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EXPERIMENTAL PLANNING AND SUPPORT DIVISION

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Introduction :

The primary purpose of this note is to address the evaluation of the linear functional,

$$s(t) = \int_0^t x(\alpha)\ d\alpha \quad (1)$$

under a various set of assumptions and where **x** represents a noisy input signal and s represents the output signal. The functional above can be generated in a variety of ways using active or passive circuits. Active circuit configurations commonly include the voltage feedback amplifier (opamp) configuration[1], transimpedance configurations[2] or current conveyors[3]. For passive realizations the RC and RL low pass filters can be employed as crude approximations to an integral when only relatively short duration integrals are required.

Further, two special cases of noise signals passed thru linear time invariant low pass filters are considered.

Case Analysis :

Case I.

Let **x** be a stochastic (noise) process that is gaussian, zero mean, white and wide sense stationary (WSS) . This type of "signal" is commonly used in the analysis of electronic systems and sometimes simply referred to as gaussian white noise. However, in general white noise does not have to be gaussian, stationary nor have zero mean. WSS specifies that the mean is constant and autocorrelation depends only on the time difference between the measurements, and not the actual time they are made. Therefore, in electronics it means that the expected (average) value and average power of the process are fixed. The average power of a stationary process is the autocorrelation evaluated at a zero shift. The integral in (1) exists at least in the mean square sense because the autocorrelation function of white noise is absolutely integrable[4].

In terms of first and second order statistics the input process is describable as :

$$\mu_X = E\,\{x(t)\} = 0 \qquad (2a)$$

$$R_{xx}(t_1,t_2) = E\{x(t_1)x(t_2)\} = \beta\delta(t_1-t_2) = \beta\delta(\tau) \qquad (2b)$$

where use has been made of the white, zero mean and WSS properties. The mean and autocorrelation of the output process (the integrator output) are shown below :

$$\mu_s = E\{s(t)\} = \int_0^T E\{x(t)\}\ dt = \int_0^T \mu_x\ dt = 0 \qquad (3a)$$

$$R_{ss}(t_1,t_2) = E\{s(t_1)s(t_2)\} = \int_0^{t_1}\int_0^{t_2} E\{x(\alpha)x(\beta)\}\ d\alpha\, d\beta = \int_0^{t_1}\int_0^{t_2} \beta\delta(\alpha-\beta)\ d\alpha\, d\beta = \beta\max(t_1,t_2) \qquad (3b)$$

Therefore, the variance of the output process, which is the autocorrelation evaluated at $T=t_1=t_2$ so $R_{ss}(t_1,t_2) = \sigma^2_{ss} = T$. Now the output process, i.e. the integrator output, is still gaussian because the system is linear however WSS no longer holds because the variance of the output process grows linearly in t. This implies the probability of finding the process within ϵ of zero is getting smaller as time goes forward. Therefore, although the expected value of the output process is zero, there is a smaller and smaller probability that the output will be in the range $-\epsilon < s(t) < \epsilon$. Practically, this states there exists a small chance that the noise input will integrate to a near-zero value and a zero chance that the integral results in identically zero. Graphs of the output probability density function (pdf) are shown as a function of time in figure 1. Standard tables of normal distributions can be consulted to determine the probability of finding the output to be in any given interval of values. This

Define the Gaussian Distribution as :

$$\mathrm{guass}(x) := \frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\frac{(x-\mu)^2}{\sigma^2}}$$

Let the mean be zero, but the variance is equal to t :

$$g(x,t) := \frac{1}{t \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\frac{(x)^2}{t^2}}$$

Below are shown graphs of the Gaussian pdf for fixed times t :

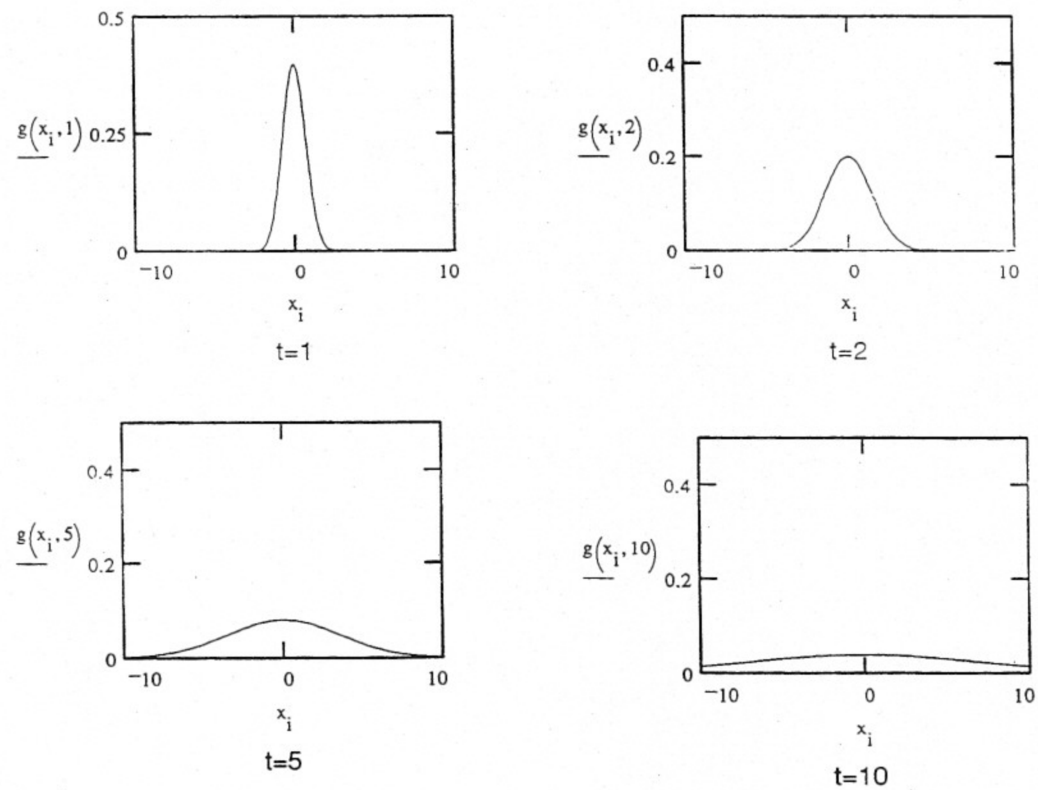

FIGURE 1

process is of special interest in stochastic studies and is called a Wiener process and is very closely related to the random walk process.

Case II.

Now let the functional be set for determining the average of the continuous process x. To perform this average, merely scale equation (1) by the gate time. Therefore the process is now described as:

$$s(t) = \frac{1}{t}\int_0^t x(\alpha)\ d\alpha \quad (4)$$

This is the direct analog of the discrete case where samples are summed and divided by the number of samples. Following the development above it is straightforward that the expected value of the output process (4) is still zero, but now the variance of the output process approaches zero as t approaches infinity. To see this consider the following :

$$R_{ss}(t_1,t_2) = E\{s(t_1)s(t_2)\} = \frac{1}{t_1 t_2}\int_0^{t_1}\int_0^{t_2} E\{x(\alpha)x(\beta)\}\ d\alpha\, d\beta = \frac{1}{t_1 t_2}\beta \max(t_1,t_2) \quad (5)$$

and note that when $T{=}t_1{=}t_2$, the average power is determined and the RHS of (5) approaches zero as T approaches infinity. Again note that the output process is not WSS.

A practical consequence of this formulation is that to estimate, within a given tolerance the average value, of say a constant embedded in additive noise, the minimum gate time is fixed. This is because the gate time implies a variance in the output process, even under perfect measurement conditions, and this variance in turn implies the "spread" of the distribution on the output. For the discrete case, the minimum number of samples is the parameter that is fixed. Further, if the process in (4) is considered as an estimate of the average of x over some interval, then the sample variance goes to zero as t approaches infinity.

Case III

In this case consider a deterministic-random process x. This type of process occurs when there is special structure to the random process. For example, measuring a 60 Hz wave when the phase is a is random variable. Such a situation occurs when integrators are placed in field applications. The gate time is generally asynchronous with the 60 Hz period. Thus the integrator output will in general not be zero at the end of the gate. If the assumed form of the process is :

$$s(t) = \int_0^T A sin(\omega t + \theta)\; dt \quad (6)$$

and if the usual assumption of uniform density of θ in $[-\pi,\pi]$ is made, then the first and second moments of the output process are given by :

$$E\{s(t)\} = \int_0^T \int_{-\pi}^{\pi} A sin(\omega t + \theta) \frac{1}{2\pi} \, d\theta \, dt = \frac{A}{\pi \omega} \sin(\omega T) \quad (7a)$$

$$R_{ss}(T_1,T_2) = \frac{A^2}{2\omega^2}[\cos(\omega T_2) - \cos(\omega (T_2 - T_1) + \cos(\omega T_1) - 1] \quad (7b)$$

However, to determine the average power of the process, the autocorrelation function is evaluated at $T = T_1 = T_2$. The result is :

$$R_{ss}(T,T) = \frac{A^2}{\omega^2}[\cos(\omega T) - 1)] \quad (8)$$

Thus if $T = 2\pi/\omega$, then the autocorrelation is zero indicating that at those times that the output power is zero, so the true value of the output must be identically zero. This makes perfect sense because we have integrated over a whole number of waveform periods.

Case IV.

Now the case of white noise passed through a linear time invariant (LTI) LPF will be considered. Specifically, the filter is the standard single pole RC type. Extensions to multipole filters are straightforward. Because the system is LTI, the input and output are related thru the convolution integral. Hence, so are the input and process statistics. To analyze a this case, let the input process be white, gaussian, zero mean and WSS. To describe the output process of the causal LTI LPF the input will be applied at t=0 is considered. To be consistent with the definition of white noise, the case where a LPF is preceded by a switch that is closed at t=0 realizes this setup.

White noise passed through a linear system will have its output mean related to its input mean through the convolution integral. The limits of the integral are set by the causality of the input and system. The manipulations are shown below , and rely on the linearity properties of expected value and convolution :

$$E\{y(t)\} = \mu_y = \int_0^t E\{x(t-\alpha)\}h(\alpha)\,d\alpha = \mu_x \int_0^t \frac{1}{\tau} e^{-\frac{\alpha}{\tau}} d\alpha = \mu_x(1-e^{-\frac{t}{\tau}}) \quad (9)$$

In (9) the $h(\alpha)$ represents the filter impulse response and τ=RC. Therefore, the output mean approaches the input mean as the exponential term approaches zero. In this case, since the input process has zero mean, the output process has zero mean, for all time.

For finding the autocorrelation two methods can be employed depending on where the transient or steady-state solutions are required. For the case where the inputs can be considered connected for a long period of time, and the input statistics are time invariant at least as WSS, then we can relate the input and output power spectral density (psd) by :

$$R_{yy}(\tau) = \mathscr{F}^{-1}\{S_{yy}(j\omega)\} = \mathscr{F}^{-1}\{S_{xx}(j\omega)H(j\omega)H(-j\omega)\} \quad (10)$$

here the indicates $\mathscr{F}^{-1}$ indicates inverse 1-D fourier transform, and H(jω) is the filter impulse response. This is known as the spectral factorization[5] method. Application of this method requires that the system be considered in the steady-state condition, otherwise the output is not a WSS process, and then the autocorrelation is not a function of τ only. Therefore the fourier relationship shown in (10) between the power spectral density and the autocorrelation does not exist. Considering the present case of causal signal input and a causal LTI LPF, the autocorrelation in both the transient and steady-state regimes can be determined from the convolution integral. Using the convolution integral expression for the output process and taking the expected values and noting the **x** process is a WSS stationary process we get :

$$R_{yy}(t_1,t_2) = \int_0^{t_1}\int_0^{t_2} h(u)h(v)R_{xx}(u-v+t_1-t_2)\,du\,dv \quad (11)$$

In general this is a difficult integral to evaluate, so in most cases only the variance is determined. In this case the process is zero mean so the average power and variance are the same quantity, by letting $t=t_1=t_2$ the result is :

$$\sigma^2 = R_{yy}(t,t) = E\{y(t)^2\} = \frac{A}{2\tau}[1-e^{\frac{-2t}{\tau}}] \quad (12)$$

and the output variance goes to a constant as t approaches infinity. The constant is determined by the product of the one-sided input power spectral density and the LPF noise equivalent bandwidth of $1/2\tau$ Hz , where τ=RC. The constant is the same value that would be obtained for the steady state solution using the spectral factorization method shown in (10). This makes reasonable sense, and the above equations show that a linear time invariant system with a stochastic input will tend to the steady state stationary condition. The development for the cases above can be used to cascade a LPF with an integrator and the resulting first and second moments can be directly determined. In fact following the development in this section, any type of system block having an analytic impulse response can be analyzed.

Case V.

Briefly, it should be mentioned that if WSS white noise is put into an ideal brickwall LPF the output autocorrelation function is a sinc function as shown in figure 2. The main lobe crossings are at 1/2B points. The significance of this is for sampling systems which would follow an anti-aliasing filter of "similar" design. If the system was designed to nyquist sample, i.e. at twice the highest frequency, then the samples are statistically uncorrelated. In addition for gaussian processes this also says they are independent. This is a useful fact in simplifying systems analysis problems and also achieving the often quoted 1/N increase in signal to noise ratio in averaging problems. Contrasting this ideal case with the case above (12), for practical filters the number of time constants for the autocorrelation function to reach a negligible value is called the decorrelation time. For single pole systems this is likely to be many time constants and hence nyquist sampling of the input would not produce uncorrelated noise samples. Even for the ideal case, sampling faster than 1/2B causes the noise samples to become more correlated, and the maximum amount of signal to noise improvement possible with a fixed number of samples cannot be realized.

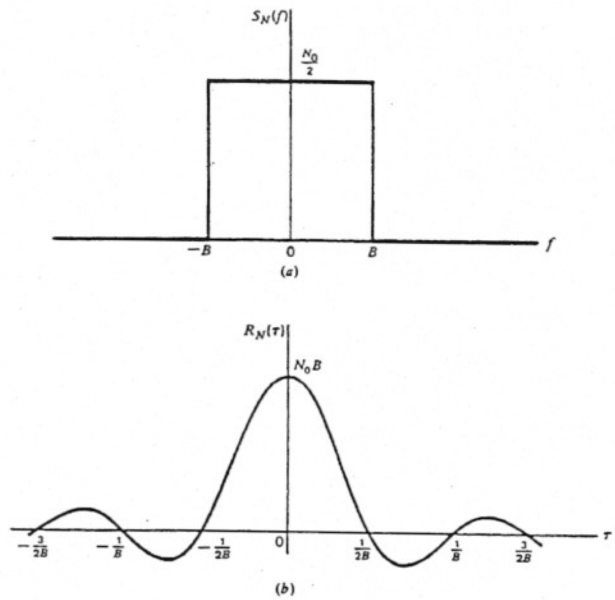

Figure 2: (a) Ideal LPF WHite Noise PSD (b) Autocorelation function

Results :

Case I shows that noise does not necessarily integrate to zero regardless of the length of the integrate gate time. In fact the variance grows linearly in t, and the possibility of finding the integral far away from zero grows as t grows.

Case II shows in contrast to the above that when the integral is scaled by the gate time then the variance will approach zero as t approaches infinity, and the result is the average value of the noise which could in fact be zero.

Case III used a sinusoidal input to show that in this case the output process actually achieves zero outputs at some equally spaced points. Also in this case the variance in periodic, with period $2\pi/\omega$.

Case IV shows the output of a LPF has changing mean and variance, however after several time constants these parameter approach the steady values of the WSS case. This is directly analogous to the transient response dying away and giving rise to the steady state solution in deterministic systems.

Case V was included merely to show a meaningful result for sampled data systems using averaging to increase signal to noise ratio.

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