# Thoughts on Healys Symplectification Algorithm 

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A nice algorithm for tweaking an almost symplectic matrix into a symplectic matrix has been given by Healy in his thesis[1] according to the literature. (I have not seen a copy of his thesis, so I am not sure of his actual wording.) Before discussing the algorithm and its limitations a lemma and theorem will first be proven.

Lemma: Given two square matrices $\mathbf{S}$ and $\mathbf{W}$ of the same rank with $\mathbf{S}^{2}=-\mathbf{I}$ where $\mathbf{I}$ is the identity matrix then

$$
\begin{equation*}
(\mathbf{I}-\mathbf{W S}) \mathbf{S}(\mathbf{I}+\mathbf{S W})=(\mathbf{I}+\mathbf{W S}) \mathbf{S}(\mathbf{I}-\mathbf{S W}) \tag{1}
\end{equation*}
$$

## Proof:

$$
\begin{aligned}
(\mathbf{I}-\mathbf{W S}) \mathbf{S}(\mathbf{I}+\mathbf{S W}) & =(\mathbf{S}+\mathbf{W})(\mathbf{I}+\mathbf{S W}) \\
& \left.=\mathbf{S}+\mathbf{W}+\mathbf{S}^{2} \mathbf{W}+\mathbf{W S W}\right) \\
& \left.=\mathbf{S}-\mathbf{W}-\mathbf{S}^{2} \mathbf{W}+\mathbf{W S W}\right) \\
& =(\mathbf{S}-\mathbf{W})(\mathbf{I}-\mathbf{S W}) \\
& =(\mathbf{I}+\mathbf{W S}) \mathbf{S}(\mathbf{I}-\mathbf{S W}) .
\end{aligned}
$$

Theorem: A symplectic matrix $\mathbf{M}$ may be written in the form

$$
\begin{equation*}
\mathbf{M}=(\mathbf{I}+\mathbf{S W})(\mathbf{I}-\mathbf{S W})^{-1} \tag{2}
\end{equation*}
$$

if and only if $\mathbf{W}$ is a symmetric matrix, and where $\mathbf{S}$ is the metric for the selected representation of the symplectic group. This statement must be qualified with the requirement that

$$
|\mathbf{I}-\mathbf{S W}| \neq 0
$$

A discussion of the restrictions will be given at the end.
In accelerator physics we usually require $\mathbf{S}$ to be a block diagonal $2 n \times 2 n$ matrix with

$$
\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)
$$

in the diagonal blocks. It is worth noting that $\mathbf{S}$ has the properties

$$
\mathbf{S}^{\mathrm{T}}=\mathbf{S}^{-1}=-\mathbf{S} .
$$

## Proof:

Show that if $\mathbf{W}$ is symmetric, then $\mathbf{M}$ is symplectic:

$$
\begin{aligned}
\mathbf{M}^{\mathrm{T}} \mathbf{S M} & =\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right)^{-1}\left(\mathbf{I}+\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right) \mathbf{S}(\mathbf{I}+\mathbf{S W})(\mathbf{I}-\mathbf{S W})^{-1} \\
& =(\mathbf{I}+\mathbf{W S})^{-1}(\mathbf{I}-\mathbf{W S}) \mathbf{S}(\mathbf{I}+\mathbf{S W})(\mathbf{I}-\mathbf{S W})^{-1} \\
& =(\mathbf{I}+\mathbf{W S})^{-1}(\mathbf{I}+\mathbf{W S}) \mathbf{S}(\mathbf{I}-\mathbf{S W})(\mathbf{I}-\mathbf{S W})^{-1} \\
& =\mathbf{S} .
\end{aligned}
$$

Therefore $\mathbf{M}$ is symplectic if $\mathbf{W}$ is symmetric.
Now let us assume that $\mathbf{W}$ is not symmetric, so it can be written as the sum of symmetric and antisymmetric matrices:

$$
\mathbf{W}=\mathbf{P}+\mathbf{Q}
$$

where $\mathbf{P}=\mathbf{P}^{\mathrm{T}}$ and $\mathbf{Q}=-\mathbf{Q}^{\mathrm{T}}$.

$$
\begin{aligned}
\mathbf{M}^{\mathrm{T}} \mathbf{S M}=\mathbf{S} & =\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right)^{-1}\left(\mathbf{I}+\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right) \mathbf{S}(\mathbf{I}+\mathbf{S W})(\mathbf{I}-\mathbf{S W})^{-1} \\
& =\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right)^{-1}(\mathbf{S}+\mathbf{P}-\mathbf{Q})(\mathbf{I}+\mathbf{S P}+\mathbf{S Q})(\mathbf{I}-\mathbf{S W})^{-1} \\
& =\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right)^{-1}\left(\mathbf{S}+\mathbf{P}-\mathbf{Q}-\mathbf{P}-\mathbf{Q}+\mathbf{W}^{\mathrm{T}} \mathbf{S W}\right)(\mathbf{I}-\mathbf{S W})^{-1} \\
& =\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right)^{-1}\left[\left(\mathbf{S}-\mathbf{P}+\mathbf{Q}+\mathbf{P}+\mathbf{Q}+\mathbf{W}^{\mathrm{T}} \mathbf{S W}\right)-4 \mathbf{Q}\right](\mathbf{I}-\mathbf{S W})^{-1} \\
& =\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right)^{-1}\left[\left(\mathbf{S}-\mathbf{W}^{\mathrm{T}}-\mathbf{S}^{2} \mathbf{W}+\mathbf{W}^{\mathrm{T}} \mathbf{S W}\right)-4 \mathbf{Q}\right](\mathbf{I}-\mathbf{S W})^{-1} \\
& =\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right)^{-1}\left[\left(\mathbf{S}-\mathbf{W}^{\mathrm{T}}\right)(\mathbf{I}-\mathbf{S W})-4 \mathbf{Q}\right](\mathbf{I}-\mathbf{S W})^{-1} \\
& =\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right)^{-1}\left[\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right) \mathbf{S}(\mathbf{I}-\mathbf{S W})-4 \mathbf{Q}\right](\mathbf{I}-\mathbf{S W})^{-1} \\
& =\mathbf{S}-4\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right)^{-1} \mathbf{Q}(\mathbf{I}-\mathbf{S W})^{-1} .
\end{aligned}
$$

So assuming that the inverses in the last line exist then $\mathbf{Q}=0$. (Actually the inverse $\left(\mathbf{I}-\mathbf{W}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}}\right)^{-1}$ must exist if its transpose $(\mathbf{I}-\mathbf{S W})^{-1}$ exists.) This proves the theorem.

Given the symplectic matrix $\mathbf{M}$, form a new matrix

$$
\begin{equation*}
\mathbf{V}=\mathbf{S}(\mathbf{I}-\mathbf{M})(\mathbf{I}+\mathbf{M})^{-1} \tag{3}
\end{equation*}
$$

Then

$$
\begin{aligned}
\mathbf{V}+\mathbf{V M} & =\mathbf{S}-\mathbf{S M} \\
(\mathbf{S}+\mathbf{V}) \mathbf{M} & =\mathbf{S}-\mathbf{V} \\
\mathbf{M} & =(\mathbf{S}+\mathbf{V})^{-1}(\mathbf{S}-\mathbf{V}) .
\end{aligned}
$$

Taking the transpose gives

$$
\mathbf{M}^{\mathrm{T}}=\left(\mathbf{S}^{\mathrm{T}}-\mathbf{V}^{\mathrm{T}}\right)\left(\mathbf{S}^{\mathrm{T}}+\mathbf{V}^{\mathrm{T}}\right)^{-1}
$$

and we can calculate the inverse via

$$
\begin{aligned}
\mathbf{M}^{-1}=\mathbf{S M}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}} & =\mathbf{S}\left(\mathbf{S}^{\mathrm{T}}-\mathbf{V}^{\mathrm{T}}\right)\left(\mathbf{S}^{\mathrm{T}}+\mathbf{V}^{\mathrm{T}}\right)^{-1} \mathbf{S}^{\mathrm{T}} \\
& =(\mathbf{I}-\mathbf{S V})(\mathbf{I}+\mathbf{S V})^{-1}
\end{aligned}
$$

Inverting this yields

$$
\mathbf{M}=(\mathbf{I}+\mathbf{S V})(\mathbf{I}-\mathbf{S V})^{-1}
$$

which is identical in form to Eq. 2.

## The symplectification algorithm

The symplectification algorithm for an almost symplectic matrix $\mathbf{M}$ is to calculate $\mathbf{V}$ by the above Eq. 3 (assuming that $|\mathbf{I}-\mathbf{M}| \neq 0$ ), then create a symmetric matrix

$$
\begin{equation*}
\mathbf{W}=\frac{\mathbf{V}+\mathbf{V}^{\mathrm{T}}}{2} \tag{4}
\end{equation*}
$$

which then may be used to calculate a new matrix

$$
\mathbf{M}^{\prime}=(\mathbf{I}+\mathbf{S W})(\mathbf{I}-\mathbf{S W})^{-1},
$$

assuming that $|\mathbf{I}-\mathbf{S W}| \neq 0$. This new matrix $\mathbf{M}^{\prime}$ must be symplectic by the previous theorem, and it should be close to the original matrix $\mathbf{M}$.

Problems with the method occur in constructing $\mathbf{V}$ when

$$
|\mathbf{I}+\mathbf{M}|=0 .
$$

This will happen when $\mathbf{M}$ has at least one eigenvalue equal to -1 . If $|\mathbf{I}-\mathbf{M}| \neq 0$, then we can define the new almost symmetric matrix by

$$
\widehat{\mathbf{V}}=\mathbf{S}(\mathbf{I}+\mathbf{M})(\mathbf{I}-\mathbf{M})^{-1}
$$

with

$$
\begin{gather*}
\widehat{\mathbf{W}}=\frac{\widehat{\mathbf{V}}+\widehat{\mathbf{V}}^{\mathrm{T}}}{\mathbf{2}} \\
\mathbf{M}^{\prime}=-(\mathbf{I}+\mathbf{S} \widehat{\mathbf{W}})(\mathbf{I}-\mathbf{S} \widehat{\mathbf{W}})^{-1} .
\end{gather*}
$$

Now we must have $|\mathbf{I}-\mathbf{S} \widehat{\mathbf{W}}| \neq 0$, and $|\mathbf{I}-\mathbf{M}| \neq 0$.
If $\mathbf{M}$ has at least one eigenvalue equal to +1 , and another equal to -1 then

$$
|\mathbf{I}-\mathbf{M}|=|\mathbf{I}+\mathbf{M}|=0
$$

and this method may not work.

For an example of this, we must be considering a matrix for at least two planes with at least four eigenvalues, since the symplectic matrix must have pairs of eigenvalues equal to 1 and -1 . The matrix

$$
\mathbf{M}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right)
$$

is symplectic and obviously has $|\mathbf{I} \pm \mathbf{M}|=0$, so we cannot hope to construct a symmetric $\mathbf{V}$ in this case.

Consider the perturbation of this matrix

$$
\mathbf{M}=\left(\begin{array}{cccc}
1+\delta & 0 & 0 & 0 \\
0 & 1-\delta & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right)
$$

then

$$
\left.\left.\begin{array}{rl}
\widehat{\mathbf{V}} & =\left(\begin{array}{cccc}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{array}\right)\left(\begin{array}{cccc}
2+\delta & 0 & 0 & 0 \\
0 & 2-\delta & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)\left(\begin{array}{ccc}
-\delta & 0 & 0 \\
0 & 0 \\
0 & \delta & 0 \\
0 \\
0 & 0 & 2
\end{array} 0\right. \\
0 & 0
\end{array} 0\right) 2\right)^{-1} .
$$

While for nonzero values of $\delta$ this exists and is symmetric, the limit of $\widehat{\mathbf{V}}$ blows up as $\delta$
goes to zero, however

$$
\begin{aligned}
\widehat{\mathbf{W}} & =\frac{2}{\delta}\left(\begin{array}{cccc}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) \\
\mathbf{S} \widehat{\mathbf{W}} & =\frac{2}{\delta}\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) \\
\mathbf{M}^{\prime} & =-\left(\begin{array}{llll}
\frac{\delta+2}{\delta} & 0 & 0 & 0 \\
0 & \frac{\delta-2}{\delta} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)\left(\begin{array}{ccc}
\frac{\delta-2}{\delta} & 0 & 0 \\
0 & \frac{\delta+2}{\delta} & 0 \\
0 \\
0 & 0 & 1 \\
0 \\
0 & 0 & 0 \\
1
\end{array}\right) \\
\mathbf{M}^{\prime} & =\left(\begin{array}{lll}
\frac{2+\delta}{2-\delta} & 0 & 0 \\
0 & \frac{2-\delta}{2+\delta} & 0 \\
0 & 0 & -1 \\
0 & 0 & 0 \\
0 & -1
\end{array}\right), \text { and } \\
\lim _{\delta \rightarrow 0} \mathbf{M}^{\prime} & =\left(\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right),
\end{aligned}
$$

as expected.
Consider a different perturbation of the matrix $\mathbf{M}$ :

$$
\mathbf{M}=\left(\begin{array}{cccc}
1+\delta & 0 & 0 & 0 \\
0 & 1+\delta & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right)
$$

then

$$
\left.\left.\begin{array}{rl}
\widehat{\mathbf{V}} & =\left(\begin{array}{cccc}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{array}\right)\left(\begin{array}{cccc}
2+\delta & 0 & 0 & 0 \\
0 & 2+\delta & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)\left(\begin{array}{ccc}
-\delta & 0 & 0 \\
0 & 0 \\
0 & -\delta & 0 \\
0 \\
0 & 0 & 2
\end{array} 0\right. \\
0 & 0 \\
0 & 2
\end{array}\right)^{-1}\right)\left(\begin{array}{cccc}
0 & -\delta-2 & 0 & 0 \\
\delta+2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) \frac{1}{2 \delta}\left(\begin{array}{cccc}
-2 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 \\
0 & 0 & \delta & 0 \\
0 & 0 & 0 & \delta
\end{array}\right) .
$$

Again for nonzero values of $\delta$ this exists, but is antisymmetric so that

$$
\widehat{\mathbf{W}}=\left(\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

This leads to $\mathbf{M}^{\prime}=-\mathbf{I}$, so

$$
\lim _{\delta \rightarrow 0} \mathbf{M}^{\prime}=\mathbf{I} \neq \mathbf{M}
$$

which might be unexpected, and is quite different from the original unperturbed matrix.
Comment on an error in Ref. [3].
In Eq. 14.13 of Ref. [3], Iselin states that a symplectic matrix $\mathbf{F}=\exp (\mathbf{S G})$ with a symmetric matrix $\mathbf{G}$ can be written in the form

$$
\begin{equation*}
\mathbf{F}=[\mathbf{I}+\tanh (\mathbf{S G} / 2)][\mathbf{I}-\tanh (\mathbf{S G} / 2)]^{-1}=(\mathbf{I}+\mathbf{W})(\mathbf{I}-\mathbf{W})^{-1} \tag{I14.13}
\end{equation*}
$$

where $\mathbf{W}$ is symmetric if and only if F is symplectic. This far right-hand side is incorrect, and is probably a typo. He should have replaced $\mathbf{W}$ by $\mathbf{S W}$ in this equation. The middle part of the equation is correct in most cases and basically comes from

$$
\begin{equation*}
e^{x}=\frac{\cosh \frac{x}{2}+\sinh \frac{x}{2}}{\cosh \frac{x}{2}-\sinh \frac{x}{2}}=\left(1+\tanh \frac{x}{2}\right)\left(1-\tanh \frac{x}{2}\right)^{-1} \tag{5}
\end{equation*}
$$

and the fact that Hamilton's equations may be written in the form

$$
\frac{d \mathbf{X}}{d s}=-\mathbf{S C X}=\mathbf{S G X}
$$

where

$$
C_{i j}=C_{j i}=\frac{\partial^{2} H}{\partial X_{i} \partial X_{j}}
$$

Hamilton's equations give the general form of the generators for this matrix representation of the symplectic group $\operatorname{Sp}(2 \mathrm{n}, \mathrm{r})$ with the metric $\mathbf{S}$. For real $x$, Eq. 5 is analytic since $|\tanh (x / 2)|<1$, however for complex $x$ the hyperbolic tangent can take on values of 1 , so that Eq. 5 has poles. In the case where $x=\mathbf{S G}$ is a generator of a symplectic matrix, then the modified equation becomes

$$
\begin{equation*}
e^{\mathbf{S G}}=[\mathbf{I}+\tanh (\mathbf{S G} / 2)][\mathbf{I}-\tanh (\mathbf{S G} / 2)]^{-1}, \tag{6}
\end{equation*}
$$

and this factorization will not work when the matrix $\tanh (\mathbf{S G} / 2)$ has an eigenvalue equal to 1 . We should also note that $\operatorname{since} \tanh (x)=-\tanh (-x)$ is an odd function it can be expanded as

$$
\tanh (x)=\sum_{j=0}^{\infty} A_{j} x^{2 j+1},
$$

so that

$$
\begin{aligned}
\tanh \left(\frac{\mathbf{S G}}{2}\right) \mathbf{S} & =\sum_{j=0}^{\infty} A_{j} \frac{(\mathbf{S G})^{2 j+1} \mathbf{S}}{2}=\mathbf{S} \tanh \left(\frac{\mathbf{G S}}{2}\right) \\
& =\sum_{j=0}^{\infty} A_{j} \frac{(\mathbf{S G})^{2 j+1} \mathbf{S}}{2}(-1)^{2 j+2} \\
& =\left[\tanh \left(\frac{\mathbf{S G}}{2}\right) \mathbf{S}\right]^{\mathrm{T}}=\left[\mathbf{S} \tanh \left(\frac{\mathbf{G S}}{2}\right)\right]^{\mathrm{T}}
\end{aligned}
$$

From this it should be obvious that the last part of Eq. I14.13 should have been written as

$$
(\mathbf{I}+\mathbf{S W})(\mathbf{I}-\mathbf{S W})^{-1}
$$

for symmetric $\mathbf{W}$.

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## References

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2. David Sagan, The Bmad Reference Manual, Rev. 3.6, (2004)
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