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# Some Useful Linear Coupling Approximations

C. J. Gardner

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Collider Accelerator Department
**Brookhaven National Laboratory**

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**Collider-Accelerator Department**
**Brookhaven National Laboratory**
**Upton, NY 11973**

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One of the hallmarks of linear coupling is the exchange of oscillation energy between the horizontal and vertical planes when the difference between the tunes is close to an integer. The standard derivation of this phenomenon can be found, for example, in the CERN reports of Guignard [1, 2]. One starts with an uncoupled lattice and adds a linear perturbation which couples the two planes. The equations of motion are expressed in Hamiltonian form. As the difference between the unperturbed tunes approaches an integer, one finds that the perturbing terms in the Hamiltonian can be divided into terms that oscillate slowly and ones that oscillate rapidly. The rapidly oscillating terms are discarded (or transformed to higher order with an appropriate canonical transformation). The resulting approximate Hamiltonian gives equations of motion that clearly exhibit the exchange of energy between the two planes.

If, instead of the Hamiltonian, one is given the four-by-four matrix for one turn around a synchrotron, then one has the complete solution for the turn-by-turn motion. However, the phenomenon of energy exchange is not obvious from a casual inspection of the matrix. What approximation to the matrix gives turn-by-turn equations that clearly exhibit the energy exchange? This approximation is identified in the following notes and applied in various instances including the analysis of turn-by-turn data from BPMs (Beam Position Monitors). The formulae are general in that no particular form of linear coupling is assumed. The only assumptions are that the one-turn matrix is symplectic and that it has distinct eigenvalues on the unit circle in the complex plane.

# 1 The One-Turn Matrix

Let $X_0$, $X_0'$, $Y_0$, $Y_0'$ be the initial horizontal and vertical positions and angles of a beam particle at some point along the equilibrium orbit of a synchrotron, and let $X$, $X'$, $Y$, $Y'$ be the positions and angles at the point on the $n$th turn around the machine. Writing

$$\mathbf{Z} = \begin{pmatrix} X \\ X' \\ Y \\ Y' \end{pmatrix}, \quad \mathbf{Z}_0 = \begin{pmatrix} X_0 \\ X_0' \\ Y_0 \\ Y_0' \end{pmatrix} \tag{1}$$

we have

$$\mathbf{Z} = \mathbf{T}^n \mathbf{Z}_0 \tag{2}$$

where

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{pmatrix} \tag{3}$$

is the four-by-four transfer matrix for one turn around the machine. It will be convenient to partition $\mathbf{Z}_0$, $\mathbf{Z}$ and $\mathbf{T}$ into two-component vectors and two-by-two matrices. Thus

$$\mathbf{Z}_0 = \begin{pmatrix} \mathbf{X}_0 \\ \mathbf{Y}_0 \end{pmatrix}, \quad \mathbf{X}_0 = \begin{pmatrix} X_0 \\ X_0' \end{pmatrix}, \quad \mathbf{Y}_0 = \begin{pmatrix} Y_0 \\ Y_0' \end{pmatrix} \tag{4}$$

$$\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} X \\ X' \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} Y \\ Y' \end{pmatrix} \tag{5}$$

and

$$\mathbf{T} = \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix} \tag{6}$$

where

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \tag{7}$$

$$\mathbf{m} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}. \tag{8}$$

The matrix elements of $\mathbf{m}$ and $\mathbf{n}$ are proportional to the skew quadrupole or solenoidal fields that give rise to linear coupling between the horizontal and vertical planes of oscillation.

## 2 Symplectic Conditions

The matrix $\mathbf{T}$ is symplectic. This means that

$$\mathbf{T}^\dagger \mathbf{S} \mathbf{T} = \mathbf{S} \tag{9}$$

where

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix} \tag{10}$$

and

$$\mathbf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{11}$$

Here we use $\mathbf{S}$ to denote both a two-by-two and a four-by-four matrix. A dagger ($\dagger$) is used throughout these notes to denote the transpose of a vector or matrix. Similarly we define

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{12}$$

The matrix $\mathbf{S}$ has the property

$$\mathbf{S}^2 = -\mathbf{I}. \tag{13}$$

Taking the inverse of both sides of (9) we obtain $\mathbf{T}^{-1}\mathbf{S}(\mathbf{T}^\dagger)^{-1} = \mathbf{S}$ and therefore

$$\mathbf{S} = \mathbf{T}\mathbf{S}\mathbf{T}^\dagger \tag{14}$$

which is an equivalent form of the symplectic condition. Following Courant and Snyder [3], we define the symplectic conjugate of a two-by-two or four-by-four matrix $\mathbf{A}$ to be

$$\overline{\mathbf{A}} = -\mathbf{S}\mathbf{A}^\dagger\mathbf{S}. \tag{15}$$

For two-by-two matrices we have

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \overline{\mathbf{A}} = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{16}$$

and it follows that

$$\mathbf{A}\overline{\mathbf{A}} = \overline{\mathbf{A}}\mathbf{A} = (A_{11}A_{22} - A_{12}A_{21})\mathbf{I} = |\mathbf{A}|\mathbf{I}, \tag{17}$$

$$\mathbf{A} + \overline{\mathbf{A}} = (A_{11} + A_{22})\mathbf{I} = (\mathrm{Tr}\,\mathbf{A})\mathbf{I}. \tag{18}$$

For the symplectic four-by-four matrix $\mathbf{T}$ we have

$$\overline{\mathbf{T}}\mathbf{T} = -\mathbf{S}\mathbf{T}^{\dagger}\mathbf{S}\mathbf{T} = -\mathbf{S}^2 = \mathbf{I}, \quad \mathbf{T}\overline{\mathbf{T}} = -\mathbf{T}\mathbf{S}\mathbf{T}^{\dagger}\mathbf{S} = -\mathbf{S}^2 = \mathbf{I} \tag{19}$$

and therefore $\overline{\mathbf{T}} = \mathbf{T}^{-1}$. Writing

$$\mathbf{T}\overline{\mathbf{T}} = \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix} \begin{pmatrix} \overline{\mathbf{M}} & \overline{\mathbf{m}} \\ \overline{\mathbf{n}} & \overline{\mathbf{N}} \end{pmatrix} = \begin{pmatrix} \mathbf{M}\overline{\mathbf{M}} + \mathbf{n}\overline{\mathbf{n}} & \mathbf{M}\overline{\mathbf{m}} + \mathbf{n}\overline{\mathbf{N}} \\ \mathbf{m}\overline{\mathbf{M}} + \mathbf{N}\overline{\mathbf{n}} & \mathbf{m}\overline{\mathbf{m}} + \mathbf{N}\overline{\mathbf{N}} \end{pmatrix} \tag{20}$$

and

$$\overline{\mathbf{T}}\mathbf{T} = \begin{pmatrix} \overline{\mathbf{M}} & \overline{\mathbf{m}} \\ \overline{\mathbf{n}} & \overline{\mathbf{N}} \end{pmatrix} \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix} = \begin{pmatrix} \overline{\mathbf{M}}\mathbf{M} + \overline{\mathbf{m}}\mathbf{m} & \overline{\mathbf{M}}\mathbf{n} + \overline{\mathbf{m}}\mathbf{N} \\ \overline{\mathbf{n}}\mathbf{M} + \overline{\mathbf{N}}\mathbf{m} & \overline{\mathbf{n}}\mathbf{n} + \overline{\mathbf{N}}\mathbf{N} \end{pmatrix} \tag{21}$$

we find that $\mathbf{T}\overline{\mathbf{T}} = \overline{\mathbf{T}}\mathbf{T} = \mathbf{I}$ implies

$$|\mathbf{M}| + |\mathbf{m}| = 1, \quad |\mathbf{N}| + |\mathbf{n}| = 1, \quad \overline{\mathbf{M}}\mathbf{n} + \overline{\mathbf{m}}\mathbf{N} = \mathbf{0} \tag{22}$$

and

$$|\mathbf{M}| + |\mathbf{n}| = 1, \quad |\mathbf{N}| + |\mathbf{m}| = 1, \quad \mathbf{M}\overline{\mathbf{m}} + \mathbf{n}\overline{\mathbf{N}} = \mathbf{0}. \tag{23}$$

Equations (22) and (23) are actually equivalent, and, as shown by Brown and Servranckx [4], they impose a total of 6 independent constraints on the 16 matrix elements of $\mathbf{T}$. The four-by-four symplectic matrix $\mathbf{T}$ is therefore specified by 10 independent parameters. Equations (22) and (23) also imply

$$|\mathbf{M}| = |\mathbf{N}|, \quad |\mathbf{m}| = |\mathbf{n}|. \tag{24}$$

## 3 Eigenvalues

It follows from the symplectic condition that if $\lambda$ is an eigenvalue of $\mathbf{T}$ then so is $1/\lambda$ [3, 5]. We shall assume that the four eigenvalues of $\mathbf{T}$ are distinct and that none of them are equal to 1 or $-1$. One then has $\lambda_1$, $1/\lambda_1$, $\lambda_2$ and $1/\lambda_2$ as the eigenvalues, with $\lambda_1 \neq \lambda_2$. The eigenvalues of

$$\mathbf{T} + \overline{\mathbf{T}} = \begin{pmatrix} \mathbf{M} + \overline{\mathbf{M}} & \mathbf{n} + \overline{\mathbf{m}} \\ \mathbf{m} + \overline{\mathbf{n}} & \mathbf{N} + \overline{\mathbf{N}} \end{pmatrix} \tag{25}$$

are then

$$\Lambda_1 = \lambda_1 + 1/\lambda_1, \quad \Lambda_2 = \lambda_2 + 1/\lambda_2 \tag{26}$$

and the corresponding characteristic equation is

$$(M - \Lambda)(N - \Lambda) - |\mathbf{m} + \overline{\mathbf{n}}| = 0 \tag{27}$$

where

$$M = \mathrm{Tr}\,\mathbf{M} = M_{11} + M_{22}, \quad N = \mathrm{Tr}\,\mathbf{N} = N_{11} + N_{22}. \tag{28}$$

Thus

$$2\Lambda = M + N \pm \sqrt{(M - N)^2 + 4|\mathbf{m} + \overline{\mathbf{n}}|} \tag{29}$$

from which one can obtain $\lambda_1$ and $\lambda_2$. Defining

$$\lambda_1 = e^{i\psi_1} = \cos\psi_1 + i\sin\psi_1, \tag{30}$$

$$\lambda_2 = e^{i\psi_2} = \cos\psi_2 + i\sin\psi_2 \tag{31}$$

where the phases $\psi_1$ and $\psi_2$ are in general complex, we have

$$4\cos\psi_1 = 2\Lambda_1 = M + N + \sqrt{(M - N)^2 + 4|\mathbf{m} + \overline{\mathbf{n}}|} \tag{32}$$

$$4\cos\psi_2 = 2\Lambda_2 = M + N - \sqrt{(M - N)^2 + 4|\mathbf{m} + \overline{\mathbf{n}}|} \tag{33}$$

Note that since the elements of $\mathbf{T}$ are real, the complex conjugate of an eigenvalue is also an eigenvalue. We shall assume that the four distinct eigenvalues of $\mathbf{T}$ lie on the unit circle in the complex plane. In this case $\psi_1$ and $\psi_2$ are real with $\cos\psi_1 \neq \cos\psi_2$. The tunes associated with the eigenvalues are

$$Q_1 = \frac{\psi_1}{2\pi}, \quad Q_2 = \frac{\psi_2}{2\pi} \tag{34}$$

and, under our assumptions, neither $Q_1$, $2Q_1$, $Q_2$, $2Q_2$, $Q_1 + Q_2$ nor $Q_1 - Q_2$ is equal to an integer.

## 4 Block-Diagonal Form

If we assume that $\mathbf{T}$ has distinct eigenvalues, then a theorem of linear algebra [6] tells us that $\mathbf{T}$ is similar to a diagonal matrix and the four eigenvectors associated with the eigenvalues are linearly independent.

Since the distinct eigenvalues are assumed to lie on the unit circle in the complex plane, the four eigenvalues are $\lambda_1$, $\lambda_1^*$, $\lambda_2$, $\lambda_2^*$, and we have

$$\mathbf{Tu} = \lambda_1 \mathbf{u}, \quad \mathbf{Tu}^* = \lambda_1^* \mathbf{u}^*, \quad \mathbf{Tv} = \lambda_2 \mathbf{v}, \quad \mathbf{Tv}^* = \lambda_2^* \mathbf{v}^* \tag{35}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \quad \mathbf{u}^* = \begin{pmatrix} u_1^* \\ u_2^* \\ u_3^* \\ u_4^* \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}, \quad \mathbf{v}^* = \begin{pmatrix} v_1^* \\ v_2^* \\ v_3^* \\ v_4^* \end{pmatrix} \tag{36}$$

are the linearly independent eigenvectors. Defining matrices

$$\mathbf{V} = \begin{pmatrix} u_1 & u_1^* & v_1 & v_1^* \\ u_2 & u_2^* & v_2 & v_2^* \\ u_3 & u_3^* & v_3 & v_3^* \\ u_4 & u_4^* & v_4 & v_4^* \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1^* & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2^* \end{pmatrix} \tag{37}$$

we then have

$$\mathbf{TV} = \mathbf{V\Lambda}. \tag{38}$$

Since the columns of $\mathbf{V}$ are linearly independent, the determinant of $\mathbf{V}$ is nonzero and the inverse $\mathbf{V}^{-1}$ exists. Thus we have

$$\mathbf{T} = \mathbf{V\Lambda V}^{-1} \tag{39}$$

and the similarity transformation $\mathbf{V}^{-1}\mathbf{TV}$ yields the diagonal matrix $\mathbf{\Lambda}$. However, the diagonal elements are complex. We can obtain a block-diagonal matrix with real elements by working with the real and imaginary parts of the eigenvectors $\mathbf{u}$ and $\mathbf{v}$. Following the treatment of Iselin [7] we let

$$\mathbf{u} = \mathbf{a} + i\mathbf{b}, \quad \mathbf{v} = \mathbf{c} + i\mathbf{d} \tag{40}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are the real vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}. \tag{41}$$

Then we have

$$\mathbf{V} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{pmatrix} = \mathcal{W}\mathcal{J} \tag{42}$$

where

$$\mathcal{W} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{pmatrix}. \tag{43}$$

The matrix $\mathbf{T}$ then becomes

$$\mathbf{T} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} = \mathcal{W}\mathcal{J}\mathbf{\Lambda}\mathcal{J}^{-1}\mathcal{W}^{-1} = \mathcal{W}\mathcal{U}\mathcal{W}^{-1} \tag{44}$$

where

$$\mathcal{U} = \mathcal{J}\mathbf{\Lambda}\mathcal{J}^{-1}. \tag{45}$$

Writing

$$\mathcal{J} = \begin{pmatrix} \mathcal{I} & \mathbf{0} \\ \mathbf{0} & \mathcal{I} \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2 \end{pmatrix} \tag{46}$$

where $\mathcal{I}$, $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$ are the two-by-two matrices

$$\mathcal{I} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad \mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}, \quad \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^* \end{pmatrix} \tag{47}$$

we then have

$$\mathcal{U} = \begin{pmatrix} \mathcal{A} & \mathbf{0} \\ \mathbf{0} & \mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathcal{I} & \mathbf{0} \\ \mathbf{0} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2 \end{pmatrix} \begin{pmatrix} \mathcal{I}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathcal{I}^{-1} \end{pmatrix}. \tag{48}$$

Carrying out the matrix multiplications we then find that

$$\mathcal{A} = \mathcal{I}\mathbf{\Lambda}_1\mathcal{I}^{-1} = \begin{pmatrix} \cos\psi_1 & \sin\psi_1 \\ -\sin\psi_1 & \cos\psi_1 \end{pmatrix} \tag{49}$$

and

$$\mathcal{B} = \mathcal{I}\mathbf{\Lambda}_2\mathcal{I}^{-1} = \begin{pmatrix} \cos\psi_2 & \sin\psi_2 \\ -\sin\psi_2 & \cos\psi_2 \end{pmatrix}. \tag{50}$$

The similarity transformation $\mathcal{W}^{-1}\mathbf{T}\mathcal{W}$ therefore yields the block-diagonal matrix $\mathcal{U}$ which has real elements. Note also that

$$\mathcal{A}\mathcal{A}^\dagger = \mathbf{I}, \quad \mathcal{B}\mathcal{B}^\dagger = \mathbf{I}, \quad \mathcal{U}\mathcal{U}^\dagger = \mathbf{I}. \tag{51}$$

In **Appendix I** it is shown that the normalization of the eigenvectors can be chosen so that the matrix $\mathcal{W}$ is symplectic. If there is another symplectic matrix $\widehat{\mathcal{W}}$ for which

$$\mathbf{T} = \widehat{\mathcal{W}}\mathcal{U}\widehat{\mathcal{W}}^{-1} \tag{52}$$

then we have

$$\widehat{\mathcal{W}} = \mathcal{W}\mathcal{O} \tag{53}$$

where, as shown in **Appendix II**, $\mathcal{O}$ must be of the form

$$\mathcal{O} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{pmatrix} \tag{54}$$

with

$$\mathbf{P} = \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{55}$$

and $c_1^2 + s_1^2 = 1$ and $c_2^2 + s_2^2 = 1$.

## 5 The Matched Ellipsoid

Writing the one-turn matrix in the block-diagonal form

$$\mathbf{T} = \mathcal{W}\mathcal{U}\mathcal{W}^{-1} \tag{56}$$

we consider the matrix

$$\mathbf{E} = \mathcal{W}\mathcal{W}^{\dagger}. \tag{57}$$

One finds that

$$\mathbf{T}\mathbf{E}\mathbf{T}^{\dagger} = \mathcal{W}\mathcal{U}\mathcal{U}^{\dagger}\mathcal{W}^{\dagger} \tag{58}$$

and since $\mathcal{U}\mathcal{U}^{\dagger} = \mathbf{I}$ we have

$$\mathbf{T}\mathbf{E}\mathbf{T}^{\dagger} = \mathbf{E}. \tag{59}$$

By construction the matrix $\mathbf{E}$ has unit determinant and is real, symmetric and positive definite. (A real symmetric matrix $\mathbf{E}$ is positive definite if and only if the quadratic form $\mathbf{Z}^{\dagger}\mathbf{E}\mathbf{Z} > 0$ for every vector $\mathbf{Z} \neq \mathbf{0}$.) It follows that the set of initial positions and angles $X_0$, $X_0'$, $Y_0$, $Y_0'$ defined by

$$\mathbf{Z}_0^{\dagger}\mathbf{E}^{-1}\mathbf{Z}_0 = \epsilon \tag{60}$$

is a four dimensional ellipsoid. On the $n$th turn around the machine we have

$$\mathbf{Z} = \mathbf{T}^n\mathbf{Z}_0 \tag{61}$$

and

$$\mathbf{Z}^{\dagger}\mathbf{E}^{-1}\mathbf{Z} = \mathbf{Z}_0^{\dagger}(\mathbf{T}^{\dagger})^n\,\mathbf{E}^{-1}\,\mathbf{T}^n\mathbf{Z}_0. \tag{62}$$

But $\mathbf{TET}^\dagger = \mathbf{E}$ implies

$$\mathbf{T}^\dagger\mathbf{E}^{-1}\mathbf{T} = \mathbf{E}^{-1} \tag{63}$$

and (by induction)

$$(\mathbf{T}^\dagger)^n\,\mathbf{E}^{-1}\,\mathbf{T}^n = \mathbf{E}^{-1}. \tag{64}$$

Thus

$$\mathbf{Z}^\dagger\mathbf{E}^{-1}\mathbf{Z} = \mathbf{Z}_0^\dagger\mathbf{E}^{-1}\mathbf{Z}_0 = \epsilon \tag{65}$$

and we see that the particle positions and angles lie on the same ellipsoid after each turn. The ellipsoid is then said to be matched to the lattice.

If there is another symplectic matrix $\widehat{\mathcal{W}}$ for which

$$\mathbf{T} = \widehat{\mathcal{W}}\mathcal{U}\widehat{\mathcal{W}}^{-1} \tag{66}$$

then, as shown in **Appendix II**, we must have

$$\widehat{\mathcal{W}} = \mathcal{W}\mathcal{O} \tag{67}$$

where

$$\mathcal{O}\mathcal{O}^\dagger = \mathbf{I}. \tag{68}$$

Thus we have

$$\widehat{\mathcal{W}}\,\widehat{\mathcal{W}}^\dagger = \mathcal{W}\mathcal{O}\mathcal{O}^\dagger\mathcal{W}^\dagger = \mathcal{W}\mathcal{W}^\dagger \tag{69}$$

and therefore

$$\mathbf{E} = \mathcal{W}\mathcal{W}^\dagger = \widehat{\mathcal{W}}\,\widehat{\mathcal{W}}^\dagger. \tag{70}$$

This shows that $\mathbf{E}$ is uniquely defined.

## 6 Edwards-Teng Parameterization

Edwards and Teng [8], and more recently Billing [9] and Roser [10], have shown that the symplectic four-by-four matrix $\mathbf{T}$ with eigenvalues

$$\lambda_1 = e^{i\psi_1}, \quad \lambda_1^* = e^{-i\psi_1}, \quad \lambda_2 = e^{i\psi_2}, \quad \lambda_2^* = e^{-i\psi_2} \tag{71}$$

can be expressed in the form

$$\mathbf{T} = \mathbf{RUR}^{-1} \tag{72}$$

where

$$\mathbf{T} = \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \tag{73}$$

$$\mathbf{R} = \begin{pmatrix} d\mathbf{I} & \overline{\mathbf{W}} \\ -\mathbf{W} & d\mathbf{I} \end{pmatrix}, \quad \mathbf{R}^{-1} = \begin{pmatrix} d\mathbf{I} & -\overline{\mathbf{W}} \\ \mathbf{W} & d\mathbf{I} \end{pmatrix}, \tag{74}$$

$$\mathbf{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \quad \overline{\mathbf{W}} = \begin{pmatrix} W_{22} & -W_{12} \\ -W_{21} & W_{11} \end{pmatrix}, \tag{75}$$

$$\mathbf{A} = \begin{pmatrix} \cos\psi_1 + \alpha_1 \sin\psi_1 & \beta_1 \sin\psi_1 \\ -\gamma_1 \sin\psi_1 & \cos\psi_1 - \alpha_1 \sin\psi_1 \end{pmatrix} = C_1\mathbf{I} + S_1\mathbf{J}_1 \tag{76}$$

$$\mathbf{B} = \begin{pmatrix} \cos\psi_2 + \alpha_2 \sin\psi_2 & \beta_2 \sin\psi_2 \\ -\gamma_2 \sin\psi_2 & \cos\psi_2 - \alpha_2 \sin\psi_2 \end{pmatrix} = C_2\mathbf{I} + S_2\mathbf{J}_2 \tag{77}$$

$$C_1 = \cos\psi_1, \quad S_1 = \sin\psi_1, \quad C_2 = \cos\psi_2, \quad S_2 = \sin\psi_2 \tag{78}$$

$$\mathbf{J}_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} \tag{79}$$

and

$$\beta_1\gamma_1 - \alpha_1^2 = 1, \quad \beta_2\gamma_2 - \alpha_2^2 = 1. \tag{80}$$

Here the four-by-four matrix $\mathbf{U}$ is block-diagonal. The two-by-two matrices $\mathbf{A}$ and $\mathbf{B}$ have unit determinant and are expressed in terms of Courant-Snyder parameters. The four-by-four matrix $\mathbf{R}$ is symplectic and the two-by-two matrix $\mathbf{W}$ has determinant given by

$$d^2 = 1 - |\mathbf{W}|. \tag{81}$$

The one-turn matrix $\mathbf{T}$ is therefore given in terms of the 10 independent parameters $\alpha_1$, $\beta_1$, $\psi_1$, $\alpha_2$, $\beta_2$, $\psi_2$, $W_{11}$, $W_{12}$, $W_{21}$ and $W_{22}$.

Writing

$$\mathbf{T} = \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix} = \begin{pmatrix} d\mathbf{I} & \overline{\mathbf{W}} \\ -\mathbf{W} & d\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \begin{pmatrix} d\mathbf{I} & -\overline{\mathbf{W}} \\ \mathbf{W} & d\mathbf{I} \end{pmatrix} \tag{82}$$

and carrying out the matrix multiplications we find

$$\mathbf{M} = d^2\mathbf{A} + \overline{\mathbf{W}}\mathbf{B}\mathbf{W}, \quad \mathbf{N} = d^2\mathbf{B} + \mathbf{W}\mathbf{A}\overline{\mathbf{W}}, \tag{83}$$

$$\mathbf{m} = d(\mathbf{B}\mathbf{W} - \mathbf{W}\mathbf{A}), \quad \mathbf{n} = d(\overline{\mathbf{W}}\mathbf{B} - \mathbf{A}\overline{\mathbf{W}}). \tag{84}$$

Note that $d = 0$ implies $\mathbf{m} = \mathbf{n} = \mathbf{0}$. Since we are considering one-turn matrices $\mathbf{T}$ for which $\mathbf{m}$ and $\mathbf{n}$ are nonzero, we must then have $d \neq 0$. Equations (83) and (84) can be inverted to obtain $\mathbf{A}$, $\mathbf{B}$, $\mathbf{W}$ and $d$ in terms of $\mathbf{M}$, $\mathbf{N}$, $\mathbf{m}$ and $\mathbf{n}$. One finds

$$\mathbf{A} = \mathbf{M} - \frac{\mathbf{nW}}{d}, \quad \mathbf{B} = \mathbf{N} + \frac{\mathbf{Wn}}{d}, \quad \mathbf{W} = -\frac{\mathbf{m} + \overline{\mathbf{n}}}{dU} \tag{85}$$

where

$$d^2 = \frac{U+T}{2U} = \frac{1}{2} + \frac{T}{2U}, \quad T = \mathrm{Tr}(\mathbf{M} - \mathbf{N}) \tag{86}$$

and

$$U = \mathrm{Tr}(\mathbf{A} - \mathbf{B}) = \pm\sqrt{T^2 + 4|\mathbf{m} + \overline{\mathbf{n}}|}. \tag{87}$$

Here the sign in front of the square root is chosen so that $U$ has the same sign as $T$. It then follows that $d^2 \geq 1/2$. If there are no magnetic elements that couple the horizontal and vertical planes of oscillation, the matrix elements of $\mathbf{m}$ and $\mathbf{n}$ are zero and we have $|\mathbf{m} + \overline{\mathbf{n}}| = 0$, $U = T$, and $d^2 = 1$.

Note that

$$U = 2\cos\psi_1 - 2\cos\psi_2, \quad \psi_1 = 2\pi Q_1, \quad \psi_2 = 2\pi Q_2 \tag{88}$$

and

$$4|\mathbf{m} + \overline{\mathbf{n}}| = U^2 - T^2 = 4U^2 d^2(1 - d^2). \tag{89}$$

These equations give the value of $|\mathbf{m} + \overline{\mathbf{n}}|$ if $Q_1$, $Q_2$ and $d^2$ are known.

# 7 The Matched Ellipsoid Matrix in Terms of Edwards-Teng Parameters

Let us compare the block-diagonal form of Section 4 with that of Edwards and Teng. We have

$$\mathbf{T} = \mathcal{W}\mathcal{U}\mathcal{W}^{-1} = \mathbf{R}\mathbf{U}\mathbf{R}^{-1} \tag{90}$$

where

$$\mathcal{U} = \begin{pmatrix} \mathcal{A} & \mathbf{0} \\ \mathbf{0} & \mathcal{B} \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \tag{91}$$

$$\mathbf{R} = \begin{pmatrix} d\mathbf{I} & \overline{\mathbf{W}} \\ -\mathbf{W} & d\mathbf{I} \end{pmatrix}, \quad \mathbf{R}^{-1} = \begin{pmatrix} d\mathbf{I} & -\overline{\mathbf{W}} \\ \mathbf{W} & d\mathbf{I} \end{pmatrix} \tag{92}$$

and

$$\mathcal{A} = \begin{pmatrix} \cos\psi_1 & \sin\psi_1 \\ -\sin\psi_1 & \cos\psi_1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \cos\psi_2 & \sin\psi_2 \\ -\sin\psi_2 & \cos\psi_2 \end{pmatrix}. \tag{93}$$

The matrices **A** and **B** can be expressed as

$$\mathbf{A} = \mathcal{F}\mathcal{A}\mathcal{F}^{-1}, \quad \mathbf{B} = \mathcal{G}\mathcal{B}\mathcal{G}^{-1} \tag{94}$$

where

$$\mathcal{F} = \frac{1}{\sqrt{\beta_1}} \begin{pmatrix} \beta_1 & 0 \\ -\alpha_1 & 1 \end{pmatrix}, \quad \mathcal{G} = \frac{1}{\sqrt{\beta_2}} \begin{pmatrix} \beta_2 & 0 \\ -\alpha_2 & 1 \end{pmatrix}. \tag{95}$$

Thus we can write

$$\mathbf{U} = \mathcal{N}\mathcal{U}\mathcal{N}^{-1} \tag{96}$$

where

$$\mathcal{N} = \begin{pmatrix} \mathcal{F} & \mathbf{0} \\ \mathbf{0} & \mathcal{G} \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} \mathcal{A} & \mathbf{0} \\ \mathbf{0} & \mathcal{B} \end{pmatrix}. \tag{97}$$

We then have

$$\mathbf{T} = \mathbf{R}\mathcal{N}\mathcal{U}\mathcal{N}^{-1}\mathbf{R}^{-1} = \widehat{\mathcal{W}}\mathcal{U}\widehat{\mathcal{W}}^{-1} = \mathcal{W}\mathcal{U}\mathcal{W}^{-1} \tag{98}$$

where

$$\widehat{W} = \mathbf{R}\mathcal{N} = \mathcal{W}\mathcal{O} \tag{99}$$

and, as shown in **Appendix II**,

$$\mathcal{O}\mathcal{O}^{\dagger} = \mathbf{I}. \tag{100}$$

Thus, we have

$$\mathbf{E} = \mathcal{W}\mathcal{W}^{\dagger} = \widehat{\mathcal{W}}\widehat{\mathcal{W}}^{\dagger} = \mathbf{R}\mathcal{N}\mathcal{N}^{\dagger}\mathbf{R}^{\dagger} = \mathbf{R}\mathbf{D}\mathbf{R}^{\dagger} \tag{101}$$

where

$$\mathbf{D} = \mathcal{N}\mathcal{N}^{\dagger} = \begin{pmatrix} \mathcal{F}\mathcal{F}^{\dagger} & \mathbf{0} \\ \mathbf{0} & \mathcal{G}\mathcal{G}^{\dagger} \end{pmatrix} = \begin{pmatrix} \mathbf{f} & \mathbf{0} \\ \mathbf{0} & \mathbf{g} \end{pmatrix} \tag{102}$$

and

$$\mathbf{f} = \mathcal{F}\mathcal{F}^{\dagger} = \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix}, \quad \mathbf{g} = \mathcal{G}\mathcal{G}^{\dagger} = \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix}. \tag{103}$$

In terms of two-by-two matrices we then have

$$\mathbf{E} = \begin{pmatrix} \mathbf{F} & \mathbf{C} \\ \mathbf{C}^{\dagger} & \mathbf{G} \end{pmatrix} = \begin{pmatrix} d\mathbf{I} & \overline{\mathbf{W}} \\ -\mathbf{W} & d\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{f} & \mathbf{0} \\ \mathbf{0} & \mathbf{g} \end{pmatrix} \begin{pmatrix} d\mathbf{I} & -\mathbf{W}^{\dagger} \\ \overline{\mathbf{W}}^{\dagger} & d\mathbf{I} \end{pmatrix} \tag{104}$$

and

$$\mathbf{F} = d^2\mathbf{f} + \overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger, \quad \mathbf{G} = d^2\mathbf{g} + \mathbf{W}\mathbf{f}\mathbf{W}^\dagger, \quad \mathbf{C} = d(\overline{\mathbf{W}}\mathbf{g} - \mathbf{f}\mathbf{W}^\dagger). \tag{105}$$

Note that since $\mathcal{A}\mathcal{A}^\dagger = \mathbf{I}$ and $\mathcal{B}\mathcal{B}^\dagger = \mathbf{I}$ we have

$$\mathbf{A}\mathbf{f}\mathbf{A}^\dagger = \mathcal{F}\mathcal{A}\mathcal{A}^\dagger\mathcal{F}^\dagger = \mathcal{F}\mathcal{F}^\dagger = \mathbf{f} \tag{106}$$

$$\mathbf{B}\mathbf{g}\mathbf{B}^\dagger = \mathcal{G}\mathcal{B}\mathcal{B}^\dagger\mathcal{G}^\dagger = \mathcal{G}\mathcal{G}^\dagger = \mathbf{g} \tag{107}$$

and

$$\mathbf{U}\mathbf{D}\mathbf{U}^\dagger = \mathcal{N}\mathcal{U}\mathcal{U}^\dagger\mathcal{N}^\dagger = \mathcal{N}\mathcal{N}^\dagger = \mathbf{D}. \tag{108}$$

# 8 Weak Coupling Fields

Let us assume that the fields responsible for coupling between the horizontal and vertical planes are weak compared to the focusing fields of the lattice. Expressing $\mathbf{T}$ in the form

$$\mathbf{T} = \mathcal{W}\mathcal{U}\mathcal{W}^{-1} \tag{109}$$

we find that

$$\mathcal{W} = \begin{pmatrix} \mathcal{W}_1 & \mathcal{C} \\ \mathcal{D} & \mathcal{W}_2 \end{pmatrix} \tag{110}$$

where $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{C}$, and $\mathcal{D}$ are two-by-two matrices and the elements of $\mathcal{C}$ and $\mathcal{D}$ are small compared to those of $\mathcal{W}_1$ and $\mathcal{W}_2$. As the coupling fields go to zero, the elements of $\mathcal{C}$ and $\mathcal{D}$ go to zero. In terms of $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{C}$, and $\mathcal{D}$, the matched ellipsoid matrix is

$$\mathbf{E} = \mathcal{W}\mathcal{W}^\dagger = \begin{pmatrix} \mathbf{F} & \mathbf{C} \\ \mathbf{C}^\dagger & \mathbf{G} \end{pmatrix} \tag{111}$$

where

$$\mathbf{F} = \mathcal{W}_1\mathcal{W}_1^\dagger + \mathcal{C}\mathcal{C}^\dagger, \quad \mathbf{G} = \mathcal{W}_2\mathcal{W}_2^\dagger + \mathcal{D}\mathcal{D}^\dagger \tag{112}$$

and

$$\mathbf{C} = \left\{\mathcal{W}_1\mathcal{D}^\dagger + \mathcal{C}\mathcal{W}_2^\dagger\right\}. \tag{113}$$

Thus, the matrix elements of $\mathbf{C}$ are small compared to those of $\mathbf{F}$ and $\mathbf{G}$ and go to zero as the coupling fields go to zero. In terms of the Edwards-Teng parameters we have

$$\mathbf{F} = d^2\mathbf{f} + \overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger, \quad \mathbf{G} = d^2\mathbf{g} + \mathbf{W}\mathbf{f}\mathbf{W}^\dagger \tag{114}$$

$$\mathbf{C} = d(\overline{\mathbf{W}}\mathbf{g} - \mathbf{f}\mathbf{W}^\dagger) \tag{115}$$

where

$$\mathbf{f} = \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix}, \quad \mathbf{W} = -\frac{\mathbf{m} + \overline{\mathbf{n}}}{dU} \tag{116}$$

$$|\mathbf{f}| = \beta_1\gamma_1 - \alpha_1^2 = 1, \quad |\mathbf{g}| = \beta_2\gamma_2 - \alpha_2^2 = 1, \quad |\mathbf{W}| = 1 - d^2 \tag{117}$$

$$d^2 = \frac{U+T}{2U} = \frac{1}{2} + \frac{T}{2U} \tag{118}$$

and

$$T = \mathrm{Tr}(\mathbf{M} - \mathbf{N}), \quad U = \pm\sqrt{T^2 + 4|\mathbf{m} + \overline{\mathbf{n}}|}. \tag{119}$$

Since the sign of $U$ is taken to be the same as the sign of $T$, we have $d^2 \geq 1/2$, and it follows that the elements of $\overline{\mathbf{W}}\mathbf{g} - \mathbf{f}\mathbf{W}^\dagger$ go to zero as the coupling fields go to zero. Thus if the coupling fields are sufficiently weak we may write

$$\overline{\mathbf{W}}\mathbf{g} - \mathbf{f}\mathbf{W}^\dagger \approx \mathbf{0}. \tag{120}$$

This approximation clearly holds if the elements of $\mathbf{W}$ are sufficiently small. It also can be true if the elements of $\mathbf{W}$ are not small, as we show in the next section.

## 9 Weak Coupling Fields with Small $T$

Weak coupling fields imply that

$$T \approx 2\cos\psi_x - 2\cos\psi_y \tag{121}$$

where

$$\psi_x = 2\pi Q_x, \quad \psi_y = 2\pi Q_y \tag{122}$$

and $Q_x$ and $Q_y$ are the tunes one would obtain with no coupling between the horizontal and vertical planes. If the difference between these tunes is close to an integer, then $T$ will be small. Let us assume that this is the case and assume further that

$$0 \leq \frac{T^2}{4|\mathbf{m} + \overline{\mathbf{n}}|} << 1. \tag{123}$$

In order to satisfy this inequality, we adjust the lattice quadrupoles so that

$$T^2 = 4|\mathbf{m} + \overline{\mathbf{n}}|\, E \tag{124}$$

where $E$ is a fixed parameter chosen to satisfy

$$0 \le E << 1. \tag{125}$$

With $T^2$ given by (124), the expressions for $U$ and $d^2$ become

$$U = \pm\sqrt{1+E}\,\sqrt{4|\mathbf{m} + \overline{\mathbf{n}}|}, \quad d^2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{E}{1+E}} \tag{126}$$

and for $0 \le E << 1$ we have

$$U \approx \pm\sqrt{4|\mathbf{m} + \overline{\mathbf{n}}|}, \quad d^2 \approx \frac{1}{2}, \quad |\mathbf{W}| \approx \frac{1}{2}. \tag{127}$$

These approximations become exact if $E = 0$. Here we see that if (123) is satisfied, the elements of $\mathbf{W}$ are finite and not small. Moreover, since $|\mathbf{g}| = 1$ and $|\mathbf{f}| = 1$, the elements of $\overline{\mathbf{W}}\mathbf{g}$ and $\mathbf{f}\mathbf{W}^\dagger$ are also finite and not small. Thus, if the coupling fields are sufficiently weak, the elements of $\overline{\mathbf{W}}\mathbf{g} - \mathbf{f}\mathbf{W}^\dagger$ will be small compared to those of $\overline{\mathbf{W}}\mathbf{g}$ and $\mathbf{f}\mathbf{W}^\dagger$. We may then write

$$\overline{\mathbf{W}}\mathbf{g} \approx \mathbf{f}\mathbf{W}^\dagger. \tag{128}$$

Multiplying this by $\mathbf{W}$ from the left or by $\overline{\mathbf{W}}^\dagger$ from the right, we have

$$\mathbf{W}\mathbf{f}\mathbf{W}^\dagger \approx (1-d^2)\mathbf{g}, \quad \overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger \approx (1-d^2)\mathbf{f}. \tag{129}$$

We also have

$$\mathbf{F} = d^2\mathbf{f} + \overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger \approx \mathbf{f}, \quad \mathbf{G} = d^2\mathbf{g} + \mathbf{W}\mathbf{f}\mathbf{W}^\dagger \approx \mathbf{g}. \tag{130}$$

We shall see that the approximation $\overline{\mathbf{W}}\mathbf{g} \approx \mathbf{f}\mathbf{W}^\dagger$ gives turn-by-turn equations that clearly exhibit the exchange of oscillation energy between the horizontal and vertical planes.

## 10 Approximate One-Turn Matrix

Let $\mathbf{H}$ be any two-by-two matrix that has determinant

$$|\mathbf{H}| = 1 - d^2 > 0 \tag{131}$$

and satisfies

$$\overline{\mathbf{H}}\mathbf{g} = \mathbf{f}\mathbf{H}^{\dagger}. \tag{132}$$

Multiplying by **H** from the left we have

$$\mathbf{H}\mathbf{f}\mathbf{H}^{\dagger} = (1-d^2)\mathbf{g}. \tag{133}$$

As shown in **Appendix III**, we must then have

$$\mathbf{H} = \mathcal{G}\Omega\mathcal{F}^{-1}\sqrt{1-d^2} \tag{134}$$

where

$$\mathcal{F} = \frac{1}{\sqrt{\beta_1}}\begin{pmatrix} \beta_1 & 0 \\ -\alpha_1 & 1 \end{pmatrix}, \quad \mathcal{G} = \frac{1}{\sqrt{\beta_2}}\begin{pmatrix} \beta_2 & 0 \\ -\alpha_2 & 1 \end{pmatrix} \tag{135}$$

and

$$\Omega = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}. \tag{136}$$

The elements of **H** are

$$H_{11}/\sqrt{1-d^2} = \sqrt{\beta_2/\beta_1}\ \{\cos\omega + \alpha_1\sin\omega\} \tag{137}$$

$$H_{12}/\sqrt{1-d^2} = \sqrt{\beta_2\beta_1}\ \sin\omega \tag{138}$$

$$H_{21}/\sqrt{1-d^2} = -\left(\frac{\alpha_2-\alpha_1}{\sqrt{\beta_2\beta_1}}\right)\cos\omega - \left(\frac{1+\alpha_2\alpha_1}{\sqrt{\beta_2\beta_1}}\right)\sin\omega \tag{139}$$

$$H_{22}/\sqrt{1-d^2} = \sqrt{\beta_1/\beta_2}\ \{\cos\omega - \alpha_2\sin\omega\} \tag{140}$$

and the angle $\omega$ must satisfy

$$\sin\omega = \frac{H_{12}}{\sqrt{\beta_1\beta_2(1-d^2)}}, \quad \cos\omega = \frac{\beta_1 H_{11} - \alpha_1 H_{12}}{\sqrt{\beta_1\beta_2(1-d^2)}}. \tag{141}$$

Thus, if the approximation $\overline{\mathbf{W}}\mathbf{g} \approx \mathbf{f}\mathbf{W}^{\dagger}$ is valid, we have

$$\mathbf{W} \approx \mathbf{H} \tag{142}$$

$$\mathbf{R} \approx \begin{pmatrix} d\mathbf{I} & \overline{\mathbf{H}} \\ -\mathbf{H} & d\mathbf{I} \end{pmatrix}, \quad \mathbf{R}^{-1} \approx \begin{pmatrix} d\mathbf{I} & -\overline{\mathbf{H}} \\ \mathbf{H} & d\mathbf{I} \end{pmatrix} \tag{143}$$

and

$$\mathbf{T} \approx \begin{pmatrix} d\mathbf{I} & \overline{\mathbf{H}} \\ -\mathbf{H} & d\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \begin{pmatrix} d\mathbf{I} & -\overline{\mathbf{H}} \\ \mathbf{H} & d\mathbf{I} \end{pmatrix} \tag{144}$$

where

$$\mathbf{H} = \sqrt{\frac{1-d^2}{\beta_1\beta_2}} \begin{pmatrix} \beta_2 & 0 \\ -\alpha_2 & 1 \end{pmatrix} \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_1 & \beta_1 \end{pmatrix} \tag{145}$$

$$\sin\omega = \frac{W_{12}}{\sqrt{\beta_1\beta_2(1-d^2)}}, \quad \cos\omega = \frac{\beta_1 W_{11} - \alpha_1 W_{12}}{\sqrt{\beta_1\beta_2(1-d^2)}} \tag{146}$$

$$\mathbf{A} = \begin{pmatrix} \cos\psi_1 + \alpha_1 \sin\psi_1 & \beta_1 \sin\psi_1 \\ -\gamma_1 \sin\psi_1 & \cos\psi_1 - \alpha_1 \sin\psi_1 \end{pmatrix} \tag{147}$$

$$\mathbf{B} = \begin{pmatrix} \cos\psi_2 + \alpha_2 \sin\psi_2 & \beta_2 \sin\psi_2 \\ -\gamma_2 \sin\psi_2 & \cos\psi_2 - \alpha_2 \sin\psi_2 \end{pmatrix}. \tag{148}$$

In this approximation the matrix $\mathbf{T}$ is symplectic and has the same eigenvalues as the exact matrix. However, it is completely specified by only eight parameters ($\alpha_1$, $\beta_1$, $Q_1$, $\alpha_2$, $\beta_2$, $Q_2$, $d$, $\omega$) rather than the ten ($\alpha_1$, $\beta_1$, $Q_1$, $\alpha_2$, $\beta_2$, $Q_2$, $W_{11}$, $W_{12}$, $W_{21}$, $W_{22}$) that specify the exact matrix. The parameters $d$ and $\omega$ are analogous to the complex coupling coefficient obtained in the Hamiltonian treatment of linear coupling.

## 11 Normal-Mode Coordinates

On the $n$th turn around the machine, a beam particle whose initial positions and angles were $X_0$, $X_0'$, $Y_0$, $Y_0'$ has positions and angles $X$, $X'$, $Y$, $Y'$ given by

$$\mathbf{Z} = \mathbf{T}^n \mathbf{Z}_0 \tag{149}$$

where

$$\mathbf{Z}_0 = \begin{pmatrix} \mathbf{X}_0 \\ \mathbf{Y}_0 \end{pmatrix}, \quad \mathbf{X}_0 = \begin{pmatrix} X_0 \\ X_0' \end{pmatrix}, \quad \mathbf{Y}_0 = \begin{pmatrix} Y_0 \\ Y_0' \end{pmatrix}, \tag{150}$$

$$\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} X \\ X' \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} Y \\ Y' \end{pmatrix} \tag{151}$$

and

$$\mathbf{T}^n = \mathbf{R}\mathbf{U}^n\mathbf{R}^{-1}. \tag{152}$$

Defining

$$\widehat{\mathbf{Z}} = \mathbf{R}^{-1}\mathbf{Z}, \quad \widehat{\mathbf{Z}}_0 = \mathbf{R}^{-1}\mathbf{Z}_0 \tag{153}$$

we then have

$$\widehat{\mathbf{Z}} = \mathbf{R}^{-1}\mathbf{T}^n\mathbf{Z}_0 = \mathbf{R}^{-1}(\mathbf{R}\mathbf{U}^n\mathbf{R}^{-1})\mathbf{Z}_0 = \mathbf{U}^n\widehat{\mathbf{Z}}_0. \tag{154}$$

Here

$$\widehat{\mathbf{Z}} = \begin{pmatrix} \widehat{X} \\ \widehat{X}' \\ \widehat{Y} \\ \widehat{Y}' \end{pmatrix}, \quad \widehat{\mathbf{Z}}_0 = \begin{pmatrix} \widehat{X}_0 \\ \widehat{X}_0' \\ \widehat{Y}_0 \\ \widehat{Y}_0' \end{pmatrix}, \quad \mathbf{U}^n = \begin{pmatrix} \mathbf{A}^n & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^n \end{pmatrix} \tag{155}$$

where $\widehat{X}_0$, $\widehat{X}_0'$, $\widehat{Y}_0$, $\widehat{Y}_0'$ and $\widehat{X}$, $\widehat{X}'$, $\widehat{Y}$, $\widehat{Y}'$ are the initial and final ($n$th turn) normal-mode positions and angles. Partitioning into two-component vectors we have

$$\widehat{\mathbf{Z}}_0 = \begin{pmatrix} \widehat{\mathbf{X}}_0 \\ \widehat{\mathbf{Y}}_0 \end{pmatrix}, \quad \widehat{\mathbf{X}}_0 = \begin{pmatrix} \widehat{X}_0 \\ \widehat{X}_0' \end{pmatrix}, \quad \widehat{\mathbf{Y}}_0 = \begin{pmatrix} \widehat{Y}_0 \\ \widehat{Y}_0' \end{pmatrix} \tag{156}$$

$$\widehat{\mathbf{Z}} = \begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}, \quad \widehat{\mathbf{X}} = \begin{pmatrix} \widehat{X} \\ X' \end{pmatrix}, \quad \widehat{\mathbf{Y}} = \begin{pmatrix} \widehat{Y} \\ \widehat{Y}' \end{pmatrix} \tag{157}$$

and $\widehat{\mathbf{Z}} = \mathbf{U}^n\widehat{\mathbf{Z}}_0$ becomes

$$\widehat{\mathbf{X}} = \mathbf{A}^n\widehat{\mathbf{X}}_0, \quad \widehat{\mathbf{Y}} = \mathbf{B}^n\widehat{\mathbf{Y}}_0. \tag{158}$$

Thus the turn-by-turn evolution of the normal-mode coordinates is decoupled into two sets of equations. The matrices $\mathbf{A}^n$ and $\mathbf{B}^n$ are

$$\mathbf{A}^n = \begin{pmatrix} \cos n\psi_1 + \alpha_1 \sin n\psi_1 & \beta_1 \sin n\psi_1 \\ -\gamma_1 \sin n\psi_1 & \cos n\psi_1 - \alpha_1 \sin n\psi_1 \end{pmatrix} \tag{159}$$

$$\mathbf{B}^n = \begin{pmatrix} \cos n\psi_2 + \alpha_2 \sin n\psi_2 & \beta_2 \sin n\psi_2 \\ -\gamma_2 \sin n\psi_2 & \cos n\psi_2 - \alpha_2 \sin n\psi_2 \end{pmatrix} \tag{160}$$

where

$$\psi_1 = 2\pi Q_1, \quad \psi_2 = 2\pi Q_2. \tag{161}$$

Writing $\mathbf{Z} = \mathbf{R}\widehat{\mathbf{Z}}$ and $\widehat{\mathbf{Z}}_0 = \mathbf{R}^{-1}\mathbf{Z}_0$ in terms of the two-component vectors defined above, we have

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = \begin{pmatrix} d\mathbf{I} & \overline{\mathbf{W}} \\ -\mathbf{W} & d\mathbf{I} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}, \tag{162}$$

$$\begin{pmatrix} \widehat{\mathbf{X}}_0 \\ \widehat{\mathbf{Y}}_0 \end{pmatrix} = \begin{pmatrix} d\mathbf{I} & -\overline{\mathbf{W}} \\ \mathbf{W} & d\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{X}_0 \\ \mathbf{Y}_0 \end{pmatrix} \tag{163}$$

and therefore

$$\mathbf{X} = d\widehat{\mathbf{X}} + \overline{\mathbf{W}}\widehat{\mathbf{Y}} = d\mathbf{A}^n\widehat{\mathbf{X}}_0 + \overline{\mathbf{W}}\mathbf{B}^n\widehat{\mathbf{Y}}_0 \tag{164}$$

$$\mathbf{Y} = -\mathbf{W}\widehat{\mathbf{X}} + d\widehat{\mathbf{Y}} = -\mathbf{W}\mathbf{A}^n\widehat{\mathbf{X}}_0 + d\mathbf{B}^n\widehat{\mathbf{Y}}_0 \tag{165}$$

where

$$\widehat{\mathbf{X}}_0 = d\mathbf{X}_0 - \overline{\mathbf{W}}\mathbf{Y}_0, \quad \widehat{\mathbf{Y}}_0 = \mathbf{W}\mathbf{X}_0 + d\mathbf{Y}_0. \tag{166}$$

These equations show that the turn-by-turn motion in each plane is the superposition of two modes of oscillation, one with tune $Q_1$ and the other with tune $Q_2$.

## 12 Courant-Snyder Invariants

It follows from the identities

$$\mathbf{f} = \mathbf{A}\mathbf{f}\mathbf{A}^\dagger, \quad \mathbf{g} = \mathbf{B}\mathbf{g}\mathbf{B}^\dagger \tag{167}$$

that

$$\mathbf{A}^\dagger \mathbf{f}^{-1}\mathbf{A} = \mathbf{f}^{-1}, \quad \mathbf{B}^\dagger \mathbf{g}^{-1}\mathbf{B} = \mathbf{g}^{-1} \tag{168}$$

and (by induction)

$$(\mathbf{A}^\dagger)^n \mathbf{f}^{-1}\mathbf{A}^n = \mathbf{f}^{-1}, \quad (\mathbf{B}^\dagger)^n \mathbf{g}^{-1}\mathbf{B}^n = \mathbf{g}^{-1}. \tag{169}$$

Using the normal-mode turn-by-turn equations

$$\widehat{\mathbf{X}} = \mathbf{A}^n\widehat{\mathbf{X}}_0, \quad \widehat{\mathbf{Y}} = \mathbf{B}^n\widehat{\mathbf{Y}}_0 \tag{170}$$

one then has

$$\widehat{\mathbf{X}}^\dagger \mathbf{f}^{-1} \widehat{\mathbf{X}} = \widehat{\mathbf{X}}_0^\dagger (\mathbf{A}^\dagger)^n \mathbf{f}^{-1}\mathbf{A}^n\widehat{\mathbf{X}}_0 = \widehat{\mathbf{X}}_0^\dagger \mathbf{f}^{-1} \widehat{\mathbf{X}}_0 = \epsilon_1 \tag{171}$$

and

$$\widehat{\mathbf{Y}}^\dagger \mathbf{g}^{-1} \widehat{\mathbf{Y}} = \widehat{\mathbf{Y}}_0^\dagger (\mathbf{B}^\dagger)^n \mathbf{g}^{-1}\mathbf{B}^n\widehat{\mathbf{Y}}_0 = \widehat{\mathbf{Y}}_0^\dagger \mathbf{g}^{-1} \widehat{\mathbf{Y}}_0 = \epsilon_2. \tag{172}$$

In terms of the matrix elements and vector components we have

$$\epsilon_1 = \gamma_1\widehat{X}_0^2 + 2\alpha_1\widehat{X}_0\widehat{X}_0' + \beta_1\widehat{X}_0'^2 = \gamma_1\widehat{X}^2 + 2\alpha_1\widehat{X}\widehat{X}' + \beta_1\widehat{X}'^2 \tag{173}$$

$$\epsilon_2 = \gamma_2\widehat{Y}_0^2 + 2\alpha_2\widehat{Y}_0\widehat{Y}_0' + \beta_2\widehat{Y}_0'^2 = \gamma_2\widehat{Y}^2 + 2\alpha_2\widehat{Y}\widehat{Y}' + \beta_2\widehat{Y}'^2. \tag{174}$$

These are the Courant-Snyder Invariants of the motion. They also can be written as

$$\epsilon_1\beta_1 = \widehat{X}_0^2 + (\alpha_1\widehat{X}_0 + \beta_1\widehat{X}_0')^2 = \widehat{X}^2 + (\alpha_1\widehat{X} + \beta_1\widehat{X}')^2 \tag{175}$$

$$\epsilon_2\beta_2 = \widehat{Y}_0^2 + (\alpha_2\widehat{Y}_0 + \beta_2\widehat{Y}_0')^2 = \widehat{Y}^2 + (\alpha_2\widehat{Y} + \beta_2\widehat{Y}')^2 \tag{176}$$

which show that each invariant is zero if and only if the corresponding normal-mode positions and angles are zero.

Note that since

$$\mathbf{D}^{-1} = \begin{pmatrix} \mathbf{f}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{g}^{-1} \end{pmatrix} \tag{177}$$

we have

$$\widehat{\mathbf{Z}}^\dagger\,\mathbf{D}^{-1}\widehat{\mathbf{Z}} = (\widehat{\mathbf{X}}^\dagger\,\mathbf{f}^{-1}\,\widehat{\mathbf{X}}) + (\widehat{\mathbf{Y}}^\dagger\,\mathbf{g}^{-1}\,\widehat{\mathbf{Y}}) = \epsilon_1 + \epsilon_2. \tag{178}$$

Writing the matched ellipsoid equation $\mathbf{Z}^\dagger\mathbf{E}^{-1}\mathbf{Z} = \epsilon$ in terms of normal-mode coordinates and using $\mathbf{E} = \mathbf{RDR}^\dagger$ we have

$$\epsilon = \mathbf{Z}^\dagger\mathbf{E}^{-1}\mathbf{Z} = \widehat{\mathbf{Z}}^\dagger\,\mathbf{R}^\dagger\mathbf{E}^{-1}\mathbf{R}\,\widehat{\mathbf{Z}} = \widehat{\mathbf{Z}}^\dagger\mathbf{D}^{-1}\widehat{\mathbf{Z}} \tag{179}$$

and therefore

$$\epsilon = \epsilon_1 + \epsilon_2. \tag{180}$$

## 13 Envelope Parameters

Constraints on the range of the turn-by-turn positions and angles $X$, $X'$, $Y$, $Y'$ follow from projections of the matched ellipsoid onto the $X$, $X'$ and $Y$, $Y'$ planes. The matched ellipsoid

$$\mathbf{Z}^\dagger\mathbf{E}^{-1}\mathbf{Z} = \epsilon \tag{181}$$

has projections [11]

$$\mathbf{X}^\dagger\mathbf{F}^{-1}\mathbf{X} \le \epsilon, \quad \mathbf{Y}^\dagger\mathbf{G}^{-1}\mathbf{Y} \le \epsilon \tag{182}$$

where

$$\mathbf{F} = d^2\mathbf{f} + \overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger, \quad \mathbf{G} = d^2\mathbf{g} + \mathbf{W}\mathbf{f}\mathbf{W}^\dagger \tag{183}$$

and

$$\mathbf{X} = \begin{pmatrix} X \\ X' \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} Y \\ Y' \end{pmatrix}. \tag{184}$$

The positions and angles in the two planes are therefore constrained to remain inside the ellipses

$$\mathbf{X}^\dagger \mathbf{F}^{-1} \mathbf{X} = \epsilon, \quad \mathbf{Y}^\dagger \mathbf{G}^{-1} \mathbf{Y} = \epsilon. \tag{185}$$

Further constraints are imposed by the Courant-Snyder invariants

$$\widehat{\mathbf{X}}^\dagger \mathbf{f}^{-1} \widehat{\mathbf{X}} = \epsilon_1, \quad \widehat{\mathbf{Y}}^\dagger \mathbf{g}^{-1} \widehat{\mathbf{Y}} = \epsilon_2. \tag{186}$$

Returning to equations (164) and (165) we have

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2, \quad \mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 \tag{187}$$

where

$$\mathbf{X}_1 = d\widehat{\mathbf{X}}, \quad \mathbf{X}_2 = \overline{\mathbf{W}}\widehat{\mathbf{Y}}, \quad \mathbf{Y}_1 = -\mathbf{W}\widehat{\mathbf{X}}, \quad \mathbf{Y}_2 = d\widehat{\mathbf{Y}}. \tag{188}$$

Now if $d \neq 0$ and $|\mathbf{W}| \neq 0$, the matrices $d^2\mathbf{f}$, $\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger$, $\mathbf{W}\mathbf{f}\mathbf{W}^\dagger$, $d^2\mathbf{g}$ all have inverses and are all positive definite. We may then write

$$\mathbf{X}_1^\dagger (d^2\mathbf{f})^{-1} \mathbf{X}_1 = \widehat{\mathbf{X}}^\dagger \mathbf{f}^{-1} \widehat{\mathbf{X}} = \epsilon_1, \tag{189}$$

$$\mathbf{X}_2^\dagger (\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)^{-1} \mathbf{X}_2 = \widehat{\mathbf{Y}}^\dagger \mathbf{g}^{-1} \widehat{\mathbf{Y}} = \epsilon_2, \tag{190}$$

$$\mathbf{Y}_1^\dagger (\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)^{-1} \mathbf{Y}_1 = \widehat{\mathbf{X}}^\dagger \mathbf{f}^{-1} \widehat{\mathbf{X}} = \epsilon_1, \tag{191}$$

$$\mathbf{Y}_2^\dagger (d^2\mathbf{g})^{-1} \mathbf{Y}_2 = \widehat{\mathbf{Y}}^\dagger \mathbf{g}^{-1} \widehat{\mathbf{Y}} = \epsilon_2 \tag{192}$$

and we see that $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ are each constrained to lie on an ellipse. The motion in each plane is therefore the superposition of motion on two ellipses. This characterization of the motion was first given by Ripken [12, 13]. The maximum possible values of $X$, $X'$, $Y$, $Y'$ are given by the diagonal elements of the ellipse matrices. Thus

$$|X| \leq \sqrt{d^2 f_{11} \epsilon_1} + \sqrt{(\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)_{11} \epsilon_2} \tag{193}$$

$$|X'| \leq \sqrt{d^2 f_{22} \epsilon_1} + \sqrt{(\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)_{22} \epsilon_2} \tag{194}$$

$$|Y| \leq \sqrt{(\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)_{11} \epsilon_1} + \sqrt{d^2 g_{11} \epsilon_2} \tag{195}$$

$$|Y'| \leq \sqrt{(\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)_{22} \epsilon_1} + \sqrt{d^2 g_{22} \epsilon_2} \tag{196}$$

where

$$f_{11} = \beta_1, \quad f_{22} = \gamma_1, \quad g_{11} = \beta_2, \quad g_{22} = \gamma_2 \tag{197}$$

$$(\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)_{11} = \frac{1}{\beta_1}\left\{W_{12}^2 + (\alpha_1 W_{12} - \beta_1 W_{11})^2\right\} \tag{198}$$

$$(\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)_{22} = \frac{1}{\gamma_1}\left\{W_{21}^2 + (\alpha_1 W_{21} - \gamma_1 W_{22})^2\right\} \tag{199}$$

$$(\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)_{11} = \frac{1}{\beta_2}\left\{W_{12}^2 + (\alpha_2 W_{12} + \beta_2 W_{22})^2\right\} \tag{200}$$

$$(\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)_{22} = \frac{1}{\gamma_2}\left\{W_{21}^2 + (\alpha_2 W_{21} + \gamma_2 W_{11})^2\right\}. \tag{201}$$

If the approximation $\overline{\mathbf{W}}\mathbf{g} \approx \mathbf{f}\mathbf{W}^\dagger$ is valid then

$$\mathbf{W}\mathbf{f}\mathbf{W}^\dagger \approx (1-d^2)\mathbf{g}, \quad \overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger \approx (1-d^2)\mathbf{f} \tag{202}$$

and equations (193–196) become

$$|X| \leq \sqrt{d^2\beta_1\epsilon_1} + \sqrt{(1-d^2)\beta_1\epsilon_2} \tag{203}$$

$$|X'| \leq \sqrt{d^2\gamma_1\epsilon_1} + \sqrt{(1-d^2)\gamma_1\epsilon_2} \tag{204}$$

$$|Y| \leq \sqrt{(1-d^2)\beta_2\epsilon_1} + \sqrt{d^2\beta_2\epsilon_2} \tag{205}$$

$$|Y'| \leq \sqrt{(1-d^2)\gamma_2\epsilon_1} + \sqrt{d^2\gamma_2\epsilon_2}. \tag{206}$$

# 14 Formulae for the Analysis of Turn-by-Turn Data Acquired from BPMs

Writing out the components of the equations

$$\mathbf{X} = d\widehat{\mathbf{X}} + \overline{\mathbf{W}}\widehat{\mathbf{Y}} = d\mathbf{A}^n\widehat{\mathbf{X}}_0 + \overline{\mathbf{W}}\mathbf{B}^n\widehat{\mathbf{Y}}_0 \tag{207}$$

$$\mathbf{Y} = -\mathbf{W}\widehat{\mathbf{X}} + d\widehat{\mathbf{Y}} = -\mathbf{W}\mathbf{A}^n\widehat{\mathbf{X}}_0 + d\mathbf{B}^n\widehat{\mathbf{Y}}_0 \tag{208}$$

one finds [14] that the horizontal and vertical positions and angles of a bunch at a given BPM (Beam Position Monitor) on the $n$th turn around the machine are given by

$$X = A_1\cos(n\psi_1 + \phi_1) + A_2\cos(n\psi_2 + \phi_2) \tag{209}$$

$$Y = B_1\cos(n\psi_1 + \eta_1) + B_2\cos(n\psi_2 + \eta_2) \tag{210}$$

$$X' = -\frac{A_1}{\beta_1}\left\{\sin\xi_1 + \alpha_1\cos\xi_1\right\} - \frac{A_2}{D_2}\left\{(1-d^2)\sin\xi_2 - E_2\cos\xi_2\right\} \tag{211}$$

$$Y' = -\frac{B_2}{\beta_2}\left\{\sin\zeta_2 + \alpha_2\cos\zeta_2\right\} - \frac{B_1}{D_1}\left\{(1-d^2)\sin\zeta_1 - E_1\cos\zeta_1\right\} \tag{212}$$

where

$$\psi_1 = 2\pi Q_1, \quad \psi_2 = 2\pi Q_2 \tag{213}$$

$$\xi_1 = n\psi_1 + \phi_1, \quad \xi_2 = n\psi_2 + \phi_2 \tag{214}$$

$$\zeta_1 = n\psi_1 + \eta_1, \quad \zeta_2 = n\psi_2 + \eta_2 \tag{215}$$

$$D_2 = (\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)_{11} = \beta_2 W_{22}^2 + 2\alpha_2 W_{22}W_{12} + \gamma_2 W_{12}^2 \tag{216}$$

$$D_1 = (\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)_{11} = \beta_1 W_{11}^2 - 2\alpha_1 W_{11}W_{12} + \gamma_1 W_{12}^2 \tag{217}$$

and

$$E_2 = (\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)_{12} = -\beta_2 W_{21}W_{22} - \alpha_2 W - \gamma_2 W_{11}W_{12} \tag{218}$$

$$E_1 = (\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)_{12} = \beta_1 W_{11}W_{21} - \alpha_1 W + \gamma_1 W_{12}W_{22} \tag{219}$$

$$W = W_{11}W_{22} + W_{12}W_{21}. \tag{220}$$

Defining parameters

$$\widehat{\beta}_1 = \frac{D_2}{1-d^2}, \quad \widehat{\alpha}_1 = \frac{-E_2}{1-d^2}, \quad \widehat{\beta}_2 = \frac{D_1}{1-d^2}, \quad \widehat{\alpha}_2 = \frac{-E_1}{1-d^2} \tag{221}$$

equations (211) and (212) become

$$X' = -\frac{A_1}{\beta_1}\left\{\sin\xi_1 + \alpha_1\cos\xi_1\right\} - \frac{A_2}{\widehat{\beta}_1}\left\{\sin\xi_2 + \widehat{\alpha}_1\cos\xi_2\right\} \tag{222}$$

$$Y' = -\frac{B_2}{\beta_2}\left\{\sin\zeta_2 + \alpha_2\cos\zeta_2\right\} - \frac{B_1}{\widehat{\beta}_2}\left\{\sin\zeta_1 + \widehat{\alpha}_2\cos\zeta_1\right\}. \tag{223}$$

If expressions (209) and (210) are fitted to the turn-by-turn positions obtained from a given BPM, then one obtains fitted parameters $A_1$, $A_2$, $B_1$, $B_2$, $Q_1$, $Q_2$, $\phi_1$, $\phi_2$, $\eta_1$, and $\eta_2$. If the turn-by-turn angles at the BPM are also known, then a fit of expressions (222) and (223) to the data yields values for parameters $\beta_1$, $\alpha_1$, $\beta_2$, $\alpha_2$, and for parameters $\widehat{\beta}_1$, $\widehat{\alpha}_1$, $\widehat{\beta}_2$, $\widehat{\alpha}_2$. We shall see that having the values of all of these parameters allows for a determination of all of the Edwards-Teng parameters. This then gives the complete one-turn matrix. We will also show that if the approximation $\overline{\mathbf{W}}\mathbf{g} \approx \mathbf{f}\mathbf{W}^\dagger$ is valid, then some of the Edwards-Teng parameters can be determined from a smaller set of fitted parameters.

## 14.1 Amplitudes and Phases

The amplitudes $A_1$, $A_2$, $B_1$, $B_2$ and phases $\phi_1$, $\phi_2$, $\eta_1$, $\eta_2$ are given by

$$A_1 \cos\phi_1 = d\widehat{X}_0, \tag{224}$$

$$-A_1 \sin\phi_1 = d(\alpha_1 \widehat{X}_0 + \beta_1 \widehat{X}'_0), \tag{225}$$

$$A_2 \cos\phi_2 = W_{22}\widehat{Y}_0 - W_{12}\widehat{Y}'_0, \tag{226}$$

$$-A_2 \sin\phi_2 = (W_{22}\alpha_2 + W_{12}\gamma_2)\widehat{Y}_0 + (W_{22}\beta_2 + W_{12}\alpha_2)\widehat{Y}'_0, \tag{227}$$

and

$$B_2 \cos\eta_2 = d\widehat{Y}_0, \tag{228}$$

$$-B_2 \sin\eta_2 = d(\alpha_2 \widehat{Y}_0 + \beta_2 \widehat{Y}'_0), \tag{229}$$

$$B_1 \cos\eta_1 = -W_{11}\widehat{X}_0 - W_{12}\widehat{X}'_0, \tag{230}$$

$$-B_1 \sin\eta_1 = -(W_{11}\alpha_1 - W_{12}\gamma_1)\widehat{X}_0 - (W_{11}\beta_1 - W_{12}\alpha_1)\widehat{X}'_0 \tag{231}$$

where $\widehat{X}_0$, $\widehat{X}'_0$, $\widehat{Y}_0$, $\widehat{Y}'_0$ are the initial normal-mode coordinates. These in turn are given by

$$\widehat{X}_0 = dX_0 - W_{22}Y_0 + W_{12}Y'_0, \quad \widehat{Y}_0 = W_{11}X_0 + W_{12}X'_0 + dY_0 \tag{232}$$

$$\widehat{X}'_0 = dX'_0 + W_{21}Y_0 - W_{11}Y'_0, \quad \widehat{Y}'_0 = W_{21}X_0 + W_{22}X'_0 + dY'_0 \tag{233}$$

where $X_0$, $X'_0$, $Y_0$, $Y'_0$ are the initial positions and angles of the bunch at the BPM. Equations (224–231) give

$$A_1 = \sqrt{d^2\beta_1\epsilon_1}, \quad A_2 = \sqrt{D_2\epsilon_2}, \quad B_1 = \sqrt{D_1\epsilon_1}, \quad B_2 = \sqrt{d^2\beta_2\epsilon_2} \tag{234}$$

where

$$\epsilon_1 = \gamma_1\widehat{X}_0^2 + 2\alpha_1\widehat{X}_0\widehat{X}'_0 + \beta_1\widehat{X}_0'^2, \quad \epsilon_2 = \gamma_2\widehat{Y}_0^2 + 2\alpha_2\widehat{Y}_0\widehat{Y}'_0 + \beta_2\widehat{Y}_0'^2. \tag{235}$$

Here we see that by forming the ratios [15]

$$R_1 = \frac{B_1}{A_1} = \sqrt{\frac{D_1}{d^2\beta_1}}, \quad R_2 = \frac{A_2}{B_2} = \sqrt{\frac{D_2}{d^2\beta_2}} \tag{236}$$

and

$$R = \frac{A_2B_1}{A_1B_2} = \frac{1}{d^2}\sqrt{\frac{D_1D_2}{\beta_1\beta_2}} \tag{237}$$

one obtains functions of the Edwards-Teng parameters that are independent of the initial conditions.

Using the identities

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \tag{238}$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \tag{239}$$

equations (224–231) also give

$$\cos(\phi_1 - \eta_1) = \frac{W_{12}\alpha_1 - W_{11}\beta_1}{\sqrt{\beta_1 D_1}}, \quad \sin(\phi_1 - \eta_1) = \frac{W_{12}}{\sqrt{\beta_1 D_1}} \tag{240}$$

$$\cos(\phi_2 - \eta_2) = \frac{W_{22}\beta_2 + W_{12}\alpha_2}{\sqrt{\beta_2 D_2}}, \quad \sin(\phi_2 - \eta_2) = \frac{-W_{12}}{\sqrt{\beta_2 D_2}} \tag{241}$$

where

$$\beta_1 D_1 = (W_{12}\alpha_1 - W_{11}\beta_1)^2 + W_{12}^2 \tag{242}$$

$$\beta_2 D_2 = (W_{22}\beta_2 + W_{12}\alpha_2)^2 + W_{12}^2. \tag{243}$$

Here again one obtains functions of the Edwards-Teng parameters that are independent of the initial conditions.

## 14.2 Determination of $d^2$ and the Coupling Strength

The fitted parameters $A_1$, $A_2$, $B_1$, $B_2$, $\beta_1$, $\beta_2$, $\widehat{\beta}_1$, $\widehat{\beta}_2$ allow for a determination of the parameters $d^2$, $D_1$, and $D_2$. Using the relations

$$\widehat{\beta}_1 = \frac{D_2}{1-d^2}, \quad \widehat{\beta}_2 = \frac{D_1}{1-d^2} \tag{244}$$

and the ratios

$$R_1 = \frac{B_1}{A_1} = \sqrt{\frac{D_1}{d^2\beta_1}}, \quad R_2 = \frac{A_2}{B_2} = \sqrt{\frac{D_2}{d^2\beta_2}} \tag{245}$$

one finds

$$\frac{1-d^2}{d^2} = \frac{\beta_1 R_1^2}{\widehat{\beta}_2}, \quad \frac{1-d^2}{d^2} = \frac{\beta_2 R_2^2}{\widehat{\beta}_1} \tag{246}$$

which give the value of $d^2$. The values of $D_1$ and $D_2$ are then given by

$$D_1 = d^2\beta_1 R_1^2, \quad D_2 = d^2\beta_2 R_2^2. \tag{247}$$

The coupling strength is determined from relation

$$4|\mathbf{m}+\overline{\mathbf{n}}| = U^2 - T^2 = 4U^2 d^2(1-d^2) \tag{248}$$

where

$$U = 2\cos\psi_1 - 2\cos\psi_2, \quad \psi_1 = 2\pi Q_1, \quad \psi_2 = 2\pi Q_2. \tag{249}$$

Here the determinant $|\mathbf{m}+\overline{\mathbf{n}}|$ is proportional to the square of the skew quadrupole or solenoidal fields that couple the motions in the horizontal and vertical planes. Note that one also can obtain $|\mathbf{m}+\overline{\mathbf{n}}|$ by adjusting the lattice quadrupoles so that $T = 0$. In this case one has

$$|\mathbf{m}+\overline{\mathbf{n}}| = U^2/4 = \{\cos(2\pi Q_1) - \cos(2\pi Q_2)\}^2 . \tag{250}$$

Now, if the approximation $\overline{\mathbf{W}}\mathbf{g} \approx \mathbf{f}\mathbf{W}^\dagger$ is valid, then

$$\mathbf{W}\mathbf{f}\mathbf{W}^\dagger \approx (1-d^2)\mathbf{g}, \quad \overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger \approx (1-d^2)\mathbf{f} \tag{251}$$

and we have

$$D_2 = (\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)_{11} \approx (1-d^2)\beta_1 \tag{252}$$

$$D_1 = (\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)_{11} \approx (1-d^2)\beta_2. \tag{253}$$

Thus

$$A_1 = \sqrt{d^2\beta_1\epsilon_1}, \quad A_2 \approx \sqrt{(1-d^2)\beta_1\epsilon_2} \tag{254}$$

$$B_1 \approx \sqrt{(1-d^2)\beta_2\epsilon_1}, \quad B_2 = \sqrt{d^2\beta_2\epsilon_2} \tag{255}$$

and the ratios $R_1$, $R_2$, $R$ become

$$R_1 = \frac{B_1}{A_1} \approx \sqrt{\frac{(1-d^2)\beta_2}{d^2\beta_1}}, \quad R_2 = \frac{A_2}{B_2} \approx \sqrt{\frac{(1-d^2)\beta_1}{d^2\beta_2}} \tag{256}$$

$$R = \frac{A_2 B_1}{A_1 B_2} \approx \frac{1-d^2}{d^2}. \tag{257}$$

The parameter $d^2$ is then given by

$$d^2 \approx \frac{1}{R+1}, \quad 1-d^2 \approx \frac{R}{R+1}. \tag{258}$$

Here we see that only the fitted parameters $A_1$, $A_2$, $B_1$, $B_2$ are needed to obtain $d^2$.

## 14.3 Determination of $W_{11}$, $W_{12}$, and $W_{22}$

Having obtained parameters $\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2$, $D_1$, $D_2$, and phases $\phi_1$, $\phi_2$, $\eta_1$, $\eta_2$, we can obtain $W_{11}$, $W_{12}$, and $W_{22}$ from the relations

$$\cos(\phi_1 - \eta_1) = \frac{W_{12}\alpha_1 - W_{11}\beta_1}{\sqrt{\beta_1 D_1}}, \quad \sin(\phi_1 - \eta_1) = \frac{W_{12}}{\sqrt{\beta_1 D_1}} \tag{259}$$

$$\cos(\phi_2 - \eta_2) = \frac{W_{22}\beta_2 + W_{12}\alpha_2}{\sqrt{\beta_2 D_2}}, \quad \sin(\phi_2 - \eta_2) = \frac{-W_{12}}{\sqrt{\beta_2 D_2}}. \tag{260}$$

If the approximation $\overline{\mathbf{W}}\mathbf{g} \approx \mathbf{f}\mathbf{W}^\dagger$ is valid, then

$$D_2 = (\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)_{11} \approx (1-d^2)\beta_1, \quad D_1 = (\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)_{11} \approx (1-d^2)\beta_2 \tag{261}$$

and the expressions for the cosine and sine of $(\phi_1 - \eta_1)$ and $(\phi_2 - \eta_2)$ become

$$\cos(\phi_1 - \eta_1) \approx \frac{W_{12}\alpha_1 - W_{11}\beta_1}{\sqrt{(1-d^2)\beta_1\beta_2}}, \quad \sin(\phi_1 - \eta_1) \approx \frac{W_{12}}{\sqrt{(1-d^2)\beta_1\beta_2}} \tag{262}$$

$$\cos(\phi_2 - \eta_2) \approx \frac{W_{22}\beta_2 + W_{12}\alpha_2}{\sqrt{(1-d^2)\beta_1\beta_2}}, \quad \sin(\phi_2 - \eta_2) \approx \frac{-W_{12}}{\sqrt{(1-d^2)\beta_1\beta_2}}. \tag{263}$$

In this case one does not need the values of $D_1$ and $D_2$ to obtain $W_{11}$, $W_{12}$, and $W_{22}$.

## 14.4 Excitation of a Single Normal Mode

If just one of the normal modes is excited, say the one with tune $Q_1$, then we have

$$X = A_1 \cos(n\psi_1 + \phi_1), \quad X' = -\frac{A_1}{\beta_1}\{\sin\xi_1 + \alpha_1 \cos\xi_1\} \tag{264}$$

$$Y = B_1 \cos(n\psi_1 + \eta_1), \quad Y' = -\frac{B_1}{\widehat{\beta}_2}\{\sin\zeta_1 + \widehat{\alpha}_2 \cos\zeta_1\} \tag{265}$$

where

$$\psi_1 = 2\pi Q_1, \quad \xi_1 = n\psi_1 + \phi_1, \quad \zeta_1 = n\psi_1 + \eta_1 \tag{266}$$

$$A_1 = \sqrt{d^2\beta_1\epsilon_1}, \quad B_1 = \sqrt{D_1\epsilon_1}, \quad \widehat{\beta}_2 = \frac{D_1}{1-d^2}, \quad \widehat{\alpha}_2 = \frac{-E_1}{1-d^2} \tag{267}$$

$$D_1 = (\mathbf{WfW}^\dagger)_{11}, \quad E_1 = (\mathbf{WfW}^\dagger)_{12}. \tag{268}$$

Thus

$$\alpha_1 X + \beta_1 X' = -A_1 \sin\xi_1, \quad \widehat{\alpha}_2 Y + \widehat{\beta}_2 Y' = -B_1 \sin\zeta_1 \tag{269}$$

and

$$X^2 + (\alpha_1 X + \beta_1 X')^2 = A_1^2 = d^2\beta_1\epsilon_1, \tag{270}$$

$$Y^2 + (\widehat{\alpha}_2 Y + \widehat{\beta}_2 Y')^2 = B_1^2 = D_1\epsilon_1. \tag{271}$$

Here we see that the turn-by-turn positions and angles in the $X$, $X'$ and $Y$, $Y'$ planes lie on ellipses. If turn-by-turn data are available for $X$ and $X'$, then a fit of equations (264) to the data yields values for parameters $Q_1$, $A_1$, $\phi_1$, $\beta_1$, and $\alpha_1$. Similarly, a fit of equations (265) to turn-by-turn data for $Y$ and $Y'$ yields values for $Q_1$, $B_1$, $\eta_1$, $\widehat{\beta}_2$, and $\widehat{\alpha}_2$.

The turn-by-turn positions in the $X$, $Y$ plane also lie on an ellipse. Here we have

$$X = A_1 \cos\xi_1, \quad Y = B_1 \cos(\xi_1 - \omega_1) \tag{272}$$

where

$$\omega_1 = \phi_1 - \eta_1. \tag{273}$$

Thus

$$Y = B_1 \{\cos\xi_1 \cos\omega_1 + \sin\xi_1 \sin\omega_1\} \tag{274}$$

and assuming $\sin\omega_1 \neq 0$, we can write

$$Y = B_1 \sin\omega_1 \left\{\sin\xi_1 + \left(\frac{\cos\omega_1}{\sin\omega_1}\right)\cos\xi_1\right\} \tag{275}$$

and

$$Y = -\frac{A_1}{\beta}\{\sin\xi_1 + \alpha\cos\xi_1\} \tag{276}$$

where

$$\beta = -\frac{A_1}{B_1 \sin\omega_1}, \quad \alpha = \frac{\cos\omega_1}{\sin\omega_1}. \tag{277}$$

Thus

$$X^2 + (\alpha X + \beta Y)^2 = A_1^2 \tag{278}$$

and we see that the horizontal and vertical positions lie on an ellipse with Courant-Snyder parameters $\beta$ and $\alpha$.

## 15 Turn-by-Turn "Action" Variables

As mentioned at the beginning of these notes, one of the hallmarks of linear coupling is the exchange of oscillation energy between the horizontal and vertical planes when the difference between the tunes is close to an integer. Here we exhibit this phenomenon by examining appropriately defined "action" or amplitude variables under the approximation

$$\overline{\mathbf{W}}\mathbf{g} \approx \mathbf{f}\mathbf{W}^\dagger. \tag{279}$$

Consider the projections

$$\mathbf{X}^\dagger \mathbf{F}^{-1} \mathbf{X} \le \epsilon, \quad \mathbf{Y}^\dagger \mathbf{G}^{-1} \mathbf{Y} \le \epsilon \tag{280}$$

of the matched ellipsoid

$$\mathbf{Z}^\dagger \mathbf{E}^{-1} \mathbf{Z} = \epsilon \tag{281}$$

onto the $X$, $X'$ and $Y$, $Y'$ planes. As already mentioned, these show that the positions and angles in the two planes are constrained to remain inside the ellipses

$$\mathbf{X}^\dagger \mathbf{F}^{-1} \mathbf{X} = \epsilon, \quad \mathbf{Y}^\dagger \mathbf{G}^{-1} \mathbf{Y} = \epsilon \tag{282}$$

where

$$\mathbf{F} = d^2\mathbf{f} + \overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger, \quad \mathbf{G} = d^2\mathbf{g} + \mathbf{W}\mathbf{f}\mathbf{W}^\dagger. \tag{283}$$

This suggests that we define "action" variables $J_x$ and $J_y$ such that

$$2J_x b_x = X^2 + (a_x X + b_x X')^2, \quad 2J_y b_y = Y^2 + (a_y Y + b_y Y')^2 \tag{284}$$

where

$$b_x = F_{11}/F, \quad a_x = -F_{12}/F, \quad g_x = F_{22}/F, \quad F = |\mathbf{F}|^{1/2}, \tag{285}$$

$$b_y = G_{11}/G, \quad a_y = -G_{12}/G, \quad g_y = G_{22}/G, \quad G = |\mathbf{G}|^{1/2} \tag{286}$$

are the Courant-Snyder parameters of the two ellipses.

The turn-by-turn positions and angles $X$, $X'$, $Y$, $Y'$ are given by

$$X = A_1 \cos\xi_1 + A_2 \cos\xi_2, \quad \xi_1 = n\psi_1 + \phi_1, \quad \xi_2 = n\psi_2 + \phi_2 \tag{287}$$

$$Y = B_1 \cos\zeta_1 + B_2 \cos\zeta_2, \quad \zeta_1 = n\psi_1 + \eta_1, \quad \zeta_2 = n\psi_2 + \eta_2 \tag{288}$$

$$X' = -\frac{A_1}{\beta_1}\{\sin\xi_1 + \alpha_1 \cos\xi_1\} - \frac{A_2}{\widehat{\beta}_1}\{\sin\xi_2 + \widehat{\alpha}_1 \cos\xi_2\} \tag{289}$$

$$Y' = -\frac{B_2}{\beta_2}\left\{\sin\zeta_2 + \alpha_2\cos\zeta_2\right\} - \frac{B_1}{\widehat{\beta}_2}\left\{\sin\zeta_1 + \widehat{\alpha}_2\cos\zeta_1\right\}. \tag{290}$$

where

$$\psi_1 = 2\pi Q_1, \quad \psi_2 = 2\pi Q_2 \tag{291}$$

and

$$\widehat{\beta}_1 = \frac{D_2}{1-d^2}, \quad \widehat{\alpha}_1 = \frac{-E_2}{1-d^2}, \quad \widehat{\beta}_2 = \frac{D_1}{1-d^2}, \quad \widehat{\alpha}_2 = \frac{-E_1}{1-d^2}. \tag{292}$$

If the approximation $\overline{\mathbf{W}}\mathbf{g} \approx \mathbf{f}\mathbf{W}^\dagger$ is valid, then

$$\mathbf{W}\mathbf{f}\mathbf{W}^\dagger \approx (1-d^2)\mathbf{g}, \quad \overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger \approx (1-d^2)\mathbf{f} \tag{293}$$

and

$$D_2 = (\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)_{11} \approx (1-d^2)\beta_1, \quad \widehat{\beta}_1 \approx \beta_1 \tag{294}$$

$$D_1 = (\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)_{11} \approx (1-d^2)\beta_2, \quad \widehat{\beta}_2 \approx \beta_2 \tag{295}$$

$$E_2 = (\overline{\mathbf{W}}\mathbf{g}\overline{\mathbf{W}}^\dagger)_{12} \approx -(1-d^2)\alpha_1, \quad \widehat{\alpha}_1 \approx \alpha_1 \tag{296}$$

$$E_1 = (\mathbf{W}\mathbf{f}\mathbf{W}^\dagger)_{12} \approx -(1-d^2)\alpha_2, \quad \widehat{\alpha}_2 \approx \alpha_2. \tag{297}$$

The expressions for $X'$ and $Y'$ then become

$$X' \approx -\frac{A_1}{\beta_1}\left\{\sin\xi_1 + \alpha_1\cos\xi_1\right\} - \frac{A_2}{\beta_1}\left\{\sin\xi_2 + \alpha_1\cos\xi_2\right\} \tag{298}$$

$$Y' \approx -\frac{B_1}{\beta_2}\left\{\sin\zeta_1 + \alpha_2\cos\zeta_1\right\} - \frac{B_2}{\beta_2}\left\{\sin\zeta_2 + \alpha_2\cos\zeta_2\right\}. \tag{299}$$

We also have

$$\mathbf{F} \approx \mathbf{f}, \quad \mathbf{G} \approx \mathbf{g} \tag{300}$$

and therefore

$$a_x \approx \alpha_1, \quad b_x \approx \beta_1, \quad a_y \approx \alpha_2, \quad b_y \approx \beta_2. \tag{301}$$

Using these approximations we then have

$$a_xX + b_xX' \approx \alpha_1 X + \beta_1 X' \approx -A_1\sin\xi_1 - A_2\sin\xi_2 \tag{302}$$

$$a_yY + b_yY' \approx \alpha_2 Y + \beta_2 Y' \approx -B_1\sin\zeta_1 - B_2\sin\zeta_2 \tag{303}$$

and

$$X^2 + (a_xX + b_xX')^2 \approx A_1^2 + A_2^2 + 2A_1A_2\cos(\xi_1 - \xi_2) \tag{304}$$

$$Y^2 + (a_yY + b_yY')^2 \approx B_1^2 + B_2^2 + 2B_1B_2\cos(\zeta_1 - \zeta_2). \tag{305}$$

Thus

$$2b_x J_x \approx A_1^2 + A_2^2 + 2A_1 A_2 \cos(\xi_1 - \xi_2) \tag{306}$$

$$2b_y J_y \approx B_1^2 + B_2^2 + 2B_1 B_2 \cos(\zeta_1 - \zeta_2) \tag{307}$$

and

$$J_x \approx \frac{A_1^2 + A_2^2}{2\beta_1} + \frac{A_1 A_2}{\beta_1}\cos(\xi_1 - \xi_2) \tag{308}$$

$$J_y \approx \frac{B_1^2 + B_2^2}{2\beta_2} + \frac{B_1 B_2}{\beta_2}\cos(\zeta_1 - \zeta_2). \tag{309}$$

Here

$$A_1 A_2 = \sqrt{d^2 \beta_1 D_2 \epsilon_1 \epsilon_2} \;\approx\; \beta_1 \sqrt{d^2(1-d^2)\epsilon_1\epsilon_2} \tag{310}$$

$$B_1 B_2 = \sqrt{d^2 \beta_2 D_1 \epsilon_1 \epsilon_2} \;\approx\; \beta_2 \sqrt{d^2(1-d^2)\epsilon_1\epsilon_2} \tag{311}$$

and

$$\xi_1 - \xi_2 = 2\pi n(Q_1 - Q_2) + (\phi_1 - \phi_2), \tag{312}$$

$$\zeta_1 - \zeta_2 = 2\pi n(Q_1 - Q_2) + (\eta_1 - \eta_2). \tag{313}$$

The approximation $\overline{\mathbf{W}}\mathbf{g} \approx \mathbf{f}\mathbf{W}^\dagger$ also implies

$$\cos(\phi_1 - \eta_1) \approx \frac{W_{12}\alpha_1 - W_{11}\beta_1}{\sqrt{(1-d^2)\beta_1\beta_2}}, \quad \sin(\phi_1 - \eta_1) \approx \frac{W_{12}}{\sqrt{(1-d^2)\beta_1\beta_2}} \tag{314}$$

$$\cos(\phi_2 - \eta_2) \approx \frac{W_{22}\beta_2 + W_{12}\alpha_2}{\sqrt{(1-d^2)\beta_1\beta_2}}, \quad \sin(\phi_2 - \eta_2) \approx \frac{-W_{12}}{\sqrt{(1-d^2)\beta_1\beta_2}} \tag{315}$$

and

$$W_{22}\beta_2 + W_{12}\alpha_2 \approx W_{11}\beta_1 - W_{12}\alpha_1. \tag{316}$$

It follows that

$$\sin(\phi_2 - \eta_2) \approx -\sin(\phi_1 - \eta_1), \quad \cos(\phi_2 - \eta_2) \approx -\cos(\phi_1 - \eta_1) \tag{317}$$

and therefore

$$(\phi_2 - \eta_2) \approx (\phi_1 - \eta_1) \pm \pi, \quad (\phi_2 - \phi_1) \approx (\eta_2 - \eta_1) \pm \pi. \tag{318}$$

Thus

$$\cos(\zeta_1 - \zeta_2) \approx -\cos(\xi_1 - \xi_2) \tag{319}$$

and we have

$$J_x \approx \frac{A_1^2 + A_2^2}{2\beta_1} + \sqrt{d^2(1-d^2)\epsilon_1\epsilon_2}\,\cos(\xi_1 - \xi_2) \tag{320}$$

$$J_y \approx \frac{B_1^2 + B_2^2}{2\beta_2} - \sqrt{d^2(1-d^2)\epsilon_1\epsilon_2}\,\cos(\xi_1 - \xi_2) \tag{321}$$

$$J_x + J_y \approx \frac{A_1^2 + A_2^2}{2\beta_1} + \frac{B_1^2 + B_2^2}{2\beta_2}. \tag{322}$$

Here we see clearly the periodic exchange of oscillation amplitude between the two planes. Since $\xi_1 - \xi_2 = 2\pi n(Q_1 - Q_2) + (\phi_1 - \phi_2)$, the period of the exchange is

$$P = 1/(Q_1 - Q_2) \tag{323}$$

turns. The sum of the "action" variables is approximately constant, a result that also follows from the standard Hamiltonian treatment of difference resonances. Note that we also have

$$\frac{A_1^2 + A_2^2}{2\beta_1} \approx \frac{d^2\beta_1\epsilon_1}{2\beta_1} + \frac{(1-d^2)\beta_1\epsilon_2}{2\beta_1} = \frac{1}{2}\left\{d^2\epsilon_1 + (1-d^2)\epsilon_2)\right\} \tag{324}$$

$$\frac{B_1^2 + B_2^2}{2\beta_2} \approx \frac{(1-d^2)\beta_2\epsilon_1}{2\beta_2} + \frac{d^2\beta_2\epsilon_2}{2\beta_2} = \frac{1}{2}\left\{(1-d^2)\epsilon_1 + d^2\epsilon_2)\right\} \tag{325}$$

and therefore

$$2(J_x + J_y) \approx \epsilon_1 + \epsilon_2 \tag{326}$$

as one might expect.

# 16 Propagation of Edwards-Teng Parameters

We include here for completeness a derivation of the formulae for propagating the Edwards-Teng parameters from one point in a synchrotron to another. The derivation is similar to that given in Ref. [16] except that an alternate proof is given for the result that the four-by-four matrix (**L**) which transforms **U** from one point to another is block diagonal.

Let $\mathbf{T}_0$ and $\mathbf{T}$ be the transfer matrices for one turn around the machine starting at $s_0$ and $s$ respectively, and let $\mathcal{M}$ be the transfer matrix from $s_0$ to $s$. Then we have

$$\mathbf{T} = \mathcal{M}\mathbf{T}_0\mathcal{M}^{-1}, \tag{327}$$

and it follows that $\mathbf{T}_0$ and $\mathbf{T}$ have the same eigenvalues which we have assumed are distinct and all lie on the unit circle in the complex plane. In terms of the Edwards-Teng matrices we have

$$\mathbf{T}_0 = \mathbf{R}_0\mathbf{U}_0\mathbf{R}_0^{-1}, \quad \mathbf{T} = \mathbf{R}\mathbf{U}\mathbf{R}^{-1} \tag{328}$$

where

$$\mathbf{R}_0 = \begin{pmatrix} d_0\mathbf{I} & \overline{\mathbf{W}}_0 \\ -\mathbf{W}_0 & d_0\mathbf{I} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} d\mathbf{I} & \overline{\mathbf{W}} \\ -\mathbf{W} & d\mathbf{I} \end{pmatrix} \tag{329}$$

$$\mathbf{U}_0 = \begin{pmatrix} \mathbf{A}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_0 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \tag{330}$$

$$\mathbf{A}_0 = \mathbf{I}\cos\psi_1 + \mathbf{K}_1\sin\psi_1, \quad \mathbf{B}_0 = \mathbf{I}\cos\psi_2 + \mathbf{K}_2\sin\psi_2 \tag{331}$$

$$\mathbf{A} = \mathbf{I}\cos\psi_1 + \mathbf{J}_1\sin\psi_1, \quad \mathbf{B} = \mathbf{I}\cos\psi_2 + \mathbf{J}_2\sin\psi_2. \tag{332}$$

$$\mathbf{K}_1 = \begin{pmatrix} a_1 & b_1 \\ -g_1 & -a_1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} a_2 & b_2 \\ -g_2 & -a_2 \end{pmatrix} \tag{333}$$

$$\mathbf{J}_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} \tag{334}$$

$$b_1g_1 - a_1^2 = 1, \quad b_2g_2 - a_2^2 = 1, \quad \beta_1\gamma_1 - \alpha_1^2 = 1, \quad \beta_2\gamma_2 - \alpha_2^2 = 1. \tag{335}$$

We have assumed here that $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}_0$ and that $\mathbf{B}$ has the same eigenvalues as $\mathbf{B}_0$. The only other possibility is that $\mathbf{A}$ has the same eigenvalues as $\mathbf{B}_0$ while $\mathbf{B}$ has the same eigenvalues as $\mathbf{A}_0$. This second possibility is discussed in Ref. [16] and will not be considered here.

### 16.1 The Matrix L

In terms of the Edwards-Teng matrices, the equation

$$\mathbf{T} = \mathcal{M}\mathbf{T}_0\mathcal{M}^{-1} \tag{336}$$

becomes

$$\mathbf{R}\mathbf{U}\mathbf{R}^{-1} = \mathcal{M}\mathbf{R}_0\mathbf{U}_0\mathbf{R}_0^{-1}\mathcal{M}^{-1}. \tag{337}$$

Multiplying by $\mathbf{R}^{-1}$ from the left and by $\mathbf{R}$ from the right we obtain

$$\mathbf{U} = \mathbf{R}^{-1}\mathcal{M}\mathbf{R}_0\mathbf{U}_0\mathbf{R}_0^{-1}\mathcal{M}^{-1}\mathbf{R} = \mathbf{L}\mathbf{U}_0\mathbf{L}^{-1} \tag{338}$$

where $\mathbf{L}$ is the four-by-four matrix

$$\mathbf{L} = \mathbf{R}^{-1}\mathcal{M}\mathbf{R}_0 = \begin{pmatrix} \mathbf{L}_1 & \mathbf{l}_2 \\ \mathbf{l}_1 & \mathbf{L}_2 \end{pmatrix} \tag{339}$$

and $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{l}_1$, $\mathbf{l}_2$ are two-by-two matrices. Since $\mathbf{R}_0$, $\mathcal{M}$, and $\mathbf{R}$ are symplectic, so is $\mathbf{L}$. These equations show that once $\mathbf{L}$ is known, we can obtain $\mathbf{U}$ and $\mathbf{R}$ from $\mathbf{U}_0$ and $\mathbf{R}_0$, i.e.

$$\mathbf{U} = \mathbf{L}\mathbf{U}_0\mathbf{L}^{-1} \tag{340}$$

and

$$\mathbf{R} = \mathcal{M}\mathbf{R}_0\mathbf{L}^{-1}. \tag{341}$$

### 16.2 Proof that L is Block-Diagonal

We now show that, under our assumptions, $\mathbf{L}$ must be block-diagonal. Writing (340) as $\mathbf{UL} = \mathbf{LU}_0$ we find

$$\mathbf{A}\mathbf{L}_1 = \mathbf{L}_1\mathbf{A}_0, \quad \mathbf{B}\mathbf{L}_2 = \mathbf{L}_2\mathbf{B}_0, \quad \mathbf{A}\mathbf{l}_2 = \mathbf{l}_2\mathbf{B}_0, \quad \mathbf{B}\mathbf{l}_1 = \mathbf{l}_1\mathbf{A}_0, \tag{342}$$

and therefore

$$\mathbf{A}|\mathbf{L}_1| = \mathbf{L}_1\mathbf{A}_0\overline{\mathbf{L}}_1, \quad \mathbf{B}|\mathbf{L}_2| = \mathbf{L}_2\mathbf{B}_0\overline{\mathbf{L}}_2, \tag{343}$$

$$\mathbf{A}|\mathbf{l}_2| = \mathbf{l}_2\mathbf{B}_0\bar{\mathbf{l}}_2, \quad \mathbf{B}|\mathbf{l}_1| = \mathbf{l}_1\mathbf{A}_0\bar{\mathbf{l}}_1. \tag{344}$$

Now, if $|\mathbf{l}_1| \neq 0$ or if $|\mathbf{l}_2| \neq 0$, then either $\mathbf{B}$ and $\mathbf{A}_0$, or $\mathbf{A}$ and $\mathbf{B}_0$ are related by a similarity transformation. It then follows that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues, which contradicts our assumption that the eigenvalues of $\mathbf{T}$ are distinct. Thus we must have

$$|\mathbf{l}_1| = |\mathbf{l}_2| = 0. \tag{345}$$

Since $\mathbf{L}$ is symplectic we then have

$$|\mathbf{L}_1| = |\mathbf{L}_2| = 1. \tag{346}$$

Equations (343) and (344) then become

$$\mathbf{A} = \mathbf{L}_1\mathbf{A}_0\overline{\mathbf{L}}_1, \quad \mathbf{B} = \mathbf{L}_2\mathbf{B}_0\overline{\mathbf{L}}_2, \quad \mathbf{l}_2\mathbf{B}_0\bar{\mathbf{l}}_2 = \mathbf{0}, \quad \mathbf{l}_1\mathbf{A}_0\bar{\mathbf{l}}_1 = \mathbf{0} \tag{347}$$

and using

$$\mathbf{A}_0 = \mathbf{I}\cos\psi_1 + \mathbf{K}_1 \sin\psi_1, \quad \mathbf{B}_0 = \mathbf{I}\cos\psi_2 + \mathbf{K}_2 \sin\psi_2 \tag{348}$$

$$\mathbf{A} = \mathbf{I}\cos\psi_1 + \mathbf{J}_1 \sin\psi_1, \quad \mathbf{B} = \mathbf{I}\cos\psi_2 + \mathbf{J}_2 \sin\psi_2 \tag{349}$$

we obtain

$$\mathbf{J}_1 = \mathbf{L}_1\mathbf{K}_1\overline{\mathbf{L}}_1, \quad \mathbf{J}_2 = \mathbf{L}_2\mathbf{K}_2\overline{\mathbf{L}}_2, \quad \mathbf{l}_2\mathbf{K}_2\bar{\mathbf{l}}_2 = \mathbf{0}, \quad \mathbf{l}_1\mathbf{K}_1\bar{\mathbf{l}}_1 = \mathbf{0}. \tag{350}$$

But

$$\mathbf{J}_1 = \mathbf{f}\,\mathbf{S}, \quad \mathbf{J}_2 = \mathbf{g}\,\mathbf{S}, \quad \mathbf{K}_1 = \mathbf{f}_0\mathbf{S}, \quad \mathbf{K}_2 = \mathbf{g}_0\mathbf{S} \tag{351}$$

where

$$\mathbf{f} = \mathcal{F}\mathcal{F}^\dagger = \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix}, \quad \mathbf{g} = \mathcal{G}\mathcal{G}^\dagger = \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} \tag{352}$$

$$\mathbf{f}_0 = \mathcal{F}_0\mathcal{F}_0^\dagger = \begin{pmatrix} b_1 & -a_1 \\ -a_1 & g_1 \end{pmatrix}, \quad \mathbf{g}_0 = \mathcal{G}_0\mathcal{G}_0^\dagger = \begin{pmatrix} b_2 & -a_2 \\ -a_2 & g_2 \end{pmatrix} \tag{353}$$

$$\mathcal{F} = \frac{1}{\sqrt{\beta_1}}\begin{pmatrix} \beta_1 & 0 \\ -\alpha_1 & 1 \end{pmatrix}, \quad \mathcal{G} = \frac{1}{\sqrt{\beta_2}}\begin{pmatrix} \beta_2 & 0 \\ -\alpha_2 & 1 \end{pmatrix} \tag{354}$$

$$\mathcal{F}_0 = \frac{1}{\sqrt{b_1}}\begin{pmatrix} b_1 & 0 \\ -a_1 & 1 \end{pmatrix}, \quad \mathcal{G}_0 = \frac{1}{\sqrt{b_2}}\begin{pmatrix} b_2 & 0 \\ -a_2 & 1 \end{pmatrix}. \tag{355}$$

Equations (350) therefore become

$$\mathbf{f} = \mathbf{L}_1\mathbf{f}_0\,\mathbf{L}_1^\dagger, \quad \mathbf{g} = \mathbf{L}_2\,\mathbf{g}_0\,\mathbf{L}_2^\dagger, \quad \mathbf{l}_2\,\mathbf{g}_0\,\mathbf{l}_2^\dagger = \mathbf{0}, \quad \mathbf{l}_1\mathbf{f}_0\,\mathbf{l}_1^\dagger = \mathbf{0} \tag{356}$$

and we have

$$(\mathbf{l}_1\mathcal{F}_0)(\mathbf{l}_1\mathcal{F}_0)^\dagger = \mathbf{0}, \quad (\mathbf{l}_2\mathcal{G}_0)(\mathbf{l}_2\mathcal{G}_0)^\dagger = \mathbf{0}. \tag{357}$$

It is easily shown that if a matrix $\mathbf{M}$ satisfies $\mathbf{M}\mathbf{M}^\dagger = \mathbf{0}$, then all of its elements must be zero. Thus we have

$$\mathbf{l}_1\mathcal{F}_0 = \mathbf{0}, \quad \mathbf{l}_2\mathcal{G}_0 = \mathbf{0}. \tag{358}$$

Writing

$$\mathbf{l}_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{359}$$

we then have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ -a_1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{360}$$

where $b_1 \neq 0$. It follows immediately that all elements of $\mathbf{l}_1$ must be zero. One finds the same for $\mathbf{l}_2$. Thus $\mathbf{L}$ is block-diagonal and we can write

$$\mathbf{L} = \mathbf{R}^{-1}\mathcal{M}\mathbf{R}_0 = \begin{pmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 \end{pmatrix}. \tag{361}$$

## 16.3 Calculation of L, W, A, and B

Let

$$\mathcal{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \tag{362}$$

where $\mathbf{M}_{ij}$ are two-by-two matrices. Writing (361) as $\mathbf{RL} = \mathcal{M}\mathbf{R}_0$ we have

$$\begin{pmatrix} d\mathbf{I} & \overline{\mathbf{W}} \\ -\mathbf{W} & d\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \begin{pmatrix} d_0\mathbf{I} & \overline{\mathbf{W}}_0 \\ -\mathbf{W}_0 & d_0\mathbf{I} \end{pmatrix} \tag{363}$$

and therefore

$$d\mathbf{L}_1 = d_0\mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{W}_0, \quad d\mathbf{L}_2 = \mathbf{M}_{21}\overline{\mathbf{W}}_0 + d_0\mathbf{M}_{22} \tag{364}$$

$$-\mathbf{W}\mathbf{L}_1 = d_0\mathbf{M}_{21} - \mathbf{M}_{22}\mathbf{W}_0, \quad \overline{\mathbf{W}}\mathbf{L}_2 = \mathbf{M}_{11}\overline{\mathbf{W}}_0 + d_0\mathbf{M}_{12}. \tag{365}$$

Then since $|\mathbf{L}_1| = |\mathbf{L}_2| = 1$, equations (364) give

$$d^2 = |d_0\mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{W}_0| = |\mathbf{M}_{21}\overline{\mathbf{W}}_0 + d_0\mathbf{M}_{22}| \tag{366}$$

and

$$\mathbf{L}_1 = (d_0\mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{W}_0)/d, \quad \mathbf{L}_2 = (\mathbf{M}_{21}\overline{\mathbf{W}}_0 + d_0\mathbf{M}_{22})/d. \tag{367}$$

Thus $\mathbf{L}_1$ and $\mathbf{L}_2$ are completely determined by $d_0$, $\mathbf{W}_0$, $\mathbf{M}_{11}$, $\mathbf{M}_{12}$, $\mathbf{M}_{21}$, and $\mathbf{M}_{22}$.

Once $\mathbf{L}_1$ and $\mathbf{L}_2$ have been calculated from equations (366) and (367), we can obtain $\mathbf{W}$ from either of equations (365). One finds

$$\mathbf{W} = -(d_0\mathbf{M}_{21} - \mathbf{M}_{22}\mathbf{W}_0)\overline{\mathbf{L}}_1. \tag{368}$$

As already shown, we also have

$$\mathbf{A} = \mathbf{L}_1\mathbf{A}_0\overline{\mathbf{L}}_1, \quad \mathbf{B} = \mathbf{L}_2\mathbf{B}_0\overline{\mathbf{L}}_2. \tag{369}$$

## 17 Appendix I

Following the treatment of Iselin [7] we have

$$\mathbf{Tu} = \lambda_1 \mathbf{u}, \quad \mathbf{Tu}^* = \lambda_1^* \mathbf{u}^*, \quad \mathbf{Tv} = \lambda_2 \mathbf{v}, \quad \mathbf{Tv}^* = \lambda_2^* \mathbf{v}^* \tag{370}$$

where

$$\mathbf{u} = \mathbf{a} + i\mathbf{b}, \quad \mathbf{v} = \mathbf{c} + i\mathbf{d} \tag{371}$$

and **a**, **b**, **c**, **d** are the real vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}. \tag{372}$$

The matrix $\mathcal{W}$ is

$$\mathcal{W} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} \tag{373}$$

and we have

$$\mathcal{W}^\dagger \mathbf{S} \mathcal{W} = \begin{pmatrix} \mathbf{a}^\dagger\mathbf{S}\mathbf{a} & \mathbf{a}^\dagger\mathbf{S}\mathbf{b} & \mathbf{a}^\dagger\mathbf{S}\mathbf{c} & \mathbf{a}^\dagger\mathbf{S}\mathbf{d} \\ \mathbf{b}^\dagger\mathbf{S}\mathbf{a} & \mathbf{b}^\dagger\mathbf{S}\mathbf{b} & \mathbf{b}^\dagger\mathbf{S}\mathbf{c} & \mathbf{b}^\dagger\mathbf{S}\mathbf{d} \\ \mathbf{c}^\dagger\mathbf{S}\mathbf{a} & \mathbf{c}^\dagger\mathbf{S}\mathbf{b} & \mathbf{c}^\dagger\mathbf{S}\mathbf{c} & \mathbf{c}^\dagger\mathbf{S}\mathbf{d} \\ \mathbf{d}^\dagger\mathbf{S}\mathbf{a} & \mathbf{d}^\dagger\mathbf{S}\mathbf{b} & \mathbf{d}^\dagger\mathbf{S}\mathbf{c} & \mathbf{d}^\dagger\mathbf{S}\mathbf{d} \end{pmatrix}. \tag{374}$$

(Here and throughout these notes, a dagger denotes the transpose of a vector or matrix, not the Hermitian conjugate.) Since $\mathbf{S}^\dagger = -\mathbf{S}$, we have for any two vectors **j** and **k**,

$$\mathbf{j}^\dagger\mathbf{S}\mathbf{k} = \mathbf{k}^\dagger\mathbf{S}^\dagger\mathbf{j} = -\mathbf{k}^\dagger\mathbf{S}\mathbf{j} \tag{375}$$

and it follows that

$$\mathbf{u}^\dagger\mathbf{S}\mathbf{u} = 0, \quad \mathbf{v}^\dagger\mathbf{S}\mathbf{v} = 0, \tag{376}$$

$$\mathbf{a}^\dagger\mathbf{S}\mathbf{a} = 0, \quad \mathbf{b}^\dagger\mathbf{S}\mathbf{b} = 0, \quad \mathbf{c}^\dagger\mathbf{S}\mathbf{c} = 0, \quad \mathbf{d}^\dagger\mathbf{S}\mathbf{d} = 0. \tag{377}$$

Thus

$$\mathcal{W}^\dagger \mathbf{S} \mathcal{W} = \begin{pmatrix} 0 & \mathbf{a}^\dagger\mathbf{S}\mathbf{b} & \mathbf{a}^\dagger\mathbf{S}\mathbf{c} & \mathbf{a}^\dagger\mathbf{S}\mathbf{d} \\ -\mathbf{a}^\dagger\mathbf{S}\mathbf{b} & 0 & \mathbf{b}^\dagger\mathbf{S}\mathbf{c} & \mathbf{b}^\dagger\mathbf{S}\mathbf{d} \\ -\mathbf{a}^\dagger\mathbf{S}\mathbf{c} & -\mathbf{b}^\dagger\mathbf{S}\mathbf{c} & 0 & \mathbf{c}^\dagger\mathbf{S}\mathbf{d} \\ -\mathbf{a}^\dagger\mathbf{S}\mathbf{d} & -\mathbf{b}^\dagger\mathbf{S}\mathbf{d} & -\mathbf{c}^\dagger\mathbf{S}\mathbf{d} & 0 \end{pmatrix}. \tag{378}$$

Now, using $\mathbf{T}^\dagger\mathbf{S}\mathbf{T} = \mathbf{S}$ we have

$$(\mathbf{Tu})^\dagger\mathbf{STv} = \mathbf{u}^\dagger\mathbf{T}^\dagger\mathbf{STv} = \mathbf{u}^\dagger\mathbf{Sv} \tag{379}$$

and therefore

$$\lambda_1\lambda_2\mathbf{u}^\dagger\mathbf{Sv} = \mathbf{u}^\dagger\mathbf{Sv}. \tag{380}$$

Similarly, one finds

$$\lambda_1^*\lambda_2\mathbf{u}^{*\dagger}\mathbf{Sv} = \mathbf{u}^{*\dagger}\mathbf{Sv} \tag{381}$$

and

$$\lambda_1^*\lambda_1\mathbf{u}^{*\dagger}\mathbf{Su} = \mathbf{u}^{*\dagger}\mathbf{Su}, \quad \lambda_2^*\lambda_2\mathbf{v}^{*\dagger}\mathbf{Sv} = \mathbf{v}^{*\dagger}\mathbf{Sv}. \tag{382}$$

Since the eigenvalues are assumed to be distinct, we have $\lambda_1\lambda_2 \neq 1$ and $\lambda_1^*\lambda_2 \neq 1$, and we see that (380) and (381) imply

$$\mathbf{u}^\dagger\mathbf{Sv} = 0, \quad \mathbf{u}^{*\dagger}\mathbf{Sv} = 0. \tag{383}$$

Furthermore, since the eigenvalues are assumed to lie on the unit circle in the complex plane, we have $\lambda_1^*\lambda_1 = 1$ and $\lambda_2^*\lambda_2 = 1$, and therefore (382) imposes no constraint on the values of $\mathbf{u}^{*\dagger}\mathbf{Su}$ and $\mathbf{v}^{*\dagger}\mathbf{Sv}$.

Let us now write out the products $\mathbf{u}^\dagger\mathbf{Sv}$, $\mathbf{u}^{*\dagger}\mathbf{Sv}$, $\mathbf{u}^{*\dagger}\mathbf{Su}$, $\mathbf{v}^{*\dagger}\mathbf{Sv}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$. We have

$$\mathbf{u}^\dagger\mathbf{Sv} = \mathbf{a}^\dagger\mathbf{Sc} - \mathbf{b}^\dagger\mathbf{Sd} + i(\mathbf{a}^\dagger\mathbf{Sd} + \mathbf{b}^\dagger\mathbf{Sc}) = 0, \tag{384}$$

$$\mathbf{u}^{*\dagger}\mathbf{Sv} = \mathbf{a}^\dagger\mathbf{Sc} + \mathbf{b}^\dagger\mathbf{Sd} + i(\mathbf{a}^\dagger\mathbf{Sd} - \mathbf{b}^\dagger\mathbf{Sc}) = 0, \tag{385}$$

$$\mathbf{u}^{*\dagger}\mathbf{Su} = \mathbf{a}^\dagger\mathbf{Sa} + \mathbf{b}^\dagger\mathbf{Sb} + i(\mathbf{a}^\dagger\mathbf{Sb} - \mathbf{b}^\dagger\mathbf{Sa}) = 2i(\mathbf{a}^\dagger\mathbf{Sb}), \tag{386}$$

$$\mathbf{v}^{*\dagger}\mathbf{Sv} = \mathbf{c}^\dagger\mathbf{Sc} + \mathbf{d}^\dagger\mathbf{Sd} + i(\mathbf{c}^\dagger\mathbf{Sd} - \mathbf{d}^\dagger\mathbf{Sc}) = 2i(\mathbf{c}^\dagger\mathbf{Sd}), \tag{387}$$

and it follows from the first two of these equations that

$$\mathbf{a}^\dagger\mathbf{Sc} = 0, \quad \mathbf{a}^\dagger\mathbf{Sd} = 0, \quad \mathbf{b}^\dagger\mathbf{Sc} = 0, \quad \mathbf{b}^\dagger\mathbf{Sd} = 0. \tag{388}$$

Thus we have

$$\mathcal{W}^\dagger\mathbf{S}\mathcal{W} = \begin{pmatrix} 0 & \mathbf{a}^\dagger\mathbf{Sb} & 0 & 0 \\ -\mathbf{a}^\dagger\mathbf{Sb} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{c}^\dagger\mathbf{Sd} \\ 0 & 0 & -\mathbf{c}^\dagger\mathbf{Sd} & 0 \end{pmatrix}. \tag{389}$$

Since the matrix $\mathcal{W}$ is by construction nonsingular, $\mathcal{W}^\dagger\mathbf{S}\mathcal{W}$ is nonsingular and neither $\mathbf{a}^\dagger\mathbf{Sb}$ nor $\mathbf{c}^\dagger\mathbf{Sd}$ can be zero. Moreover, equations (386) and (387) show that we may choose the normalization of $\mathbf{u}$ and $\mathbf{v}$ such that

$$\mathbf{a}^\dagger\mathbf{Sb} = 1, \quad \mathbf{c}^\dagger\mathbf{Sd} = 1. \tag{390}$$

Thus we have

$$\mathcal{W}^\dagger \mathbf{S} \mathcal{W} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \mathbf{S}. \tag{391}$$

# 18 Appendix II

Suppose that

$$\mathbf{T} = \mathcal{W}\mathcal{U}\mathcal{W}^{-1} = \widehat{\mathcal{W}}\mathcal{U}\widehat{\mathcal{W}}^{-1} \tag{392}$$

where $\mathcal{W}$ and $\widehat{\mathcal{W}}$ are symplectic. Multiplying this by $\mathcal{W}^{-1}$ from the left and by $\mathcal{W}$ from the right we have

$$\mathcal{U} = \mathcal{W}^{-1}\,\widehat{\mathcal{W}}\mathcal{U}\widehat{\mathcal{W}}^{-1}\,\mathcal{W} = \mathcal{O}\mathcal{U}\mathcal{O}^{-1} \tag{393}$$

where

$$\mathcal{O} = \mathcal{W}^{-1}\widehat{\mathcal{W}}. \tag{394}$$

Thus

$$\mathcal{U}\mathcal{O} = \mathcal{O}\mathcal{U} \tag{395}$$

and writing

$$\mathcal{O} = \begin{pmatrix} \mathbf{P} & \mathbf{V} \\ \mathbf{L} & \mathbf{Q} \end{pmatrix} \tag{396}$$

where $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{L}$, $\mathbf{V}$ are two-by-two matrices, we have

$$\begin{pmatrix} \mathcal{A} & \mathbf{0} \\ \mathbf{0} & \mathcal{B} \end{pmatrix}\begin{pmatrix} \mathbf{P} & \mathbf{V} \\ \mathbf{L} & \mathbf{Q} \end{pmatrix} = \begin{pmatrix} \mathbf{P} & \mathbf{V} \\ \mathbf{L} & \mathbf{Q} \end{pmatrix}\begin{pmatrix} \mathcal{A} & \mathbf{0} \\ \mathbf{0} & \mathcal{B} \end{pmatrix}. \tag{397}$$

Carrying out the matrix multiplications one then finds

$$\mathcal{A}\mathbf{P} = \mathbf{P}\mathcal{A}, \quad \mathcal{B}\mathbf{Q} = \mathbf{Q}\mathcal{B}, \quad \mathcal{B}\mathbf{L} = \mathbf{L}\mathcal{A}, \quad \mathcal{A}\mathbf{V} = \mathbf{V}\mathcal{B}. \tag{398}$$

Suppose now that $|\mathbf{L}| \neq 0$. Then we have

$$\mathcal{B} = \mathbf{L}\mathcal{A}\mathbf{L}^{-1} \tag{399}$$

and it follows that $\mathcal{B}$ and $\mathcal{A}$ have the same eigenvalues. This contradicts our assumption that the eigenvalues of $\mathbf{T}$ are distinct. Thus we must have $|\mathbf{L}| = 0$. Similarly, one finds that $|\mathbf{V}| = 0$. Thus the equations $\mathcal{B}\mathbf{L} = \mathbf{L}\mathcal{A}$ and $\mathcal{A}\mathbf{V} = \mathbf{V}\mathcal{B}$ imply

$$\overline{\mathbf{L}}\mathcal{B}\mathbf{L} = \mathbf{0}, \quad \mathbf{L}\mathcal{A}\overline{\mathbf{L}} = \mathbf{0}, \quad \overline{\mathbf{V}}\mathcal{A}\mathbf{V} = \mathbf{0}, \quad \mathbf{V}\mathcal{B}\overline{\mathbf{V}} = \mathbf{0} \tag{400}$$

where

$$\mathbf{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad \overline{\mathbf{L}} = \begin{pmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{pmatrix} \tag{401}$$

$$\mathbf{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad \overline{\mathbf{V}} = \begin{pmatrix} V_{22} & -V_{12} \\ -V_{21} & V_{11} \end{pmatrix} \tag{402}$$

and

$$\mathcal{A} = \begin{pmatrix} \cos\psi_1 & \sin\psi_1 \\ -\sin\psi_1 & \cos\psi_1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \cos\psi_2 & \sin\psi_2 \\ -\sin\psi_2 & \cos\psi_2 \end{pmatrix}. \tag{403}$$

Carrying out the matrix multiplications, one finds

$$(L_{12}^2 + L_{22}^2)\sin\psi_2 = 0, \quad (L_{11}^2 + L_{21}^2)\sin\psi_2 = 0 \tag{404}$$

$$(L_{12}^2 + L_{11}^2)\sin\psi_1 = 0, \quad (L_{22}^2 + L_{21}^2)\sin\psi_1 = 0 \tag{405}$$

with similar equations for the elements of $\mathbf{V}$. Since having both $\sin\psi_1 = 0$ and $\sin\psi_2 = 0$ would contradict our assumption of distinct eigenvalues, at least one of these must be nonzero. It then follows that all elements of $\mathbf{L}$ must be zero. The same is true for $\mathbf{V}$. Thus

$$\mathbf{L} = \mathbf{0}, \quad \mathbf{V} = \mathbf{0}. \tag{406}$$

Now, writing out the matrix elements of $\mathbf{P}\mathcal{A} = \mathcal{A}\mathbf{P}$ and $\mathbf{Q}\mathcal{B} = \mathcal{B}\mathbf{Q}$, one finds

$$(P_{12} + P_{21})\sin\psi_1 = 0, \quad (P_{11} - P_{22})\sin\psi_1 = 0 \tag{407}$$

$$(Q_{12} + Q_{21})\sin\psi_2 = 0, \quad (Q_{11} - Q_{22})\sin\psi_2 = 0. \tag{408}$$

Here our assumption that no eigenvalue is equal to 1 or $-1$ implies that neither $\sin\psi_1$ nor $\sin\psi_2$ can be zero. Thus we have

$$P_{21} = -P_{12}, \quad P_{22} = P_{11}, \quad Q_{21} = -Q_{12}, \quad Q_{22} = Q_{11}. \tag{409}$$

The matrices $\mathcal{O}$, $\mathbf{P}$, and $\mathbf{Q}$ therefore must be of the form

$$\mathcal{O} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix}. \tag{410}$$

Furthermore, since $\mathcal{O}$ is symplectic, we must have $c_1^2 + s_1^2 = 1$ and $c_2^2 + s_2^2 = 1$, and therefore

$$\mathbf{P}\mathbf{P}^\dagger = \mathbf{I}, \quad \mathbf{Q}\mathbf{Q}^\dagger = \mathbf{I}, \quad \mathcal{O}\mathcal{O}^\dagger = \mathbf{I}. \tag{411}$$

# 19 Appendix III

Let $\mathbf{M}$ be a two-by-two matrix that has unit determinant and satisfies

$$\mathbf{M}\mathbf{f}\mathbf{M}^{\dagger} = \mathbf{g} \tag{412}$$

where

$$\mathbf{f} = \mathcal{F}\mathcal{F}^{\dagger} = \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix}, \quad \mathbf{g} = \mathcal{G}\mathcal{G}^{\dagger} = \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} \tag{413}$$

$$\mathcal{F} = \frac{1}{\sqrt{\beta_1}} \begin{pmatrix} \beta_1 & 0 \\ -\alpha_1 & 1 \end{pmatrix}, \quad \mathcal{G} = \frac{1}{\sqrt{\beta_2}} \begin{pmatrix} \beta_2 & 0 \\ -\alpha_2 & 1 \end{pmatrix} \tag{414}$$

and

$$\beta_1\gamma_1 - \alpha_1^2 = 1, \quad \beta_2\gamma_2 - \alpha_2^2 = 1. \tag{415}$$

Then we have

$$\mathbf{M}\mathcal{F}\mathcal{F}^{\dagger}\mathbf{M}^{\dagger} = \mathcal{G}\mathcal{G}^{\dagger} \tag{416}$$

$$(\mathcal{G}^{-1}\mathbf{M}\mathcal{F})(\mathcal{F}^{\dagger}\mathbf{M}^{\dagger}\mathcal{G}^{\dagger -1}) = \mathbf{I} \tag{417}$$

and

$$\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{I} \tag{418}$$

where

$$\mathbf{U} = \mathcal{G}^{-1}\mathbf{M}\mathcal{F}. \tag{419}$$

Thus

$$\mathbf{M} = \mathcal{G}\mathbf{U}\mathcal{F}^{-1} \tag{420}$$

where

$$\mathbf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad |\mathbf{U}| = ad - bc = 1 \tag{421}$$

and

$$a^2 + b^2 = 1, \quad c^2 + d^2 = 1, \quad ac + bd = 0. \tag{422}$$

It follows that

$$c = -b, \quad d = a \tag{423}$$

and therefore

$$\mathbf{U} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \tag{424}$$

for some angle $\omega$.

## References

[1] G. Guignard, “The General Theory of All Sum and Difference Resonances in a Three-Dimensional Magnetic Field in a Synchrotron”, CERN 76-06, 23 March 1976.

[2] G. Guignard, “A General Treatment of Resonances in Accelerators”, CERN 78-11, 10 November 1978.

[3] E. D. Courant and H. S. Snyder, Annals of Physics 3, 1–48 (1958).

[4] K. L. Brown and R. V. Servranckx, Particle Accelerators 36, 121–139 (1991).

[5] A.J. Dragt, “Lectures on Nonlinear Orbit Dynamics”, Physics of High Energy Particle Accelerators, AIP Conference Proceedings 87, 147–313 (1982).

[6] L. Mirsky, “An Introduction to Linear Algebra”, Dover, New York, 1990, pp. 290–296

[7] F. Christoph Iselin, “The MAD Program Physical Methods Manual”, CERN, Geneva, September 15, 1994.

[8] D. A. Edwards and L. C. Teng, “Parameterization of Linear Coupled Motion in Periodic Systems”, IEEE Trans. on Nucl. Sc. 20, 885 (1973). L. C. Teng, “Concerning n-Dimensional Coupled Motions”, NAL Report FN-229, May 3, 1971.

[9] M. Billing, “The Theory of Weakly Coupled Transverse Motion in Storage Rings”, Cornell University Report CBN 85–2.

[10] T. Roser, “Generalized Treatment of Linear Motion in Two-Dimensional Periodic Systems”, University of Michigan Report UM-HE 88-8 (1988).

[11] C.J. Gardner, “Projections of Beam Distributions”, AGS/AD/Tech Note 439, July 11, 1996.

[12] G. Ripken, DESY Report R1–70/04 (1970).

[13] F. Willeke and G. Ripken, “Methods of Beam Optics”, Physics of Particle Accelerators, AIP Conference Proceedings No. 184, pp. 785–819 (1989).

[14] C.J. Gardner, “Formulae for the PIP Analysis of Linear Coupling”, AGS/AD/Tech Note 476, March 30, 1998.

[15] I am indebted to M. Blaskiewicz for pointing out the utility of the parameter $R$.

[16] D. Sagan and D. Rubin, “Propagation of Twiss and Coupling Parameters”, Cornell University Report CBN 96–20.