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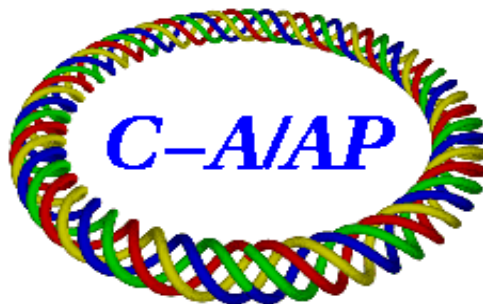
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# Thoughts on Healy's Symplectification Algorithm

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## Thoughts on Healy's Symplectification Algorithm

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A nice algorithm for tweaking an almost symplectic matrix into a symplectic matrix has been given by Healy in his thesis[1] according to the literature. (I have not seen a copy of his thesis, so I am not sure of his actual wording.) Before discussing the algorithm and its limitations a lemma and theorem will first be proven.

**Lemma:** Given two square matrices  $\mathbf{S}$  and  $\mathbf{W}$  of the same rank with  $\mathbf{S}^2 = -\mathbf{I}$  where  $\mathbf{I}$  is the identity matrix then

$$(\mathbf{I} - \mathbf{W}\mathbf{S})\mathbf{S}(\mathbf{I} + \mathbf{S}\mathbf{W}) = (\mathbf{I} + \mathbf{W}\mathbf{S})\mathbf{S}(\mathbf{I} - \mathbf{S}\mathbf{W}). \quad (1)$$

**Proof:**

$$\begin{aligned} (\mathbf{I} - \mathbf{W}\mathbf{S})\mathbf{S}(\mathbf{I} + \mathbf{S}\mathbf{W}) &= (\mathbf{S} + \mathbf{W})(\mathbf{I} + \mathbf{S}\mathbf{W}) \\ &= \mathbf{S} + \mathbf{W} + \mathbf{S}^2\mathbf{W} + \mathbf{W}\mathbf{S}\mathbf{W} \\ &= \mathbf{S} - \mathbf{W} - \mathbf{S}^2\mathbf{W} + \mathbf{W}\mathbf{S}\mathbf{W} \\ &= (\mathbf{S} - \mathbf{W})(\mathbf{I} - \mathbf{S}\mathbf{W}) \\ &= (\mathbf{I} + \mathbf{W}\mathbf{S})\mathbf{S}(\mathbf{I} - \mathbf{S}\mathbf{W}). \end{aligned}$$

**Theorem:** A symplectic matrix  $\mathbf{M}$  may be written in the form

$$\mathbf{M} = (\mathbf{I} + \mathbf{S}\mathbf{W})(\mathbf{I} - \mathbf{S}\mathbf{W})^{-1}, \quad (2)$$

if and only if  $\mathbf{W}$  is a symmetric matrix, and where  $\mathbf{S}$  is the metric for the selected representation of the symplectic group. This statement must be qualified with the requirement that

$$|\mathbf{I} - \mathbf{S}\mathbf{W}| \neq 0.$$

A discussion of the restrictions will be given at the end.

In accelerator physics we usually require  $\mathbf{S}$  to be a block diagonal  $2n \times 2n$  matrix with

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in the diagonal blocks. It is worth noting that  $\mathbf{S}$  has the properties

$$\mathbf{S}^T = \mathbf{S}^{-1} = -\mathbf{S}.$$

**Proof:**

Show that if  $\mathbf{W}$  is symmetric, then  $\mathbf{M}$  is symplectic:

$$\begin{aligned}
\mathbf{M}^T \mathbf{S} \mathbf{M} &= (\mathbf{I} - \mathbf{W}^T \mathbf{S}^T)^{-1} (\mathbf{I} + \mathbf{W}^T \mathbf{S}^T) \mathbf{S} (\mathbf{I} + \mathbf{S} \mathbf{W}) (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1} \\
&= (\mathbf{I} + \mathbf{W} \mathbf{S})^{-1} (\mathbf{I} - \mathbf{W} \mathbf{S}) \mathbf{S} (\mathbf{I} + \mathbf{S} \mathbf{W}) (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1} \\
&= (\mathbf{I} + \mathbf{W} \mathbf{S})^{-1} (\mathbf{I} + \mathbf{W} \mathbf{S}) \mathbf{S} (\mathbf{I} - \mathbf{S} \mathbf{W}) (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1} \\
&= \mathbf{S}.
\end{aligned}$$

Therefore  $\mathbf{M}$  is symplectic if  $\mathbf{W}$  is symmetric.

Now let us assume that  $\mathbf{W}$  is not symmetric, so it can be written as the sum of symmetric and antisymmetric matrices:

$$\mathbf{W} = \mathbf{P} + \mathbf{Q}$$

where  $\mathbf{P} = \mathbf{P}^T$  and  $\mathbf{Q} = -\mathbf{Q}^T$ .

$$\begin{aligned}
\mathbf{M}^T \mathbf{S} \mathbf{M} = \mathbf{S} &= (\mathbf{I} - \mathbf{W}^T \mathbf{S}^T)^{-1} (\mathbf{I} + \mathbf{W}^T \mathbf{S}^T) \mathbf{S} (\mathbf{I} + \mathbf{S} \mathbf{W}) (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1} \\
&= (\mathbf{I} - \mathbf{W}^T \mathbf{S}^T)^{-1} (\mathbf{S} + \mathbf{P} - \mathbf{Q}) (\mathbf{I} + \mathbf{S} \mathbf{P} + \mathbf{S} \mathbf{Q}) (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1} \\
&= (\mathbf{I} - \mathbf{W}^T \mathbf{S}^T)^{-1} (\mathbf{S} + \mathbf{P} - \mathbf{Q} - \mathbf{P} - \mathbf{Q} + \mathbf{W}^T \mathbf{S} \mathbf{W}) (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1} \\
&= (\mathbf{I} - \mathbf{W}^T \mathbf{S}^T)^{-1} [(\mathbf{S} - \mathbf{P} + \mathbf{Q} + \mathbf{P} + \mathbf{Q} + \mathbf{W}^T \mathbf{S} \mathbf{W}) - 4\mathbf{Q}] (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1} \\
&= (\mathbf{I} - \mathbf{W}^T \mathbf{S}^T)^{-1} [(\mathbf{S} - \mathbf{W}^T - \mathbf{S}^2 \mathbf{W} + \mathbf{W}^T \mathbf{S} \mathbf{W}) - 4\mathbf{Q}] (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1} \\
&= (\mathbf{I} - \mathbf{W}^T \mathbf{S}^T)^{-1} [(\mathbf{S} - \mathbf{W}^T) (\mathbf{I} - \mathbf{S} \mathbf{W}) - 4\mathbf{Q}] (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1} \\
&= (\mathbf{I} - \mathbf{W}^T \mathbf{S}^T)^{-1} [(\mathbf{I} - \mathbf{W}^T \mathbf{S}^T) \mathbf{S} (\mathbf{I} - \mathbf{S} \mathbf{W}) - 4\mathbf{Q}] (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1} \\
&= \mathbf{S} - 4(\mathbf{I} - \mathbf{W}^T \mathbf{S}^T)^{-1} \mathbf{Q} (\mathbf{I} - \mathbf{S} \mathbf{W})^{-1}.
\end{aligned}$$

So assuming that the inverses in the last line exist then  $\mathbf{Q} = 0$ . (Actually the inverse  $(\mathbf{I} - \mathbf{W}^T \mathbf{S}^T)^{-1}$  must exist if its transpose  $(\mathbf{I} - \mathbf{S} \mathbf{W})^{-1}$  exists.) This proves the theorem.

Given the symplectic matrix  $\mathbf{M}$ , form a new matrix

$$\mathbf{V} = \mathbf{S}(\mathbf{I} - \mathbf{M})(\mathbf{I} + \mathbf{M})^{-1} \tag{3}.$$

Then

$$\begin{aligned}
\mathbf{V} + \mathbf{V} \mathbf{M} &= \mathbf{S} - \mathbf{S} \mathbf{M}. \\
(\mathbf{S} + \mathbf{V}) \mathbf{M} &= \mathbf{S} - \mathbf{V} \\
\mathbf{M} &= (\mathbf{S} + \mathbf{V})^{-1} (\mathbf{S} - \mathbf{V}).
\end{aligned}$$

Taking the transpose gives

$$\mathbf{M}^T = (\mathbf{S}^T - \mathbf{V}^T)(\mathbf{S}^T + \mathbf{V}^T)^{-1},$$

and we can calculate the inverse via

$$\begin{aligned}\mathbf{M}^{-1} &= \mathbf{S}\mathbf{M}^T\mathbf{S}^T = \mathbf{S}(\mathbf{S}^T - \mathbf{V}^T)(\mathbf{S}^T + \mathbf{V}^T)^{-1}\mathbf{S}^T \\ &= (\mathbf{I} - \mathbf{S}\mathbf{V})(\mathbf{I} + \mathbf{S}\mathbf{V})^{-1}.\end{aligned}$$

Inverting this yields

$$\mathbf{M} = (\mathbf{I} + \mathbf{S}\mathbf{V})(\mathbf{I} - \mathbf{S}\mathbf{V})^{-1},$$

which is identical in form to Eq. 2.

### The symplectification algorithm

The symplectification algorithm for an almost symplectic matrix  $\mathbf{M}$  is to calculate  $\mathbf{V}$  by the above Eq. 3 (assuming that  $|\mathbf{I} - \mathbf{M}| \neq 0$ ), then create a symmetric matrix

$$\mathbf{W} = \frac{\mathbf{V} + \mathbf{V}^T}{2} \quad (4)$$

which then may be used to calculate a new matrix

$$\mathbf{M}' = (\mathbf{I} + \mathbf{S}\mathbf{W})(\mathbf{I} - \mathbf{S}\mathbf{W})^{-1},$$

assuming that  $|\mathbf{I} - \mathbf{S}\mathbf{W}| \neq 0$ . This new matrix  $\mathbf{M}'$  must be symplectic by the previous theorem, and it should be close to the original matrix  $\mathbf{M}$ .

Problems with the method occur in constructing  $\mathbf{V}$  when

$$|\mathbf{I} + \mathbf{M}| = 0.$$

This will happen when  $\mathbf{M}$  has at least one eigenvalue equal to  $-1$ . If  $|\mathbf{I} - \mathbf{M}| \neq 0$ , then we can define the new almost symmetric matrix by

$$\widehat{\mathbf{V}} = \mathbf{S}(\mathbf{I} + \mathbf{M})(\mathbf{I} - \mathbf{M})^{-1} \quad (3')$$

with

$$\widehat{\mathbf{W}} = \frac{\widehat{\mathbf{V}} + \widehat{\mathbf{V}}^T}{2}. \quad (4')$$

$$\mathbf{M}' = -(\mathbf{I} + \mathbf{S}\widehat{\mathbf{W}})(\mathbf{I} - \mathbf{S}\widehat{\mathbf{W}})^{-1}. \quad (2')$$

Now we must have  $|\mathbf{I} - \mathbf{S}\widehat{\mathbf{W}}| \neq 0$ , and  $|\mathbf{I} - \mathbf{M}| \neq 0$ .

If  $\mathbf{M}$  has at least one eigenvalue equal to  $+1$ , and another equal to  $-1$  then

$$|\mathbf{I} - \mathbf{M}| = |\mathbf{I} + \mathbf{M}| = 0.$$

and this method may not work.

For an example of this, we must be considering a matrix for at least two planes with at least four eigenvalues, since the symplectic matrix must have pairs of eigenvalues equal to 1 and  $-1$ . The matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is symplectic and obviously has  $|\mathbf{I} \pm \mathbf{M}| = 0$ , so we cannot hope to construct a symmetric  $\mathbf{V}$  in this case.

Consider the perturbation of this matrix

$$\mathbf{M} = \begin{pmatrix} 1 + \delta & 0 & 0 & 0 \\ 0 & 1 - \delta & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

then

$$\begin{aligned} \widehat{\mathbf{V}} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 + \delta & 0 & 0 & 0 \\ 0 & 2 - \delta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & \delta - 2 & 0 & 0 \\ \delta + 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{2\delta} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \\ &= \frac{1}{\delta} \begin{pmatrix} 0 & \delta - 2 & 0 & 0 \\ -\delta - 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

While for nonzero values of  $\delta$  this exists and is symmetric, the limit of  $\widehat{\mathbf{V}}$  blows up as  $\delta$

goes to zero, however

$$\begin{aligned}
\widehat{\mathbf{W}} &= \frac{2}{\delta} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathbf{s}\widehat{\mathbf{W}} &= \frac{2}{\delta} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathbf{M}' &= - \begin{pmatrix} \frac{\delta+2}{\delta} & 0 & 0 & 0 \\ 0 & \frac{\delta-2}{\delta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\delta-2}{\delta} & 0 & 0 & 0 \\ 0 & \frac{\delta+2}{\delta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \\
\mathbf{M}' &= \begin{pmatrix} \frac{2+\delta}{2-\delta} & 0 & 0 & 0 \\ 0 & \frac{2-\delta}{2+\delta} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{and} \\
\lim_{\delta \rightarrow 0} \mathbf{M}' &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\end{aligned}$$

as expected.

Consider a different perturbation of the matrix  $\mathbf{M}$ :

$$\mathbf{M} = \begin{pmatrix} 1+\delta & 0 & 0 & 0 \\ 0 & 1+\delta & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

then

$$\begin{aligned}
\widehat{\mathbf{V}} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2+\delta & 0 & 0 & 0 \\ 0 & 2+\delta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\delta & 0 & 0 & 0 \\ 0 & -\delta & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0 & -\delta-2 & 0 & 0 \\ \delta+2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{2\delta} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \\
&= \frac{1}{\delta} \begin{pmatrix} 0 & \delta+2 & 0 & 0 \\ -\delta-2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$



Again for nonzero values of  $\delta$  this exists, but is antisymmetric so that

$$\widehat{\mathbf{W}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This leads to  $\mathbf{M}' = -\mathbf{I}$ , so

$$\lim_{\delta \rightarrow 0} \mathbf{M}' = \mathbf{I} \neq \mathbf{M}$$

which might be unexpected, and is quite different from the original unperturbed matrix.

**Comment on an error in Ref. [3].**

In Eq. 14.13 of Ref. [3], Iselin states that a symplectic matrix  $\mathbf{F} = \exp(\mathbf{S}\mathbf{G})$  with a symmetric matrix  $\mathbf{G}$  can be written in the form

$$\mathbf{F} = [\mathbf{I} + \tanh(\mathbf{S}\mathbf{G}/2)][\mathbf{I} - \tanh(\mathbf{S}\mathbf{G}/2)]^{-1} = (\mathbf{I} + \mathbf{W})(\mathbf{I} - \mathbf{W})^{-1}, \quad (\text{I14.13})$$

where  $\mathbf{W}$  is symmetric if and only if  $\mathbf{F}$  is symplectic. This far right-hand side is incorrect, and is probably a typo. He should have replaced  $\mathbf{W}$  by  $\mathbf{S}\mathbf{W}$  in this equation. The middle part of the equation is correct in most cases and basically comes from

$$e^x = \frac{\cosh \frac{x}{2} + \sinh \frac{x}{2}}{\cosh \frac{x}{2} - \sinh \frac{x}{2}} = \left(1 + \tanh \frac{x}{2}\right) \left(1 - \tanh \frac{x}{2}\right)^{-1} \quad (5)$$

and the fact that Hamilton's equations may be written in the form

$$\frac{d\mathbf{X}}{ds} = -\mathbf{S}\mathbf{C}\mathbf{X} = \mathbf{S}\mathbf{G}\mathbf{X},$$

where

$$C_{ij} = C_{ji} = \frac{\partial^2 H}{\partial X_i \partial X_j}.$$

Hamilton's equations give the general form of the generators for this matrix representation of the symplectic group  $\text{Sp}(2n, r)$  with the metric  $\mathbf{S}$ . For real  $x$ , Eq. 5 is analytic since  $|\tanh(x/2)| < 1$ , however for complex  $x$  the hyperbolic tangent can take on values of 1, so that Eq. 5 has poles. In the case where  $x = \mathbf{S}\mathbf{G}$  is a generator of a symplectic matrix, then the modified equation becomes

$$e^{\mathbf{S}\mathbf{G}} = [\mathbf{I} + \tanh(\mathbf{S}\mathbf{G}/2)][\mathbf{I} - \tanh(\mathbf{S}\mathbf{G}/2)]^{-1}, \quad (6)$$

and this factorization will not work when the matrix  $\tanh(\mathbf{S}\mathbf{G}/2)$  has an eigenvalue equal to 1. We should also note that since  $\tanh(x) = -\tanh(-x)$  is an odd function it can be expanded as

$$\tanh(x) = \sum_{j=0}^{\infty} A_j x^{2j+1},$$

so that

$$\begin{aligned}
\tanh\left(\frac{\mathbf{SG}}{2}\right)\mathbf{S} &= \sum_{j=0}^{\infty} A_j \frac{(\mathbf{SG})^{2j+1}\mathbf{S}}{2} = \mathbf{S} \tanh\left(\frac{\mathbf{GS}}{2}\right) \\
&= \sum_{j=0}^{\infty} A_j \frac{(\mathbf{SG})^{2j+1}\mathbf{S}}{2} (-1)^{2j+2} \\
&= \left[ \tanh\left(\frac{\mathbf{SG}}{2}\right)\mathbf{S} \right]^T = \left[ \mathbf{S} \tanh\left(\frac{\mathbf{GS}}{2}\right) \right]^T
\end{aligned}$$

From this it should be obvious that the last part of Eq. I14.13 should have been written as

$$(\mathbf{I} + \mathbf{SW})(\mathbf{I} - \mathbf{SW})^{-1}$$

for symmetric  $\mathbf{W}$ .

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### References

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3. F. Christoph Iselin, *The MAD Program Physical Methods Manual*, CERN/SL/92-?? (AP), unfinished report (1994).