Intrabeam scattering growth rates for a bi-gaussian distribution

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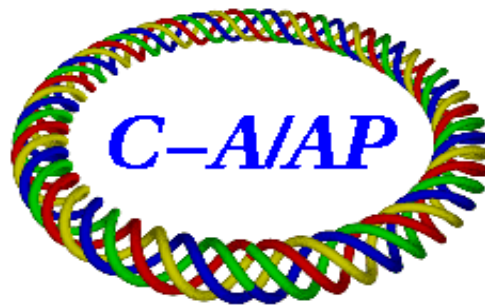# Intrabeam scattering growth rates for a bi-gaussian distribution

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Abstract

This note finds results for the intrabeam scattering growth rates for a bi-gaussian distribution. The bi-gaussian distribution is interesting for studying the possibility of using electron cooling in RHIC. Experiments and computer studies indicate that in the presence of electron cooling, the beam distribution changes so that it developes a strong core and a long tail which is not described well by a gaussian, but may be better described by a bi-gaussian. Being able to compute the effects of intrabeam scattering for a bi-gaussian distribution would be useful in computing the effects of electron cooling, which depend critically on the details of the intrabeam scattering. The calculation is done using the reformulation of intrabeam scattering theory given in [1] based on the treatments given by A. Piwinski [2] and J. Bjorken and S.K. Mtingwa [3]. The bi-gaussian distribution is defined below as the sum of two gaussians in the particle coordinates x, y, s, p_x, p_y, p_s. The gaussian with the smaller dimensions produces most of the core of the beam, and the gaussian with the larger dimensions largely produces the long tail of the beam. The final result for the growth rates are expressed as the sum of three terms which can be interperted respectively as the contribution to the growth rates due to the scattering of the particles in the first gaussian from themselves, the scattering of the particles in the second gaussian from themselves, and the scattering of the particles in the first gaussian from the particles in the second gaussian.

1　Introduction

This note finds results for the intrabeam scattering growth rates for a bi-gaussian distribution.

The bi-gaussian distribution is interesting for studying the possibility of using electron cooling in RHIC. Experiments and computer studies indicate that in the presence of electron cooling, the beam distribution changes so that it developes a strong core and a long tail which is not described well by a gaussian, but may be better described by a bi-gaussian. Being able to compute the effects of intrabeam scattering for a bi-gaussian distribution would be useful in computing the effects of electron cooling, which depend critically on the details of the intrabeam scattering. The calculation is done using the reformulation of intrabeam scattering theory given in [1] based on the treatments given by A. Piwinski [2] and by J. Bjorken and S. Mtingwa [3]. The bi-gaussian distribution is defined below as the sum of two gaussians in the particle coordinates x, y, s, p_x, p_y, p_s. The gaussian with the smaller dimensions produces most of the core of the beam, and the gaussian with the larger dimensions largely produces the long tail of the beam. The final result for the growth rates are expressed as the sum of three terms which can be interperted respectively as the contribution to the growth rates due to the scattering of the particles in the first gaussian from themselves, the scattering of the particles in the second gaussian from themselves, and the scattering of the particles in the first gaussian from the particles in the second gaussian.

2　Basic results for intrabeam scattering

This section lists some general results which can be used to find growth rates for a beam with any particle distribution $f(x.p)$. Following [3], growth rates will be computed for $< p_i p_j >$, where the $<>$ indicates an average over all the particles in the bunch. From these one can compute the growth rates for the emittances, $< \epsilon_i >$. A result that holds in any coordinate system and for any particle distribution $f(x.p)$ is given in [1] as

$$\delta < (p_i p_j) > \;\; = \;\; N \int \; d^3 x \frac{d^3 p_1}{\gamma_1} \frac{d^3 p_2}{\gamma_2} f(x, p_1) f(x, p_2) F(p_1, p_2) C_{ij} dt$$

$$C_{ij} \;\; = \;\; \pi \int_0^\pi d\theta \sigma(\theta) \sin^3 \theta \; \Delta^2 [\delta_{ij} - 3\frac{\Delta_i \Delta_j}{\Delta^2} + \frac{W_i W_j}{W^2}] \;\; i, j = 1, 3$$

$$\begin{aligned}
\Delta_i &= \frac{1}{2}(p_{1i} - p_{2i}) \\
W_i &= p_{1i} + p_{2i}
\end{aligned} \tag{1}$$

$Nf(x, p)$ gives the number of particles in $d^3x d^3p$, where N is the number of particles in a bunch. $\delta < (p_i p_j) >$ is the change in $< (p_i p_j) >$ due to all particle collisions in the time interval dt. The invariants $F(p_1, p_2), \Delta^2, W^2$ are given by

$$\begin{aligned}
F(p_1, p_2) &= c\frac{[(p_1 p_2)^2 - m_1^2 m_2^2 c^4]^{1/2}}{m_1 m_2 c^2} \\
F(p_1, p_2) &= \gamma_1 \gamma_2 c[(\vec{\beta_1} - \vec{\beta_2})^2 - (\vec{\beta_1} \times \vec{\beta_2})^2]^{1/2} \\
\Delta^2 &= \vec{\Delta}^2 - \Delta_0^2, \quad \Delta_0 = (E_1 - E_2)/(2c) \\
W^2 &= \vec{W}^2 - W_0^2, \quad W_0 = (E_1 + E_2)/c
\end{aligned}$$

Eq.(1) is considerably simplified by going to the rest CS , which is the CS moving along with the bunch and the particle motion is non-relativistic, and putting σ equal to the Coulomb cross section. One gets

$$\begin{aligned}
\frac{1}{p_0^2} < \delta(p_{1i} p_{1j}) > &= N \int d^3x d^3p_1 d^3p_2 f(x, p_1) f(x, p_2) 2\bar{\beta}c \, C_{ij} \, dt \\
\Delta_i &= \frac{1}{2}(p_{1i} - p_{2i}) \\
\bar{\beta}c &= |\vec{\Delta}|/m \\
C_{ij} &= \frac{2\pi}{p_0^2}(r_0/2\bar{\beta}^2)^2 \ln(1 + (2\bar{\beta}^2 b_{max}/r_0)^2) \\
&\quad [|\vec{\Delta}|^2 \delta_{ij} - 3\Delta_i \Delta_j] \quad i, j = 1, 3 \\
r_0 &= Z^2 e^2/mc^2 \\
\sigma(\theta) &= [\frac{r_0}{2\bar{\beta}^2}]^2 \frac{1}{(1 - \cos\theta)^2} \\
\cot(\theta_{min}/2) &= 2\bar{\beta}^2 b_{max}/r_0
\end{aligned} \tag{2}$$

b_{max} is the largest allowed impact parameter in the center of mass CS. It has been asumed that one can replace $\ln(1 + (2\bar{\beta}^2 b_{max}/r_0) - 1$ by $\ln(1 + (2\bar{\beta}^2 b_{max}/r_0)$.

In Eq.(1), the original 11-dimensional integral which arises from intra-beam scattering theory has been reduced in [1] to a 9-dimensional integral

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by integrating over all possible scattering angles. In [1] this reduction was done for any particle distribution, $f(x, p)$. In [3], Bjorken and Mtingwa first do the integration over x, p_1, p_2 using a simple gaussian distribution before doing the integration over the scattering angles and no general result for doing this reduction for any $f(x, p)$ is given. In [2] Piwinski computes the growth rates for the emittances $< \epsilon_i >$ instead of for $< p_i p_j >$. A general result for reducing the integral by integrating over all possible scattering angles, for any $f(x, p)$, for the growth rates of $< \epsilon_i >$ is given. However, using this result for a complicated distribution like the bi-gaussian would be difficult.

3 Gaussian distribution

We will first consider the case of a gaussian particle distribution. This will provide a more simple example of using the results in the reformulation given in [1] and of the methods used to evaluate the integrals. Afterwards, the same procedures will be applied to the case of the bi-gaussian distribution.

Let $N f(x, p)$ gives the number of particles in $d^3 x d^3 p$, where N is the number of particles in a bunch. For a gaussian distribution, $f(x, p)$ ls given by

$$
\begin{aligned}
f(x, p) &= \frac{1}{\Gamma} exp[-S(x, p)] \\
\Gamma &= \int d^3 x d^3 p \, exp[-S(x, p)]
\end{aligned}
\tag{3}
$$

$$
S = S_x + S_y + S_s
$$

$$
\begin{aligned}
S_x &= \frac{1}{\epsilon_x} \epsilon_x(x_\beta, x'_\beta) \\
x_\beta &= x - D(p - p_0)/p_0 \\
x'_\beta &= x' - D'(p - p_0)/p_0 \quad x' = p_x/p_0 \\
\epsilon_x(x, x') &= [x^2 + (\beta_x x' + \alpha_x x)^2]/\beta_x
\end{aligned}
$$

$$
\begin{aligned}
S_y &= \frac{1}{\epsilon_y} \epsilon_y(y, y') \quad y' = p_y/p_0 \\
\epsilon_y(y, y') &= [y^2 + (\beta_y y' + \alpha_y y)^2]/\beta_y
\end{aligned}
$$

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$$S_s = \frac{1}{\bar{\epsilon}_s}\epsilon_s(s - s_c, (p - p_0)/p_0)$$

$$\epsilon_s(s - s_c, (p - p_0)/p_0) = \frac{(s - s_c)^2}{2\sigma_s^2} + \frac{((p - p_0)/p_0)^2}{2\sigma_p^2}$$

$$\epsilon_s(s - s_c, (p - p_0)/p_0) = \frac{1}{\beta_s}(s - s_c)^2 + \beta_s((p - p_0)/p_0)^2$$

$$\epsilon_s(s - s_c, (p - p_0)/p_0) = [(s - s_c)^2 + (\beta_s((p - p_0)/p_0))^2]/\beta_s$$

$$\beta_s = \sigma_s/\sigma_p$$

$$\bar{\epsilon}_s = 2\sigma_s\sigma_p \quad (4)$$

D is the horizontal dispersion. $D' = dD/ds$. A longitudinal emittance has been introduced so that the longitudinal motion and the transverse motions can be treated in a similar manner. s_c locates the center of the bunch.

Γ can now be computed using Eq.(1).This will provide an example how the integrals are done in this paper. The integration methods used here are somewhat more complicated than those used in [3] but they will also work for the more complicated bi-gaussian distribution.

$$\Gamma = \int d^3x d^3p \; exp[-S_x - S_y - S_s]$$

Writing Γ as $\Gamma = \Gamma_y\Gamma_{xs}$ and computing Γ_y first because this part is simpler,

$$\Gamma_y = \int dy dp_y \; exp[-S_y]$$

$$S_y = \frac{1}{\bar{\epsilon}_y}\epsilon_y(y, y') \quad y' = p_y/p_0$$

$$\epsilon_y(y, y') = [y^2 + 2 + (\beta_y y' + \alpha_y y)^2]/\beta_y$$

$$\eta_y = y/\sqrt{\beta_y}, \quad p_{\eta y} = (\beta_y y' + \alpha_y y)/\sqrt{\beta_y}$$

$$dy dp_y = p_0 d\eta_y dp_{\eta y}$$

$$\Gamma_y = p_0 \int d\eta_y dp_{\eta y} \; exp[-(\eta_y^2 + p_{\eta y}^2)/\bar{\epsilon}_y]$$

$$\Gamma_y = \pi\bar{\epsilon}_y p_0 \quad (5)$$

Now for the remaining integral we have

$$\Gamma_{xs} = \int dx dp_x ds dp_s \; exp[-S_x - S_s]$$

$$\Gamma_{xs} \quad = \quad \int ds dp_s \, exp[-S_s] \int dx dp_x \, exp[-S_x]$$

Make the transformation

$$x_\beta \quad = \quad x - D(p - p_0)/p_0$$
$$x'_\beta \quad = \quad x' - D'(p - p_0)/p_0$$
$$x' \quad = \quad p_x/p_0, \quad x'_\beta = p_{\beta x}/p_0$$
$$dx dp_x \quad = \quad p_0 dx_\beta dx'_\beta$$

$$\int dx dp_x \, exp[-S_x] \quad = \quad p_0 \int dx_\beta dx'_\beta \, exp[-S_x]$$

$$S_x \quad = \quad \frac{1}{\epsilon_x} \epsilon_x(x_\beta, x'_\beta)$$

$$\int dx dp_x \, exp[-S_x] \quad = \quad \pi \bar\epsilon_x p_0 \text{ as in evaluating } \Gamma_y$$

$p \sim p_s$ in the Lab. CS and

$$\Gamma_{xs} \quad = \quad \pi^2 \bar\epsilon_s \bar\epsilon_x p_0^2$$

$$\Gamma \quad = \quad \pi^3 \bar\epsilon_s \bar\epsilon_x \bar\epsilon_y p_0^3 \qquad (6)$$

4   Growth rates for a Gaussian distribution

In the following,the growth rates are given in the Rest Coordinate System, which is the coordinate system moving along with the bunch. Growth rates are given for $< p_i p_j >$. From these one can compute the growth rates for $< \epsilon_i >$. Using the general result, Eq.(2), one gets

$$\frac{1}{p_0^2} < \delta(p_i p_j) > \quad = \quad \frac{N}{\Gamma^2} \int \, d^3 x d^3 p_1 d^3 p_2 exp[-S(x, p_1) - S(x, p_2)] 2\bar\beta c \, C_{ij} \, dt$$

$$\vec\Delta \quad = \quad \frac{1}{2}(\vec{p_1} - \vec{p_2})$$

$$\bar\beta c \quad = \quad |\vec\Delta|/m$$

$$C_{ij} \quad = \quad \frac{2\pi}{p_0^2}(r_0/2\bar\beta^2)^2 \ln(1 + (2\bar\beta^2 b_{max}/r_0)^2) \, [|\vec\Delta|^2 \delta_{ij} - 3\Delta_i \Delta_j] \, i, j = 1, 3$$

$$r_0 \quad = \quad Z^2 e^2/mc^2$$

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$$\Gamma \quad = \quad \pi^3 \bar{\epsilon}_s \bar{\epsilon}_x \bar{\epsilon}_y p_0^3 \tag{7}$$

Transform to W, Δ

$$p_1 \quad = \quad \frac{W}{2} + \Delta, \qquad p_2 = \frac{W}{2} - \Delta$$

$$W \quad = \quad p_1 + p_2, \qquad \Delta = \frac{p_1 - p_2}{2}$$

$$d^3 p_1 d^3 p_2 \quad = \quad d^3 W d^3 \Delta \tag{8}$$

We will first do the integral over d^3x and over d^3W. For the y part of the integral

$$S_y(y, p_{1y}) \quad = \quad \frac{1}{\bar{\epsilon}_y} \epsilon_y(y, y_1'), \quad y_1' = p_{1y}/p_0$$

$$\epsilon_y(y, y_1') \quad = \quad [y^2 + (\beta_y y_1' + \alpha_y y)^2]/\beta_y$$

$$S_y(y, p_{1y}) \quad = \quad [y^2 + (\beta_y(\frac{W_y}{2} + \Delta_y)/p_0 + \alpha_y y)^2]/(\beta_y \bar{\epsilon}_y)$$

$$S_y(y, p_{1y}) + S_y(y, p_{2y}) \quad = \quad (2y^2/\beta_y + 2(\beta_y(W_y/p_0)/2 + \alpha_y y)^2/\beta_y$$
$$+ 2\beta_y^2(\Delta_y/p_0)^2/\beta_y))/\bar{\epsilon}_y$$

Make the transformation

$$\eta_y \quad = \quad \sqrt{2}y/\sqrt{\beta_y}, \qquad p_{\eta y} = \sqrt{2}(\beta_y(W_y/p_0)/2 + \alpha_y y)/\sqrt{\beta_y}$$

$$dy dW_y \quad = \quad p_0 d\eta_y dp_{\eta y} \tag{9}$$

Integrate over dy, dW_y

$$\int dy dW_y exp[-S_y(y, p_{1y}) - S_y(y, p_{2y})] \quad = \quad p_0 \int d\eta_y dp_{\eta y}$$

$$exp[-\frac{\eta_y^2 + p_{\eta y}^2 + 2\beta_y^2(\Delta_y/p_0)^2/\beta_y}{\bar{\epsilon}_y}]$$

$$= \quad p_0 \pi \bar{\epsilon}_y exp[-\frac{2\beta_y}{\bar{\epsilon}_y}(\Delta_y/p_0)^2]$$

$$= \quad p_0 \pi \bar{\epsilon}_y sxp[-R_y]$$

$$R_y \quad = \quad \frac{2\beta_y}{\bar{\epsilon}_y}(\Delta_y/p_0)^2 \tag{10}$$

In doing the remainder of the integral, the integral over $dx dW_x ds dW_s$ we will do the integral over $dx dW_x$ first and then the integral over $ds dW_s$. Note that the integral is being done in the Rest CS and in the expression for S_x one has to replace $p - p_0 \sim p_s - p_0$ in the Lab CS by γp_s in the Rest CS. Remember also that $f(x, p)$ is an invariant (see [1]) One finds for $S_x(x, p_{1x})$

$$S_x(x, p_{1x}) = \{[x - \gamma D \bar{W}_s/2 - \gamma D \bar{\Delta}_s]^2 + [\beta_x(\bar{W}_x/2 + \bar{\Delta}_x - \gamma D' \bar{W}_s/2 - \gamma D' \bar{\Delta}_s) + \alpha_x(x - \gamma D \bar{W}_s/2 - \gamma D \bar{\Delta}_s)]^2\}/(\beta_x \bar{\epsilon}_x)$$

$$\bar{W}_i = W_i/p_0 \qquad \bar{\Delta}_i = \Delta/p_0$$

$$S_x(y, p_{1x}) = \{[x - \gamma D \bar{W}_s/2 - \gamma D \bar{\Delta}_s]^2 + [\beta_x(\bar{W}_x/2 - \gamma D' \bar{W}_s/2) + \alpha_x(x - \gamma D \bar{W}_s/2) + (\beta_x \bar{\Delta}_x - \gamma \bar{D} \bar{\Delta}_s)]^2\}/(\beta_x \bar{\epsilon}_x)$$

$$\bar{D} = \beta_x D' + \alpha_x D \tag{11}$$

we then find for $S_x(x, p_{1x}) + S_x(x, p_{2x})$

$$S_x(x, p_{1x}) + S_x(x, p_{2x}) = \{2[x - \gamma D \bar{W}_s/2]^2 + 2\gamma^2 D^2 \bar{\Delta}_s^2 + 2[\beta_x(\bar{W}_x/2 - \gamma D' \bar{W}_s/2) + \alpha_x(x - \gamma D \bar{W}_s/2)]^2 + 2[\beta_x \bar{\Delta}_x - \gamma \bar{D} \bar{\Delta}_s]^2\}/(\beta_x \bar{\epsilon}_x) \tag{12}$$

Now make the transformations

$$x^* = \sqrt{2}x - \gamma D \bar{W}_s/\sqrt{2} \qquad p_x^* = \bar{W}_x/\sqrt{2} - \gamma D' \bar{W}_s/\sqrt{2}$$

$$\eta_x = x^*/\sqrt{\beta_x} \qquad p_{\eta_x x} = (\beta_x p_x^* + \alpha_x x^*)/\sqrt{\beta_x}$$

$$dx dW_x = p_0 dx^* dp_x^* = p_0 d\eta_x dp_{\eta_x x} \tag{13}$$

Doing the integral over $dx dW_x$ one finds

$$\int dx dW_x \exp[-S_x(x, p_{1x}) - S_x(x, p_{2x})] = p_0 \int d\eta_x dp_{\eta_x x}$$
$$\exp[-\{\eta_x^2 + p_{\eta_x}^2 + 2[\gamma^2 D^2 \bar{\Delta}_s^2 + (\beta_x \bar{\Delta}_x - \gamma \bar{D} \bar{\Delta}_s)^2]/\beta_x)\}/\bar{\epsilon}_x]$$
$$= p_0 \pi \bar{\epsilon}_x \exp[-R_x]$$

$$R_x = 2[\gamma^2 D^2 \bar{\Delta}_s^2 + (\beta_x \bar{\Delta}_x - \gamma \bar{D} \bar{\Delta}_s)^2]/(\beta_x \bar{\epsilon}_x) \tag{14}$$

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Now do the integral over $dsdW_s$. One may note that the form of the intgral here is similar to the integral done over $dydW_y$. The result is then the same with the proper sustitutions of s for y.

$$\int dxdW_s exp[-S_s(s,p_{1s}) - S_s(s,p_{2s})] = p_0\pi\bar{\epsilon}_s exp[-R_s]$$

$$R_s = \frac{2\gamma^2\beta_s}{\bar{\epsilon}_s}(\Delta_s/p_0)^2 \qquad (15)$$

Note that the term $\beta_s((p-p_0)/p_0)^2$ in S_s in the Lab. CS has to be replaced by $\gamma^2\beta_s(p_s/p_0)^2$ in the Rest CS.

Using Eq.(7), one gets the result for the growth rates in the Rest CS for a gauusian distribution.

$$\frac{1}{p_0^2}\frac{d}{dt} < p_ip_j > = \frac{N}{\Gamma}\int d^3\Delta\ exp[-R]C_{ij}$$

$$C_{ij} = \frac{2\pi}{p_0^2}(r_0/2\bar{\beta}^2)^2(|\Delta|^2\delta_{ij} - 3\Delta_i\Delta_j)2\bar{\beta}c\ ln[1 + (2\bar{\beta}^2 b_{max}/r_0)^2]$$

$$\bar{\beta} = \beta_0\gamma_0|\Delta/p_0|$$

$$r_0 = Z^2e^2/Mc^2$$

$$\Gamma = \pi^3\bar{\epsilon}_s\bar{\epsilon}_x\bar{\epsilon}_y p_0^3$$

$$R = R_x + R_y + R_s$$

$$R_x = \frac{2}{\beta_x\epsilon_x}[\gamma^2 D^2\Delta_s^2 + (\beta_x\Delta_x - \gamma\tilde{D}\Delta_s)^2]/p_0^2$$

$$\tilde{D} = \beta_x D' + \alpha_x D$$

$$R_y = \frac{2\beta_y}{\bar{\epsilon}_y}\Delta_y^2/p_0^2$$

$$R_s = \frac{2\beta_s}{\bar{\epsilon}_s}\gamma^2\Delta_s^2/p_0^2 \qquad (16)$$

The integral over $d^3\Delta$ is an integral over all possible values of the relative momemtum for any two particles in a bunch. β_0, γ_0 are the beta and gamma corresponding to p_0, the central momemtum of the bunch in the Laboratory Coordinate System. $\gamma = \gamma_0$

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The above 3-dimensional integral can be reduced to a 2-dimensional integral by integrating over $|\Delta|$ and using $d^3\Delta = |\Delta|^2 d|\Delta| sin\theta d\theta d\phi$. This gives

$$\frac{1}{p_0^2}\frac{d}{dt} < p_i p_j > = \frac{N}{\Gamma} 2\pi p_0^3 \left(\frac{r_0}{2\gamma_0^2 \beta_0^2}\right)^2 2\beta_0\gamma_0 c \int sin\theta d\theta d\phi \ (\delta_{ij} - 3g_i g_j)$$

$$\frac{1}{F} ln\left[\frac{\hat{C}}{F}\right]$$

$$g_3 = cos\theta = g_s$$

$$g_1 = sin\theta cos\phi = g_x$$

$$g_2 = sin\theta sin\phi = g_y$$

$$\hat{C} = 2\gamma_0^2 \beta_0^2 b_{max}/r_0$$

$$F = R/(|\Delta|/p_0)^2$$

$$F = F_x + F_y + F_s$$

$$F_x = \frac{2}{\beta_x \bar{\epsilon}_x}[\gamma^2 D^2 g_s^2 + (\beta_x g_x - \gamma \bar{D} g_s)^2]$$

$$F_y = \frac{2}{\bar{\epsilon}_y}\beta_y g_y^2$$

$$F_s = \frac{2}{\bar{\epsilon}_s}\beta_s \gamma^2 g_s^2 \qquad (17)$$

In obtaining the above, one uses $z = |\bar{\Delta}|^2, dz = 2|\bar{\Delta}|d|\bar{\Delta}|$ and

$$\int_0^\infty dz \ exp[-Fz]ln[\hat{C}z] = \frac{1}{F}[ln\left[\frac{\hat{C}}{F}\right] - .5772]$$

For $Z = 80, A = 200, \gamma = 100, b_{max} = 1cm$, $\log_{10} \hat{C} = 18.6$

5   Bi-Gaussian distribution

The bi-gaussian distribution will be assumed to have the form given by the following.

$Nf(x,p)$ gives the number of particles in d^3xd^3p, where N is the number of particles in a bunch. For a bi-gaussian distribution, $f(x,p)$ ls given by

$$f(x,p) = \frac{N_a}{N}\frac{1}{\Gamma_a}exp[-S_a(x,p)] + \frac{N_b}{N}\frac{1}{\Gamma_b}exp[-S_b(x,p)]$$

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$$\begin{aligned}
\Gamma_a &= \pi^3 \bar{\epsilon}_{sa} \bar{\epsilon}_{xa} \bar{\epsilon}_{ya} p_0^3 \\
\Gamma_b &= \pi^3 \bar{\epsilon}_{sb} \bar{\epsilon}_{xb} \bar{\epsilon}_{yb} p_0^3
\end{aligned} \tag{18}$$

In the first gaussian, to find Γ_a, S_a then in the expressions for Γ, S, given above for the gaussian distribution, replace $\bar{\epsilon}_x, \bar{\epsilon}_y, \bar{\epsilon}_s$ by $\bar{\epsilon}_{xa}, \bar{\epsilon}_{ya}, \bar{\epsilon}_{sa}$. In the second gaussian, in the expressions for Γ, S, replace $\bar{\epsilon}_x, \bar{\epsilon}_y, \bar{\epsilon}_s$ by $\bar{\epsilon}_{xb}, \bar{\epsilon}_{yb}, \bar{\epsilon}_{sb}$. In addition. $N_a + N_b = N$. This bi-gaussian has 7 parameters instead of the three parameters of a gaussian.

6 Growth rates for a Bi- Gaussian distribution

In the following, the growth rates are given in the Rest Coordinate System, which is the coordinate system moving along with the bunch. Growth rates are given for $< p_i p_j >$. From these one can compute the growth rates for $< \epsilon_i >$. Starting with Eq.2 and using the $f(x,p)$ from Eq.18, one gets

$$\begin{aligned}
\frac{1}{p_0^2} < \delta(p_i p_j) > &= \int d^3 x \, d^3 p_1 d^3 p_2 \left[\frac{N_a}{N} \frac{1}{\Gamma_a} exp[-S_a(x,p_1)] + \frac{N_b}{N} \frac{1}{\Gamma_b} exp[-S_b(x,p_1)] \right] \\
&\qquad \left[\frac{N_a}{N} \frac{1}{\Gamma_a} exp[-S_a(x,p_2)] + \frac{N_b}{N} \frac{1}{\Gamma_b} exp[-S_b(x,p_2)] \right] \\
&\qquad 2 \bar{\beta} c \, C_{ij} \, dt \\
\vec{\Delta} &= \frac{1}{2}(\vec{p_1} - \vec{p_2}) \\
\bar{\beta} c &= |\vec{\Delta}|/m \\
C_{ij} &= \frac{2\pi}{p_0^2} (r_0/2\bar{\beta}^2)^2 \ln(1 + (2\bar{\beta}^2 b_{max}/r_0)^2) \; [|\vec{\Delta}|^2 \delta_{ij} - 3\Delta_i \Delta_j] \;\; i,j = 1,3 \\
r_0 &= Z^2 e^2/mc^2
\end{aligned} \tag{19}$$

The term in the integrand which contains $exp[-S_a(x,p_1) - S_a(x,p_2)]$ is similar to the integrand for the gaussian distribution except that $\bar{\epsilon}_i$ are replaced by $\bar{\epsilon}_{ia}$ and leads to the same result as that given by Eq.(16) for the gaussian beam except that R has to be replaced by R_a where R_a is obtained from R by replacing $\bar{\epsilon}_i$ by $\bar{\epsilon}_{ia}$. The term containing $exp[-S_b(x,p_1) - S_b(x,p_2)]$ can be evaluated in the same way leading to the same result as that given by Eq.(16) for the gaussian beam except that R has to be replaced by R_b where R_b is

obtained from R by replacing $\bar{\epsilon}_i$ by $\bar{\epsilon}_{ib}$. The only terms that need further evaluation are the the two cross product terms. The two cross product terms are equal because of the symmetry of p_1 and p_2 in the rest of the integrand. This leads to the remaining integral to be evaluated

$$\int d^3x d^3p_1 d^3p_2 \frac{2N_a N_b}{N^2} \frac{1}{\Gamma_a \Gamma_b} exp[-S_a(x,p_1) - S_b(x,p_2)] \, 2\bar{\beta}c \, C_{ij}$$

In evaluating this integral, we will use the same procedure as was used for the gaussian distribution. We will first transform to W, Δ from p_1, p_2 (see Eq.(8). We will then do the integral over d^3x and over d^3W. For the y part of the integral one finds ,

$$\begin{aligned} S_{ya}(y,p_{1y}) &= \{y^2 + [\beta_y(\bar{W}_y/2 + \bar{\Delta}_y) + \alpha_y y]^2\}/(\beta_y \bar{\epsilon}_{ya}) \\ \bar{W}_y &= W_y/p_0 \qquad \bar{\Delta}_y = \Delta_y/p_0 \end{aligned} \qquad (20)$$

One then finds that

$$\begin{aligned} S_{ya}(y,p_{1y}) + S_{yb}(y,p_{2y}) &= \{2y^2/\beta_y + 2[\beta_y(\bar{W}_y/2 + \alpha_y y)]^2/\beta_y \\ &\quad + 2\beta_y^2 \bar{\Delta}_y^2/\beta_y)\}/\bar{\epsilon}_{yc} + \\ &\quad \{4(\beta_y \bar{\Delta}_y)(\beta_y \bar{W}_y/2 + \alpha_y y)/\beta_y\}/\bar{\epsilon}_{yd} \\ \frac{1}{\bar{\epsilon}_{yc}} &= \frac{1}{2}(\frac{1}{\bar{\epsilon}_{ya}} + \frac{1}{\bar{\epsilon}_{yb}}) \\ \frac{1}{\bar{\epsilon}_{yd}} &= \frac{1}{2}(\frac{1}{\bar{\epsilon}_{ya}} - \frac{1}{\bar{\epsilon}_{yb}}) \end{aligned}$$

$$(21)$$

Make the transformation

$$\begin{aligned} \eta_y &= \sqrt{2}y/\sqrt{\beta_y}, \qquad p_{\eta y} = \sqrt{2}(\beta_y \bar{W}_y/2 + \alpha_y y)/\sqrt{\beta_y} \\ dy dW_y &= p_0 d\eta_y dp_{\eta y} \end{aligned} \qquad (22)$$

Integrate over dy, dW_y

$$\int dy dW_y ecp[-S_{ya}(y,p_{1y}) - S_{yb}(y,p_{2y})] = p_0 \int d\eta_y dp_{\eta y}$$

$$exp[-\frac{\eta_y^2 + p_{\eta y}^2 + 2\beta_y^2(\Delta_y/p_0)^2/\beta_y}{\bar{\epsilon}_{yc}} +$$

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$$4\beta_y\bar{\Delta}_y\frac{p_{\eta y}/(\sqrt{2}\sqrt{\beta_y})}{\bar{\epsilon}_{yd}}]$$

$$= p_0\pi\bar{\epsilon}_{yc}exp[-\frac{2\beta_y}{\bar{\epsilon}_{yc}}\bar{\Delta}_y^2 + \frac{2\beta_y}{\bar{\epsilon}_{yd}^2/\bar{\epsilon}_{yc}}\bar{\Delta}_y^2]$$

$$= p_0\pi\bar{\epsilon}_{yc}exp[-R_{yc} + R_{yd}]$$

$$R_{yc} = \frac{2\beta_y}{\bar{\epsilon}_{yc}}\bar{\Delta}_y^2$$

$$R_{yd} = \frac{2\beta_y}{\bar{\epsilon}_{yd}^2/\bar{\epsilon}_{yc}}\bar{\Delta}_y^2 \qquad (23)$$

The exponent $R_{yc} - R_{yd}$ has to be positive. This can be made more obvious by noting that

$$\frac{1}{\bar{\epsilon}_{yc}^2} - \frac{1}{\bar{\epsilon}_{yd}^2} = \frac{1}{\bar{\epsilon}_{ya}\bar{\epsilon}_{yb}}$$

In doing the remainder of the integral, the integral over $dxdW_xdsdW_s$ we will do the integral over $dxdW_x$ first and then the integral over $dsdW_s$. Note that the integral is being done in the Rest CS and in the expression for S_x one has to replace $p - p_0 \sim p_s - p_0$ in the Lab. CS by γp_s in the Rest CS. Remember also that $f(x,p)$ is an invariant (see [1]) One finds for $S_{xa}(x, p_{1x})$

$$S_{xa}(x, p_{1x}) = \{[x - \gamma D\bar{W}_s/2 - \gamma D\bar{\Delta}_s]^2 + [\beta_x(\bar{W}_x/2 + \bar{\Delta}_x - \gamma D'\bar{W}_s/2 - \gamma D'\bar{\Delta}_s) + \alpha_x(x - \gamma D\bar{W}_s/2 - \gamma D\bar{\Delta}_s)]^2\}/(\beta_x\bar{\epsilon}_x a)$$

$$\bar{W}_i = W_i/p_0 \qquad \bar{\Delta}_i = \Delta/p_0$$

$$S_{xa}(y, p_{1x}) = \{[x - \gamma D\bar{W}_s/2 - \gamma D\bar{\Delta}_s]^2 + [\beta_x(\bar{W}_x/2 - \gamma D'\bar{W}_s/2) + \alpha_x(x - \gamma D\bar{W}_s/2) + (\beta_x\bar{\Delta}_x - \gamma\bar{D}\bar{\Delta}_s)]^2\}/(\beta_x\bar{\epsilon}_{xa})$$

$$\bar{D} = \beta_x D' + \alpha_x D \qquad (24)$$

we then find for $S_{xa}(x, p_{1x}) + S_{xb}(x, p_{2x})$

$$S_{xa}(x, p_{1x}) + S_{xb}(x, p_{2x}) = \{2[x - \gamma D\bar{W}_s/2]^2 + 2\gamma^2 D^2\bar{\Delta}_s^2 + 2[\beta_x(\bar{W}_x/2 - \gamma D'\bar{W}_s/2) + \alpha_x(x - \gamma D\bar{W}_s/2)]^2 + 2[\beta_x\bar{\Delta}_x - \gamma\bar{D}\bar{\Delta}_s]^2\}/(\beta_x\bar{\epsilon}_{xc}) +$$

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$$\{-4\gamma D\bar{\Delta}_s[x - \gamma D\bar{W}_s/2]/\beta_x$$
$$+4(\beta_x\bar{\Delta}_x - \gamma\bar{D}\bar{\Delta}_s)[\beta_x(\bar{W}_x/2 - \gamma D'\bar{W}_s/2) +$$
$$\alpha_x(x - \gamma D\bar{W}_s/2)]/\beta_x\}/\bar{\epsilon}_{xd}$$

$$(25)$$

Now make the transformations

$$x^* = \sqrt{2}x - \gamma D\bar{W}_s/\sqrt{2} \quad p_x^* = \bar{W}_x/\sqrt{2} - \gamma D'\bar{W}_s/\sqrt{2}$$
$$\eta_x = x^*/\sqrt{\beta_x} \quad p_{\eta_x x} = (\beta_x p_x^* + \alpha_x x^*)/\sqrt{\beta_x}$$
$$dx dW_x = p_0 dx^* dp_x^* = p_0 d\eta_x dp_{\eta_x x}$$

$$(26)$$

Doing the integral over $dx dW_x$ one finds

$$\int dx dW_x exp[-S_{xa}(x, p_{1x}) - S_{xb}(x, p_{2x})] = p_0 \int d\eta_x dp_{\eta_x x}$$
$$exp[-\{\eta_x^2 + p_{\eta_x}^2$$
$$+2[\gamma^2 D^2\bar{\Delta}_s^2 + (\beta_x\bar{\Delta}_x - \gamma\bar{D}\bar{\Delta}_s)^2]/\beta_x\}/\bar{\epsilon}_{xc}$$

$$+\{-4\gamma D\bar{\Delta}_s\eta_x/\sqrt{2\beta_x}$$
$$+4(\beta_x\bar{\Delta}_x - \gamma\bar{D}\bar{\Delta}_s)p_{\eta_x x}/\sqrt{2\beta_x}\}/\bar{\epsilon}_{xd}]$$

$$= p_0\pi\epsilon_{xc}^- exp[-R_{xc} + R_{xd}]$$

$$R_{xc} = 2[\gamma^2 D^2\bar{\Delta}_s^2 + (\beta_x\bar{\Delta}_x - \gamma\bar{D}\bar{\Delta}_s)^2]/(\beta_x\bar{\epsilon}_{xc})$$
$$R_{xd} = 2\{[-\gamma D\bar{\Delta}_s]^2$$
$$+[(\beta_x\bar{\Delta}_x - \gamma\bar{D}\bar{\Delta}_s)]^2\}$$
$$/(\beta_x\bar{\epsilon}_{xd}^2/\bar{\epsilon}_{xc})$$

$$(27)$$

Now do the integral over $ds dW_s$. One may note that the form of the intgral here is similar to the integral done over $dy dW_y$. The result is then the same with the proper sustitutions of s for y.

$$\int dx dW_s exp[-S_{sa}(s, p_{1s}) - S_{sb}(s, p_{2s})] = p_0\pi\bar{\epsilon}_{sc} exp[-R_{sc} + R_{sd}]$$

$$R_{sc} = \frac{2\beta_s}{\bar{\epsilon}_{sc}}\bar{\Delta}_s^2$$

$$R_{sd} = \frac{2\beta_s}{\bar{\epsilon}_{sd}^2/\bar{\epsilon}_{sc}}\bar{\Delta}_s^2 \qquad (28)$$

Note that the term $\beta_s((p-p_0)/p_0)^2$ in S_s in the Lab. CS has to be replaced by $\gamma^2\beta_s(p_s/p_0)^2$ in the Rest CS.

Putting all the above results, for the bi-gaussian distribution, together one gets the final result

$$\frac{1}{p_0^2}\frac{d}{dt}<p_ip_j> = N\int d^3\Delta \, C_{ij}[\left(\frac{N_a}{N}\right)^2\frac{exp(-R_a)}{\Gamma_a} + \left(\frac{N_b}{N}\right)^2\frac{exp(-R_b)}{\Gamma_b}$$
$$+2\frac{N_aN_b}{N^2}\frac{\Gamma_c}{\Gamma_a\Gamma_b}exp(-T)]$$

$$C_{ij} = \frac{2\pi}{p_0^2}(r_0/2\bar{\beta}^2)^2(|\Delta|^2\delta_{ij} - 3\Delta_i\Delta_j)2\bar{\beta}c \, \, ln[1 + (2\bar{\beta}^2 b_{max}/r_0)^2]$$

$$\bar{\beta} = \beta_0\gamma_0|\Delta/p_0|$$

$$r_0 = Z^2e^2/Mc^2$$

$$\frac{1}{\bar{\epsilon}_{ic}} = \frac{1}{2}\left(\frac{1}{\bar{\epsilon}_{ia}} + \frac{1}{\bar{\epsilon}_{ib}}\right) \, \, i = x, y, s$$

$$\frac{1}{\bar{\epsilon}_{id}} = \frac{1}{2}(\frac{1}{\bar{\epsilon}_{ia}} - \frac{1}{\bar{\epsilon}_{ib}})$$

$$r_0 = Z^2e^2/Mc^2$$

$$\Gamma_a = \pi^3\bar{\epsilon}_{sa}\bar{\epsilon}_{xa}\bar{\epsilon}_{ya}p_0^3$$

$$R_a = R_{xa} + R_{ya} + R_{sa}$$

$$R_{xa} = \frac{2}{\beta_x\bar{\epsilon}_{xa}}[\gamma^2D^2\Delta_s^2 + (\beta_x\Delta_x - \gamma\tilde{D}\Delta_s)^2]/p_0^2$$

$$\tilde{D} = \beta_xD' + \alpha_xD$$

$$R_{ya} = \frac{2}{\beta_y\bar{\epsilon}_{ya}}\beta_y^2\Delta_y^2/p_0^2$$

$$R_{sa} = \frac{2}{\beta_s\bar{\epsilon}_{sa}}\beta_s^2\gamma^2\Delta_s^2/p_0^2$$

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$$
\begin{aligned}
T &= T_x + T_y + T_s \\
T_x &= R_{xc} - R_{xd} \\
T_y &= R_{yc} - R_{yd} \\
T_s &= R_{sc} - R_{sd}
\end{aligned}
$$

$$
\begin{aligned}
R_{xd} &= 2\{[-\gamma D\bar{\Delta}_s]^2 \\
&\quad +[(\beta_x\bar{\Delta}_x - \gamma\bar{D}\bar{\Delta}_s)]^2\} \\
&\quad /(\beta_x\bar{\epsilon}_{xd}^2/\bar{\epsilon}_{xc}) \\
R_{yd} &= \frac{2\beta_y}{\bar{\epsilon}_{yd}^2/\bar{\epsilon}_{yc}}\bar{\Delta}_y^2 \\
R_{sd} &= \frac{2\beta_s}{\bar{\epsilon}_{sd}^2/\bar{\epsilon}_{sc}}\bar{\Delta}_s^2 \\
\bar{\Delta}_i &= \Delta_i/p_0 \quad\quad\quad\quad\quad\quad\quad\quad (29)
\end{aligned}
$$

R_a, R_b, R_c are each the same as R_a given above except that $\bar{\epsilon}_{ia}$ are replaced by $\bar{\epsilon}_{ia}, \bar{\epsilon}_{ib}, \bar{\epsilon}_{ic}$ respectively. The same remarks apply to $\Gamma_a, \Gamma_b, \Gamma_c$

The above 3-dimensional integral can be reduced to a 2-dimensional integral by integrating over $|\Delta|$ and using $d^3\Delta = |\Delta|^2 d|\Delta| sin\theta d\theta d\phi$. This gives

$$
\begin{aligned}
\frac{1}{p_0^2}\frac{d}{dt} < p_i p_j > &= 2\pi p_0^3 \left(\frac{r_0}{2\gamma_0^2\beta_0^2}\right)^2 2\beta_0\gamma_0 c \int sin\theta d\theta d\phi \, (\delta_{ij} - 3g_i g_j) \\
&\quad N[\left(\frac{N_a}{N}\right)^2 \frac{1}{\Gamma_a F_a}ln[\frac{\hat{C}}{F_a}] + \left(\frac{N_b}{N}\right)^2 \frac{1}{\Gamma_b F_b}ln[\frac{\hat{C}}{F_b}] \\
&\quad +2\frac{N_a N_b}{N^2}\frac{\Gamma_c}{\Gamma_a\Gamma_b}\frac{1}{G}ln[\frac{\hat{C}}{G}]]
\end{aligned}
$$

$$
\begin{aligned}
g_3 &= cos\theta = g_s \\
g_1 &= sin\theta cos\phi = g_x \\
g_2 &= sin\theta sin\phi = g_y \\
\hat{C} &= 2\gamma_0^2\beta_0^2 b_{max}/r_0
\end{aligned}
$$

$$
F_i = R_i/(|\Delta|/p_0)^2 \quad i = a, b, c
$$

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$$G = T/(|\Delta|/p_0)^2 \tag{30}$$

F_a, F_b, F_c are each the same F that was defined for the Gaussian distribution except that the $\bar{\epsilon}_i$ are replaced by $\bar{\epsilon}_{ia}, \bar{\epsilon}_{ib}, \bar{\epsilon}_{ic}$ respectively.

The above results for the growth rates for a bi-gaussian distribution are expressed as an integral which contains 3 terms, each of which is similar to the one term in the results for the gaussian distribution. These three terms may be given a simple interpertation. The first term represents the contribution to the growth rates due to the scattering of the N_a particles of the first gaussian from themselves, the seond term the contribution due to the scattering of the N_b particles of the second gaussian from themselves, and the third term the contribution due to the scattering of the N_a particles of the first gaussian from the N_b partcles of the second gaussian.

7 Emittance growth rates

One can compute growth rates for the average emittances, $< \epsilon_i >$ in the Laboratory Coordinate System, from the growth rates for $< p_i p_j >$ in the Rest Coordinate System.In the following , dt is the time interval in the Laboratory System and $d\tilde{t}$ is the time interval in the Rest System. $dt = \gamma d\tilde{t}$

$$\frac{d}{dt} < \epsilon_x > = \frac{\beta_x}{\gamma} \frac{d}{d\tilde{t}} < p_x^2/p_0^2 > + \frac{D^2 + \tilde{D}^2}{\beta_x} \gamma \frac{d}{d\tilde{t}} < p_s^2/p_0^2 > -2\tilde{D} \frac{d}{d\tilde{t}} < p_x p_s/p_0^2 >$$

$$\frac{d}{dt} < \epsilon_y > = \frac{\beta_y}{\gamma} \frac{d}{d\tilde{t}} < p_y^2/p_0^2 >$$

$$\frac{d}{dt} < \epsilon_s > = \beta_s \gamma \frac{d}{d\tilde{t}} < p_s^2/p_0^2 > \tag{31}$$

To derive the above results, the simplest case to treat is that of the vertical emittance. The verical emmitance is given by

$$\epsilon_y(y, y') = [y^2 + (\beta_y y' + \alpha_y y)^2]/\beta_y$$

$$\delta \epsilon_y = \beta_y \delta(y'^2)$$

$$\frac{d}{dt} < \epsilon_y > = \frac{\beta_y}{\gamma} \frac{d}{d\tilde{t}} < p_y^2/p_0^2 > \tag{32}$$

In Eq.(32), $y' = p_y/p_0$, $\delta\epsilon_y$ is the change in ϵ_y in a scattering event.

For the longitudinal emittance one finds

$$
\begin{aligned}
\epsilon_s &= [s^2/\gamma^2 + (\beta_s\gamma p_s/p_0)^2]/\beta_s \\
\delta\epsilon_s &= \beta_s\delta(\gamma p_s/p_0)^2 \\
\frac{d}{dt} < \epsilon_s > &= \beta_s\gamma\frac{d}{d\tilde{t}} < p_s^2/p_0^2 >
\end{aligned}
$$

(33)

In Eq.(33), s, p_s are the coordinates in the rest system and I have used the relationship $(p - p_0)_{LAB} = (\gamma p_s)_{REST}$

For the horizontal emittance one finds

$$
\begin{aligned}
\epsilon_x &= \{[x - \gamma Dp_s/p_0]^2 + [\beta_x(p_x/p_0 - \gamma D'p_s/p_0) + \alpha_x(x - \gamma Dp_s/p_0)]^2\}/\beta_x \\
&= \{[x - \gamma Dp_s/p_0]^2 + [\beta_xp_x/p_0 + \alpha_xx - \bar{D}\gamma p_s/p_0]^2\}/\beta_x \\
&= \{x^2 + (\gamma Dp_s/p_0)^2 - 2x\gamma Dp_s/p_0 + (\beta_xp_x/p_0 + \alpha_xx)^2 + (\bar{D}\gamma p_s/p_0)^2 - \\
& \quad 2(\beta_xp_x/p_0 + \alpha_xx)(\bar{D}\gamma p_s/p_0)\}/\beta_x \\
\delta\epsilon_x &= \delta\{\beta_x^2(p_x/p_0)^2 + \gamma^2(D^2 + \bar{D}^2)(p_s/p_0)^2 - 2\beta_x\bar{D}\gamma(p_x/p_0)(p_s/p_0)\}/\beta_x \\
\frac{d}{dt} < \epsilon_x > &= \frac{\beta_x}{\gamma}\frac{d}{d\tilde{t}} < p_x^2/p_0^2 > + \frac{D^2 + \tilde{D}^2}{\beta_x}\gamma\frac{d}{d\tilde{t}} < p_s^2/p_0^2 > -2\tilde{D}\frac{d}{d\tilde{t}} < p_xp_s/p_0^2 >
\end{aligned}
$$

(34)

In the result for $\delta\epsilon_x$, the terms that are linear in p_x or p_s have been dropped as they do not contribute to $< \delta\epsilon_x >$. In a scattering event involving two particles , the δp_x of one particle is equal and opposite to the δp_x of the other particle. This is also true for p_s.

Acknowledgements

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