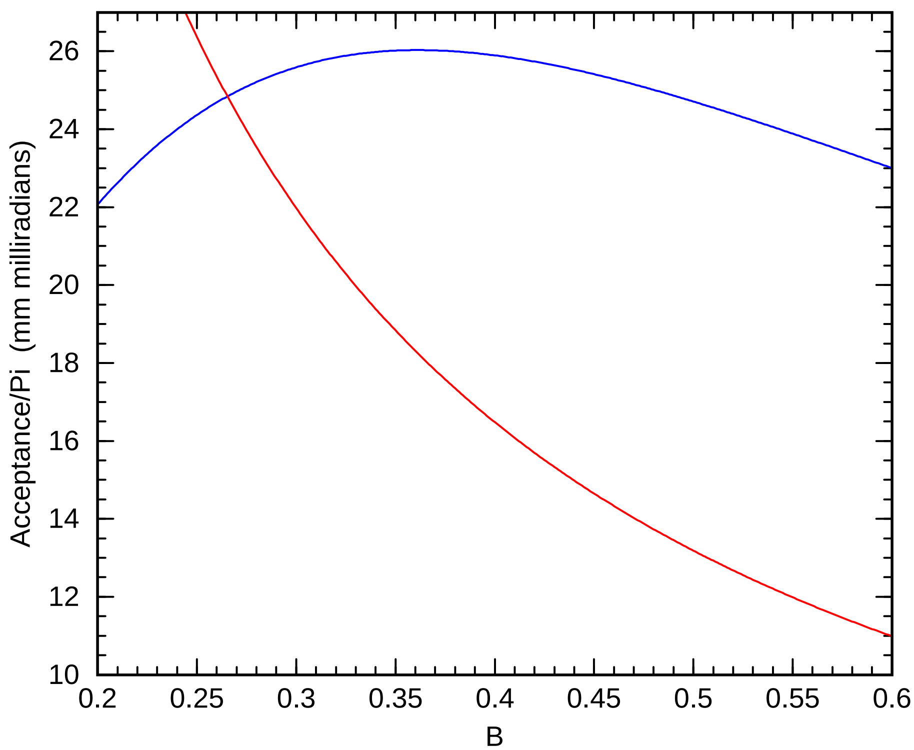

Figure 12: Inflector Acceptance versus B for $0.26 < B < 0.58$. Here we see that for this range of B, a beam ellipse with emittance $\pi\epsilon_0 = 11\pi$ mm milliradians will fit through the inflector channel.