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G. Parzen

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Collider Accelerator Department
Brookhaven National Laboratory

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Results for the intrabeam scattering growth Rates for a bi-gaussian distribution

George Parzen



**Collider-Accelerator Department
Brookhaven National Laboratory
Upton, NY 11973**

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Abstract

This note lists results for the intrabeam scattering growth rates for a bi-gaussian distribution. The derivation of these results will be given in a future note.

Introduction

This note finds results for the intrabeam scattering growth rates for a bi-gaussian distribution.

The bi-gaussian distribution is interesting for studying the possibility of using electron cooling in RHIC. Studies done using the SIMCOOL program [1] indicate that in the presence of electron cooling, the beam distribution changes so that it develops a strong core and a long tail which is not described well by a gaussian, but may be better described by a bi-gaussian. Being able to compute the effects of intrabeam scattering for a bi-gaussian distribution would be useful in computing the effects of electron cooling, which depend critically on the details of the intrabeam scattering.

Gaussian distribution

Before defining the bi-gaussian distribution, the gaussian distribution will be reviewed.

$Nf(x, p)$ gives the number of particles in $d^3x d^3p$, where N is the number of particles in a bunch. For a gaussian distribution, $f(x, p)$ is given by

$$f(x, p) = \frac{1}{\Gamma} \exp[-S(x, p)] \quad (1)$$

$$\begin{aligned}
S &= S_x + S_y + S_s \\
S_x &= \frac{1}{\bar{\epsilon}_x} \epsilon_x(x_\beta, p_{x\beta}/p_0) \\
x_\beta &= x - D(p - p_0)/p_0 \\
p_{x\beta}/p_0 &= p_x/p_0 - D'(p - p_0)/p_0 \\
\epsilon_x(x, x') &= \gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 \\
S_y &= \frac{1}{\bar{\epsilon}_y} \epsilon_y(y, p_y/p_0) \\
\epsilon_y(y, y') &= \gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2 \\
S_s &= \frac{1}{2\sigma_s^2} (s - s_c)^2 + \frac{1}{2\sigma_p^2} ((p - p_0)/p_0)^2 \\
S_s &= \frac{1}{\bar{\epsilon}_s} \left(\frac{1}{\beta_s} (s - s_c)^2 + \beta_s ((p - p_0)/p_0)^2 \right) \\
\beta_s &= \sigma_s / \sigma_p \\
\bar{\epsilon}_s &= 2\sigma_s \sigma_p \\
S_s &= \frac{1}{\bar{\epsilon}_s} \epsilon_s(s - s_c, (p - p_0)/p_0) \\
\Gamma &= \int d^3x d^3p \exp[-S(x, p)] \\
\Gamma &= \pi^3 \bar{\epsilon}_x \bar{\epsilon}_y \bar{\epsilon}_s p_0^3 \\
\bar{\epsilon}_i &= \langle \epsilon_i(x, p) \rangle \quad i = x, y, s
\end{aligned} \quad (2)$$

D is the horizontal dispersion. $D' = dD/ds$. $\langle \rangle$ indicates an average over all the particles in a bunch.

Growth rates for a Gaussian distribution

In the following, the growth rates are given in the Rest Coordinate System, which is the coordinate system moving along with the bunch. Growth rates

are given for $\langle p_i p_j \rangle$. From these one can compute the growth rates for $\langle \epsilon_i \rangle$ using the relations given at the end of this note.

$$\begin{aligned}
\frac{1}{p_0^2} \frac{d}{dt} \langle p_i p_j \rangle &= \frac{N}{\Gamma} \int d^3 \Delta \exp[-R] C_{ij} \\
C_{ij} &= \frac{2\pi}{p_0^2} (r_0/2\bar{\beta}^2)^2 (|\Delta|^2 \delta_{ij} - 3\Delta_i \Delta_j) 2\bar{\beta} c \ln[1 + (2\bar{\beta}^2 b_{max}/r_0)^2] \\
\bar{\beta} &= \beta_0 \gamma_0 |\Delta/p_0| \\
r_0 &= Z^2 e^2 / M c^2 \\
R &= R_x + R_y + R_s \\
R_x &= \frac{2}{\beta_x \bar{\epsilon}_x} [\gamma^2 D^2 \Delta_s^2 + (\beta_x \Delta_x - \gamma \tilde{D} \Delta_s)^2] / p_0^2 \\
\tilde{D} &= \beta_x D' + \alpha_x D \\
R_y &= \frac{2}{\beta_y \bar{\epsilon}_y} \beta_y^2 \Delta_y^2 / p_0^2 \\
R_s &= \frac{2}{\beta_s \bar{\epsilon}_s} \beta_s^2 \gamma^2 \Delta_s^2 / p_0^2
\end{aligned} \tag{3}$$

The integral over $d^3 \Delta$ is an integral over all possible values of the relative momentum for any two particles in a bunch. β_0, γ_0 are the beta and gamma corresponding to p_0 , the central momentum of the bunch in the Laboratory Coordinate System. $\gamma = \gamma_0$

The above 3-dimensional integral can be reduced to a 2-dimensional integral by integrating over $|\Delta|$ and using $d^3 \Delta = |\Delta|^2 d|\Delta| \sin\theta d\theta d\phi$.

$$\begin{aligned}
\frac{1}{p_0^2} \frac{d}{dt} \langle p_i p_j \rangle &= \frac{N}{\Gamma} 2\pi p_0^3 \left(\frac{r_0}{2\gamma_0^2 \beta_0^2} \right)^2 2\beta_0 \gamma_0 c \int \sin\theta d\theta d\phi (\delta_{ij} - 3g_i g_j) \\
&\quad \frac{1}{F} \ln \left[\frac{\hat{C}}{F} \right] \\
g_3 &= \cos\theta = g_s \\
g_1 &= \sin\theta \cos\phi = g_x \\
g_2 &= \sin\theta \sin\phi = g_y \\
\hat{C} &= 2\gamma_0^2 \beta_0^2 b_{max}/r_0 \\
F &= R/(|\Delta|/p_0)^2
\end{aligned}$$

$$\begin{aligned}
F &= F_x + F_y + F_s \\
F_x &= \frac{2}{\beta_x \bar{\epsilon}_x} [\gamma^2 D^2 g_s^2 + (\beta_x g_x - \gamma \tilde{D} g_s)^2] \\
F_y &= \frac{2}{\beta_y \bar{\epsilon}_y} \beta_y^2 g_y^2 \\
F_s &= \frac{2}{\beta_s \bar{\epsilon}_s} \beta_s^2 \gamma^2 g_s^2
\end{aligned} \tag{4}$$

Bi-Gaussian distribution

The bi-gaussian distribution will be assumed to have the form given by the following.

$Nf(x, p)$ gives the number of particles in $d^3x d^3p$, where N is the number of particles in a bunch. For a bi-gaussian distribution, $f(x, p)$ is given by

$$f(x, p) = \frac{N_a}{N} \frac{1}{\Gamma_a} \exp[-S_a(x, p)] + \frac{N_b}{N} \frac{1}{\Gamma_b} \exp[-S_b(x, p)] \tag{5}$$

In the first gaussian, to find Γ_a, S_a then in the expressions for Γ, S , given above for the gaussian distribution, replace $\bar{\epsilon}_x, \bar{\epsilon}_y, \bar{\epsilon}_s$ by $\epsilon_{xa}, \epsilon_{ya}, \epsilon_{sa}$. In the second gaussian, in the expressions for Γ, S , replace $\bar{\epsilon}_x, \bar{\epsilon}_y, \bar{\epsilon}_s$ by $\epsilon_{xb}, \epsilon_{yb}, \epsilon_{sb}$. In addition. $N_a + N_b = N$. This bi-gaussian has 7 parameters instead of the three parameters of a gaussian.

Growth rates for a Bi- Gaussian distribution

In the following, the growth rates are given in the Rest Coordinate System, which is the coordinate system moving along with the bunch. Growth rates are given for $\langle p_i p_j \rangle$. From these one can compute the growth rates for $\langle \epsilon_i \rangle$ using the relations given at the end of this note.

$$\begin{aligned}
\frac{1}{p_0^2} \frac{d}{dt} \langle p_i p_j \rangle &= N \int d^3\Delta C_{ij} \left[\left(\frac{N_a}{N} \right)^2 \frac{\exp(-R_a)}{\Gamma_a} + \left(\frac{N_b}{N} \right)^2 \frac{\exp(-R_b)}{\Gamma_b} \right. \\
&\quad \left. + 2 \frac{N_a N_b}{N^2} \frac{\Gamma_c}{\Gamma_a \Gamma_b} \exp(-T) \right] \\
C_{ij} &= \frac{2\pi}{p_0^2} (r_0/2\bar{\beta}^2)^2 (|\Delta|^2 \delta_{ij} - 3\Delta_i \Delta_j) 2\bar{\beta} c \ln[1 + (2\bar{\beta}^2 b_{max}/r_0)^2]
\end{aligned}$$

$$\begin{aligned}\bar{\beta} &= \beta_0 \gamma_0 |\Delta/p_0| \\ r_0 &= Z^2 e^2 / M c^2\end{aligned}$$

$$\begin{aligned}\frac{1}{\bar{\epsilon}_{ic}} &= \frac{1}{2} \left(\frac{1}{\bar{\epsilon}_{ia}} + \frac{1}{\bar{\epsilon}_{ib}} \right) \quad i = x, y, s \\ \frac{1}{\bar{\epsilon}_{id}} &= \frac{1}{2} \left(\frac{1}{\bar{\epsilon}_{ia}} - \frac{1}{\bar{\epsilon}_{ib}} \right)\end{aligned}$$

$$\begin{aligned}r_0 &= Z^2 e^2 / M c^2 \\ \Gamma_a &= \pi^3 \bar{\epsilon}_{sa}^- \bar{\epsilon}_{xa}^- \bar{\epsilon}_{ya}^- p_0^3\end{aligned}$$

$$\begin{aligned}R_a &= R_{xa} + R_{ya} + R_{sa} \\ R_{xa} &= \frac{2}{\beta_x \bar{\epsilon}_x} [\gamma^2 D^2 \Delta_s^2 + (\beta_x \Delta_x - \gamma \tilde{D} \Delta_s)^2] / p_0^2 \\ \tilde{D} &= \beta_x D' + \alpha_x D \\ R_{ya} &= \frac{2}{\beta_y \bar{\epsilon}_y} \beta_y^2 \Delta_y^2 / p_0^2 \\ R_{sa} &= \frac{2}{\beta_s \bar{\epsilon}_s} \beta_s^2 \gamma^2 \Delta_s^2 / p_0^2\end{aligned}$$

$$\begin{aligned}T &= T_x + T_y + T_s \\ T_x &= R_{xc} - R_{xd} \\ T_y &= R_{yc} - R_{yd} \\ T_s &= R_{sc} - R_{sd}\end{aligned}$$

$$\begin{aligned}R_{xd} &= 2\{[-\gamma D \bar{\Delta}_s]^2 \\ &\quad + [(\beta_x \bar{\Delta}_x - \gamma \bar{D} \bar{\Delta}_s)]^2\} \\ &\quad / (\beta_x \bar{\epsilon}_{xd}^2 / \bar{\epsilon}_{xc}) \\ R_{yd} &= \frac{2\beta_y}{\bar{\epsilon}_{yd}^2 / \bar{\epsilon}_{yc}} \bar{\Delta}_y^2 \\ R_{sd} &= \frac{2\beta_s}{\bar{\epsilon}_{sd}^2 / \bar{\epsilon}_{sc}} \bar{\Delta}_s^2 \\ \bar{\Delta}_i &= \Delta_i / p_0 \quad i = x, y, s\end{aligned} \tag{6}$$

R_a, R_b, R_c are each the same as R_a given above except that $\bar{\epsilon}_{ia}^-$ are replaced

by $\bar{\epsilon}_{ia}, \bar{\epsilon}_{ib}, \bar{\epsilon}_{ic}$ respectively.

The above 3-dimensional integral can be reduced to a 2-dimensional integral by integrating over $|\Delta|$.

$$\begin{aligned}
\frac{1}{p_0^2} \frac{d}{dt} \langle p_i p_j \rangle &= 2\pi p_0^3 \left(\frac{r_0}{2\gamma_0^2 \beta_0^2} \right)^2 2\beta_0 \gamma_0 c \int \sin\theta d\theta d\phi (\delta_{ij} - 3g_i g_j) \\
&\quad N \left[\left(\frac{N_a}{N} \right)^2 \frac{1}{\Gamma_a F_a} \ln \left[\frac{\hat{C}}{F_a} \right] + \left(\frac{N_b}{N} \right)^2 \frac{1}{\Gamma_b F_b} \ln \left[\frac{\hat{C}}{F_b} \right] \right. \\
&\quad \left. + 2 \frac{N_a N_b}{N^2} \frac{\Gamma_c}{\Gamma_a \Gamma_b} \frac{1}{G} \ln \left[\frac{\hat{C}}{G} \right] \right] \\
g_3 &= \cos\theta = g_s \\
g_1 &= \sin\theta \cos\phi = g_x \\
g_2 &= \sin\theta \sin\phi = g_y \\
\hat{C} &= 2\gamma_0^2 \beta_0^2 b_{max}/r_0 \\
F_i &= R_i / (|\Delta|/p_0)^2 \quad i = a, b, c \\
G &= T / (|\Delta|/p_0)^2
\end{aligned} \tag{7}$$

F_a, F_b, F_c are each the same F that was defined for the Gaussian distribution except that the $\bar{\epsilon}_i$ are replaced by $\bar{\epsilon}_{ia}, \bar{\epsilon}_{ib}, \bar{\epsilon}_{ic}$ respectively.

The above results for the growth rates for a bi-gaussian distribution are expressed as an integral which contains 3 terms, each of which is similar to the one term in the results for the gaussian distribution. These three terms may be given a simple interpretation. The first term represents the contribution to the growth rates due to the scattering of the N_a particles of the first gaussian from themselves, the second term the contribution due to the scattering of the N_b particles of the second gaussian from themselves, and the third term the contribution due to the scattering of the N_a particles of the first gaussian from the N_b particles of the second gaussian.

Emittance growth rates

One can compute growth rates for the average emittances, $\langle \epsilon_i \rangle$ in the Laboratory Coordinate System, from the growth rates for $\langle p_i p_j \rangle$ in the Rest Coordinate System. In the following, dt is the time interval in the Laboratory System and $d\tilde{t}$ is the time interval in the Rest System. $dt = \gamma d\tilde{t}$

$$\begin{aligned}\frac{d}{dt}\epsilon_x &= \frac{\beta_x}{\gamma} \frac{d}{d\tilde{t}} \langle p_x^2/p_0^2 \rangle + \frac{D^2 + \tilde{D}^2}{\beta_x} \gamma \frac{d}{d\tilde{t}} \langle p_s^2/p_0^2 \rangle - 2\tilde{D} \frac{d}{d\tilde{t}} \langle p_x p_s/p_0^2 \rangle \\ \frac{d}{dt}\epsilon_y &= \frac{\beta_y}{\gamma} \frac{d}{d\tilde{t}} \langle p_y^2/p_0^2 \rangle \\ \frac{d}{dt}\epsilon_s &= \beta_s \gamma \frac{d}{d\tilde{t}} \langle p_s^2/p_0^2 \rangle\end{aligned}\tag{8}$$

I thank I. Ben-Zvi for his comments and encouragement. The results given above were found using the results given in references [2,3,4,]. The derivation of the results is given in Ref.[5]

References

1. V.Parhomchhuk and I. Ben-Zvi, BNL report C-A/AP/47, April 2001; A. Fedotov, Y. Eidelman (Private Communication 2004)
2. G.Parzen, BNL report C-A/AP/No.150 (2004)
3. A. Piwinski Proc. 9th Int. Conf. on High Energy Accelerators (1974) 405, M. Martini CERN PS/84-9 (1984), A. Piwinski Cern 87-03 (1987) 402, A. Piwinski CERN 92-01 (1992) 226
4. J.D. Bjorken and S.K. Mtingwa, Part. Accel.13 (1983) 115, K. Kubo and K. Oide Phys. Rev. S.T.A.B., 4, (2001) 124401
5. G. Parzen, BNL report C-A/AP/No.169 (2004)