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A reformulation of intrabeam scattering theory

G. Parzen

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Collider Accelerator Department
Brookhaven National Laboratory

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George Parzen

Collider-Accelerator Department
Brookhaven National Laboratory
Upton, NY 11973

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Abstract

The motivation for the treatment of intrabeam scattering theory given in this paper was to find results which would be convenient for computing the intrabeam scattering growth rates for particle distributions which are more complicated than a simple gaussian. It was shown by A. Piwinski that beam growth rates due to intrabeam scattering can be expressed as a multidimensional integral [1]. It was pointed out by J. Bjorken and S. Mtingwa [2] that the reduction of the multidimensional integral to a 3-dimensional untegral is made easier by writing the integral so that its relativistic transformation properties are more obvious. The starting point in [2] was a result from the treatment of the two body scattering problem in relativistic quantum theory . In this paper the starting point is the relativistic transformation properties of the scattering cross section which may be a more familiar starting point. The resulting expression for the multidimensional integral is simpler to reduce. In addition, the results do not depend on the particular form of the Coulomb cross section that was used in [2] and are valid for any collision cross section.

1 Introduction

The motivation for the treatment of intrabeam scattering theory given in this paper was to find results which would be convenient for computing the intrabeam scattering growth rates for particle distributions which are more complicated than a simple gaussian. It was shown by A. Piwinski that beam growth rates due to intrabeam scattering can be expressed as a multidimensional integral [1]. It was pointed out by J. Bjorken and S. Mtingwa [2] that the reduction of the multidimensional integral to a 3-dimensional untegral is made easier by writing the integral so that its relativistic transformation properties are more obvious. The starting point in [2] was a result from the treatment of the two body scattering problem in relativistic quantum theory . In this paper the starting point is the relativistic transformation properties of the scattering cross section which may be a more familiar starting point. The resulting expression for the multidimensional integral is simpler to reduce. In addition, the results do not depend on the particular form of the Coulomb cross section that was used in [2] and are valid for any collision cross section.The final result is given by Eq.(14), which can be used for computing the intrabeam scattering growth rates for particle distributions which are more complicated than a simple gaussian.

2 Transformation properties of the cross section

The cross section, σ, which describes the the scattering of particles with the momentum p_1 from the target particles with momentum p_2 is first defined in the CS (coordinate system) where the target particles are at rest, $p_2 = 0$. In a scattering event, the particle momenta change from p_1, p_2 to p_1', p_2'. As we are assuming that both momentum and energy are conserved , the final momenta, p_1', p_2' are determined by the direction of p_1' which is indicated by the unit vector $\hat{p_1'}$. In this CS where $p_2 = 0$, σ is defined so that the number of incident particles which are scattered by the target particles with momemtum p_2 which are in the volume element, d^3x, in the time interval dt, into the solid angle $d\Omega'$ corresponding to the direction $\hat{p_1'}$ is given by

$$\delta N = \sigma d\Omega' \rho_1(x) v_1 \rho_2(x) d^3x dt \tag{1}$$

where $\rho_1(x), \rho_2(x)$ are the density functions and v_1 is the velocity of the incident particle.

Now let us go to a CS where $p_2 \neq 0$. In this CS, σ is defined by requiring $\sigma d\Omega'$ to be invariant, that is to have the same value in all coordinate systems. A simple way to find the relationship between δN and σ in this CS is to write δN as (see [3])

$$\delta N = \sigma d\Omega' \frac{\rho_1(x)}{\gamma_1} \frac{\rho_2(x)}{\gamma_2} F(p_1, p_2) d^3x dt \tag{2}$$

where $\rho_1(x)/\gamma_1$ is an invariant as it is just the density function for particle 1 in the CS where $p_1 = 0$. Similarly for $\rho_2(x)/\gamma_2$. If one can find an invariant $F(p_1, p_2)$ which for $p_2 = 0$ gives $F = \gamma_1 v_1$, then this expression for δN gives the correct result when $p_2 = 0$ and also gives the correct result when $p_2 \neq 0$. $F(p_1, p_2)$ that satifies these requirements is

$$F(p_1, p_2) \quad = \quad c\frac{[(p_1 p_2)^2 - m_1^2 m_2^2 c^4]^{.5}}{m_1 m_2 c^2} \tag{3}$$

Here, p_1, p_2 are 4-vectors whose first three components are the components of the momemtum and the fourth component is iE/c, $E = (p^2c^2 + m^2c^4)^{.5}$. $F(p_1, p_2)$ is an invariant and when $p_2 = 0$, $F = \gamma_1 v_1$. The result for $F(p_1, p_2)$ given by Eq.(3) can also be written as

$$F(p_1, p_2) \quad = \quad \gamma_1 \gamma_2 c[(\vec{\beta_1} - \vec{\beta_2})^2 - (\vec{\beta_1} \times \vec{\beta_2})^2]^{.5} \tag{4}$$

Here, $\vec{\beta_1}, \vec{\beta_2}$ are vectors in 3-space corresponding to the velocities of the particles divide by c.

3 The $f(x, p)$ distribution and the scattering rate δN

Let us now treat the case where the particles are contained within a bunch and their distibution is given by $f(x, p)$ where $Nf(x, p)$ is the number of particles in d^3xd^3p. N is the number of particles in the bunch, all particles have the same rest mass m and

$$\int d^3xd^3p \; f(x, p) = 1$$

Let δN be the number of particles with momentum p_1 in d^3p_1 and space coordinate x in d^3x which are scattered by the particles with momentum p_2 in d^3p_2 which are also in d^3x, in the time interval dt , into the solid angle $d\Omega'$ corresponding to the direction $\hat{p_1'}$. Then δN can be obtained using the same procedure used in obtaining Eq.(2), provided one knows that d^3p/γ and $f(x, p)$ are invariants, which is shown in section 5. δN is given by

$$\begin{aligned} \delta N &= N^2 \sigma d\Omega' \frac{d^3p_1}{\gamma_1} \frac{d^3p_2}{\gamma_2} f(x, p_1) f(x, p_2) F(p_1, p_2) d^3x dt \\ F(p_1, p_2) &= \frac{[(p_1 p_2)^2 - m^4]^{.5}}{m^2} \end{aligned} \tag{5}$$

One may note that the right hand side of this expression for δN is an invariant. We will be putting $c = 1$ except when something may be gained by showing c explicitly.

4 Growth rates for $< p_i p_j >$

Growth rates will be given for $< p_i p_j >$. where the $<>$ indicate an average over all the particles in the bunch. From these one can compute the growth rates for the emittances, $< \epsilon_i >$. The advantage due to computing growth rates for $< p_i p_j >$ stems from the observation that if p_i, p_j are the components of the momentum 4-vector, then $p_i p_j$ is a tensor in 4-space and so is $\delta < p_i p_j >$, as will be seen below, where $\delta < p_i p_j >$ is the change in $< p_i p_j >$ in a time interval dt. The transfornation properties of a tensor can then be used to facilitate the transfer of results between two CS.

In a scattering event, where a particle with momentum p_1 scatters off a particle with momentum p_2, the momenta will change to p_1' and p_2'. Let δp_{1i} represent the change in p_{1i} in the collision, and similarly for $\delta(p_{1i} p_{1j})$. Then

$$\begin{aligned} \delta p_{1i} &= p_{1i}' - p_{1i} \\ \delta(p_{1i} p_{1j}) &= p_{1i}' p_{1j}' - p_{1i} p_{1j} \end{aligned} \tag{6}$$

Using the scattering rate given by Eq.(5), one can now compute $\delta < p_i p_j >$

$$\begin{aligned} < \delta(p_{1i} p_{1j}) > &= N \int d^3x \frac{d^3p_1}{\gamma_1} \frac{d^3p_2}{\gamma_2} f(x, p_1) f(x, p_2) F(p_1, p_2) \\ & \quad \sigma d\Omega' (p_{1i}' p_{1j}' - p_{1i} p_{1j}) dt \\ F(p_1, p_2) &= \frac{[(p_1 p_2)^2 - m^4]^{.5}}{m^2} \end{aligned} \tag{7}$$

One may note that

$$< \delta(p_{1i} p_{1j}) >= \delta < p_{1i} p_{1j} >$$

and that $\delta < p_{1i} p_{1j} >$ is a tensor in 4-space because of the transformation properties given above for the quantities appearing on the right hand side of Eq.(7). Eq.(7) is our general result for the growth rates , holds in all CS, and can be used for any particle distribution, $f(x, p)$.

This result can be further simplified by first considering the integral, for a given p_1, p_2,

$$C_{ij} = \int \sigma d\Omega' (p'_{1i}p'_{1j} - p_{1i}p_{1j}) \tag{8}$$

C_{ij} has the transformation properties of a tensor in 4-space as $\sigma d\Omega'$ is an invariant. For a given p_1, p_2, C_{ij} can be evaluated in the CMS (the center of mass CS) and if the result can be written in terms of 4-vectors and tensors in 4-space, then the result in this form. will hold in all CS. The calculation of C_{ij} can be simplified by noting that because of the symmetry in p_1 and p_2 we have

$$< \delta(p_{1i}p_{1j}) > = < \delta(p_{2i}p_{2j}) > \tag{9}$$

and we can define C_{ij} as

$$C_{ij} = \int \sigma d\Omega' \frac{1}{2}[\delta(p_{1i}p_{1j}) + \delta(p_{2i}p_{2j})] \tag{10}$$

and Eq.(7) can be written as

$$< \delta(p_{1i}p_{1j}) > = N \int d^3x \frac{d^3p_1}{\gamma_1}\frac{d^3p_2}{\gamma_2} f(x,p_1)f(x,p_2)F(p_1,p_2)C_{ij}dt \tag{11}$$

We will now further evaluate C_{ij} by first evaluating C_{ij} for some particular values of p_1,p_2 in the CMS corresponding to p_1,p_2 and and then using the tensor properties of C_{ij} to find a result that holds in any other CS. We are particularly interested in finding a result in the Rest CS, which is the CS which moves along with the bunch. In the CMS,

$$\begin{aligned} \vec{p_2} &= -\vec{p_1} \\ \vec{\Delta} &= \frac{1}{2}(\vec{p_1} - \vec{p_2}) = \vec{p_1} \\ \vec{q_1} &= \vec{p'_1} - \vec{p_1} \\ \vec{q_2} &= -\vec{q_1} \end{aligned}$$

Using $\vec{q_1} = \vec{p'_1} - \vec{p_1}$ and $\vec{q_2} = -\vec{q_1}$, one can show that

$$\frac{1}{2}(\delta(p_{1i}p_{1j}) + \delta(p_{2i}p_{2j}) = q_{1i}q_{1j} + \Delta_i q_{1j} + \Delta_j q_{1i} \tag{12}$$

In the CMS, we introduce a polar coordinate system θ, ϕ where θ is measured relative to the direction of $\vec{p_1}$ or $\vec{\Delta}$ and we assume that $\sigma(\theta,\phi)$ is a fumction of θ only. we can then write

$$\begin{aligned} \vec{p_1} &= (0,0,1)|\vec{\Delta}| \\ \vec{p_1'} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)|\vec{\Delta}| \\ \vec{q_1} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta - 1)|\vec{\Delta}| \end{aligned}$$

Considering p_1,p_2 to be 4-vectors, and $\Delta = .5(p_1 - p_2)$, $q_1 = p_1' - p_1$, then Δ, q_1 are also 4-vectors and in the CMS, $\Delta_4 = 0, q_{14} = 0$.

Using Eqs(10) and (12), one now finds for C_{ij} in the CMS

$$C_{ij} = \pi \int_0^\pi d\theta\sigma \sin^3\theta \ |\vec{\Delta}|^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To find C_{ij} in the Rest CS or in the Lab CS, we will try to find an expression for C_{ij} in terms of the 4-vectors p_{1i}, p_{2i} which gives the above result for C_{ij} in the CMS. The expression that does this is given by

$$\begin{aligned} C_{ij} &= \pi \int_0^\pi d\theta\sigma \sin^3\theta \ \Delta^2 [\delta_{ij} - 3\frac{\Delta_i \Delta_j}{\Delta^2} + \frac{W_i W_j}{W^2}] \ \ i,j = 1,4 \\ \Delta_i &= \frac{1}{2}(p_{1i} - p_{2i}) \\ W_i &= p_{1i} + p_{2i} \end{aligned} \tag{13}$$

where σ is the cross section in the CMS. Let us now verify that this expression gives the correct result for C_{ij} in the CMS. In the CMS,

$$\begin{aligned} \Delta^2 &= |\vec{\Delta}|^2 + \Delta_4^2 = |\vec{\Delta}|^2 \\ W_i &= 0 \ \ i = 1,3 \\ \Delta_i &= 0 \ \ i = 1,2 \ \ \Delta_3 = |\vec{\Delta}| \end{aligned}$$

so that Eq.(13) does give the correct result in the CMS.

An important further simplification results from the fact that the particle motion is non-relativistic in the CMS and also in the Rest CS which moves along with the bunch. For RHIC parameters, for $\gamma = 100$, one finds that $p \simeq 1e-3 \ \ mc$. One can then drop the $W_i W_j / W^2$ term. Also $\Delta^2 = |\vec{\Delta}|^2$ in the CMS and in the Rest CS and one can evaluate $F(p_1,p_2)$ using Eq.(4) as $F(p_1,p_2) = 2c\bar{\beta}$ where $\bar{\beta}c$ is the velocity of either particle in the CMS . In the Rest CS, one can write

$$\begin{aligned}
C_{ij} &= \pi \int_0^\pi d\theta \sigma \sin^3\theta \, [|\vec{\Delta}|^2 \delta_{ij} - 3\Delta_i \Delta_j] \;\; i,j = 1,3 \\
\Delta_i &= \frac{1}{2}(p_{1i} - p_{2i}) \\
< \delta(p_{1i}p_{1j}) > &= N \int \; d^3x d^3p_1 d^3p_2 f(x,p_1) f(x,p_2) 2\bar{\beta}c \; C_{ij} \; dt \\
\bar{\beta}c &= |\vec{\Delta}|/m
\end{aligned} \tag{14}$$

Eq.(14) would be a good starting point for computing growth rates for a particle distribution more complicated than a simple gaussian. For the case of the Coulomb cross section, one can write C_{ij} as

$$\begin{aligned}
C_{ij} &= 2\pi(r_0/2\bar{\beta}^2)^2 \ln(1 + (2\bar{\beta}^2 b_{max}/r_0)^2) \;\; [|\vec{\Delta}|^2 \delta_{ij} - 3\Delta_i \Delta_j] \;\; i,j = 1,3 \\
\sigma &= (r_0/2\bar{\beta}^2)^2/(1 - \cos\theta)^2 \\
r_0 &= Z^2 e^2/mc^2
\end{aligned} \tag{15}$$

b_{max} is the largest allowed impact parameters in the CMS.

5 Invariants d^3p/γ and $f(x,p)$

In order to eatablish Eq.(5), one needs to know that d^3p/γ and $f(x,p)$ are invariants. Consider a CS moving with the velocity v_0 with respect to the Laboratory CS. Let the coordinates be x, p in the Laboratory CS and $\hat{x}, \hat{p}$ in the new CS. $p, \hat{p}$ are related by

$$\begin{aligned}
\hat{p_s} &= \gamma_0(p_s - v_0 E) \\
\hat{p_x} &= p_x \\
\hat{p_y} &= p_y \\
E &= \sqrt{p^2 + m^2} \\
\gamma_0 &= 1/\sqrt{1 - v_0^2}
\end{aligned} \tag{16}$$

It then follows that

$$d\hat{p_s} = \gamma_0(dp_s - v_0 dE)$$

$$\begin{aligned}
dE &= (p_s/E)dp_s \quad p_x, p_y constant \\
d\hat{p_s} &= \gamma_0(1 - v_0 p_s/E)dp_s \\
\\
\hat{E} &= \sqrt{\hat{p}^2 + m^2} \\
\hat{E} &= \gamma_0(E - v_0 p_s) \\
\gamma_0(1 - v_0 p_s/E) &= \hat{E}/E \\
\frac{d\hat{p_s}}{\hat{E}} &= \frac{dp_s}{E}
\end{aligned} \tag{17}$$

Thus dp_s/γ is invariant under this momentum transformation and also d^3p/γ is invariant.

Now let us show that $f(x,p)$ is an invariant. Since $f(x,p)d^3xd^3p$ is an invariant, as it gives the number of particles in d^3xd^3p, we need to show that d^3xd^3p is an invariant. Consider the point x, p in some CS. In the moving CS where $\hat{p_s} = 0$, which is moving with the velocity $v = p_s/E$ with respect to the first CS,

$$\begin{aligned}
d\hat{p_s} &= \gamma(dp_s - vdE) \\
dE &= (p_s/E)dp_s \quad p_x, p_y constant \\
d\hat{p_s} &= dp_s/\gamma \\
\gamma &= 1/\sqrt{1 - v^2}
\end{aligned} \tag{18}$$

Since $dp_s/\gamma = d\hat{p_s}$ holds for any CS, dp_s/γ is an invariant and d^3p/γ is an invariant. One also has γd^3x is invariant because of the Lorentz-Fitsgerald contraction. We have then that that d^3xd^3p is an invariant.

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