BNL-104854-2014-TECH

AGS/AD/Tech Note No. 438;BNL-104854-2014-IR

Envelope Parameters for Linear Coupled Motion in Terms of the One-Turn Transfer Matrix

C. J. Gardner

July 1996

Collider Accelerator Department

Brookhaven National Laboratory

U.S. Department of Energy

USDOE Office of Science (SC)

DISCLAIMER

For Internal Distribution Only

Accelerator Division
Alternating Gradient Synchrotron Department
BROOKHAVEN NATIONAL LABORATORY
Upton, New York 11973

Accelerator Division
Technical Note

AGS/AD/Tech. Note No. 438

Envelope Parameters for Linear Coupled Motion in Terms of the One-Turn Transfer Matrix

C.J. Gardner

July 10, 1996

Envelope Parameters for Linear Coupled Motion in Terms of the One-Turn Transfer Matrix

C. J. Gardner

July 10, 1996

1 Introduction

In this note we obtain the envelope parameters for linear coupled motion in terms of the transfer matrix for one turn around a machine. The treatment is equivalent to others found in the literature, but follows more closely that of Courant and Snyder for the case of uncoupled motion.

2 The One-Turn Transfer Matrix

We shall use $\mathbf{T}$ to denote the one-turn transfer matrix, and in this section we examine its symplectic nature. By definition, the four-by-four matrix $\mathbf{T}$ is symplectic if

$$\mathbf{T}^{\dagger}\mathbf{S}\mathbf{T} = \mathbf{S}, \tag{1}$$

where

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad -\mathbf{S}^2 = \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Taking the inverse of both sides of (1) we obtain $\mathbf{T}^{-1}\mathbf{S}(\mathbf{T}^{\dagger})^{-1} = \mathbf{S}$ and therefore

$$\mathbf{S} = \mathbf{T}\mathbf{S}\mathbf{T}^{\dagger} \tag{3}$$

which is an equivalent form of the symplectic condition. Now, following Courant and Snyder [1], we define the symplectic conjugate of a two-by-two or four-by-four matrix $\mathbf{A}$ to be

$$\overline{\mathbf{A}} = -\mathbf{S}\mathbf{A}^\dagger\mathbf{S}, \tag{4}$$

where $\mathbf{S}$ is given by (2) for the case of four-by-four matricies. For two-by-two matricies we have

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \overline{\mathbf{A}} = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}, \tag{5}$$

and it follows that (for two-by-two matricies)

$$\mathbf{A}\overline{\mathbf{A}} = \overline{\mathbf{A}}\mathbf{A} = (A_{11}A_{22} - A_{12}A_{21})\mathbf{I} = |\mathbf{A}|\mathbf{I}, \tag{6}$$

$$\mathbf{A} + \overline{\mathbf{A}} = (A_{11} + A_{22})\mathbf{I} = \mathrm{Tr}(\mathbf{A})\mathbf{I}. \tag{7}$$

Using (1) and (3) we have

$$\overline{\mathbf{T}}\mathbf{T} = -\mathbf{S}\mathbf{T}^\dagger\mathbf{S}\mathbf{T} = -\mathbf{S}^2 = \mathbf{I}, \quad \mathbf{T}\overline{\mathbf{T}} = -\mathbf{T}\mathbf{S}\mathbf{T}^\dagger\mathbf{S} = -\mathbf{S}^2 = \mathbf{I} \tag{8}$$

and therefore $\overline{\mathbf{T}} = \mathbf{T}^{-1}$ if $\mathbf{T}$ is symplectic. Now writing

$$\mathbf{T} = \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix}, \tag{9}$$

where $\mathbf{M}$, $\mathbf{N}$, $\mathbf{m}$, $\mathbf{n}$ are two-by-two matricies, we have

$$\mathbf{T}\overline{\mathbf{T}} = \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix} \begin{pmatrix} \overline{\mathbf{M}} & \overline{\mathbf{m}} \\ \overline{\mathbf{n}} & \overline{\mathbf{N}} \end{pmatrix} = \begin{pmatrix} \mathbf{M}\overline{\mathbf{M}} + \mathbf{n}\overline{\mathbf{n}} & \mathbf{M}\overline{\mathbf{m}} + \mathbf{n}\overline{\mathbf{N}} \\ \mathbf{m}\overline{\mathbf{M}} + \mathbf{N}\overline{\mathbf{n}} & \mathbf{m}\overline{\mathbf{m}} + \mathbf{N}\overline{\mathbf{N}} \end{pmatrix} \tag{10}$$

and

$$\overline{\mathbf{T}}\mathbf{T} = \begin{pmatrix} \overline{\mathbf{M}} & \overline{\mathbf{m}} \\ \overline{\mathbf{n}} & \overline{\mathbf{N}} \end{pmatrix} \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix} = \begin{pmatrix} \overline{\mathbf{M}}\mathbf{M} + \overline{\mathbf{m}}\mathbf{m} & \overline{\mathbf{M}}\mathbf{n} + \overline{\mathbf{m}}\mathbf{N} \\ \overline{\mathbf{n}}\mathbf{M} + \overline{\mathbf{N}}\mathbf{m} & \overline{\mathbf{n}}\mathbf{n} + \overline{\mathbf{N}}\mathbf{N} \end{pmatrix}, \tag{11}$$

and comparing $\overline{\mathbf{T}}\mathbf{T} = \mathbf{T}\overline{\mathbf{T}} = \mathbf{I}$ with these equations we find

$$|\mathbf{M}| + |\mathbf{m}| = 1, \quad |\mathbf{N}| + |\mathbf{n}| = 1, \quad \overline{\mathbf{M}}\mathbf{n} + \overline{\mathbf{m}}\mathbf{N} = \mathbf{0} \tag{12}$$

and

$$|\mathbf{M}| + |\mathbf{n}| = 1, \quad |\mathbf{N}| + |\mathbf{m}| = 1, \quad \mathbf{M}\overline{\mathbf{m}} + \mathbf{n}\overline{\mathbf{N}} = \mathbf{0}. \tag{13}$$

Equations (12) and (13) are actually equivalent, and, as shown by Brown and Servranckx [2], they impose a total of 6 independent constraints on the 16 matrix elements of $\mathbf{T}$. The four-by-four symplectic matrix, $\mathbf{T}$, is therefore specified by 10 independent parameters. Equations (12) and (13) also imply the relations

$$|\mathbf{M}| = |\mathbf{N}|, \quad |\mathbf{m}| = |\mathbf{n}|. \tag{14}$$

Other properties of symplectic matricies, including the nature of their eigenvalues, are discussed in Refs. [1] and [3].

3 Reduction to Block-Diagonal Form

We shall assume that the four eigenvalues of $\mathbf{T}$ are distinct and lie on the unit circle in the complex plane. (This is generally true for the case of stable motion away from any linear resonances.) Then, as shown by Berz [4], the assumption of distinct eigenvalues ensures that $\mathbf{T}$ can be expressed in the form

$$\mathbf{T} = \mathbf{R}\mathbf{U}\mathbf{R}^{-1}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \tag{15}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two-by-two matricies and $\mathbf{U}$ and $\mathbf{R}$ are symplectic. Since $\mathbf{U}$ is symplectic we have $|\mathbf{A}| = |\mathbf{B}| = 1$ and it follows that $\mathbf{U}$ is specified by six independent parameters. The additional assumption that the eigenvalues lie on the unit circle in the complex plane, allows one to define Courant-Snyder parameters such that

$$\cos\psi_1 = (A_{11} + A_{22})/2, \quad \cos\psi_2 = (B_{11} + B_{22})/2, \tag{16}$$

$$\alpha_1 \sin\psi_1 = (A_{11} - A_{22})/2, \quad \alpha_2 \sin\psi_2 = (B_{11} - B_{22})/2, \tag{17}$$

$$\beta_1 \sin\psi_1 = A_{12}, \quad \beta_2 \sin\psi_2 = B_{12}, \tag{18}$$

$$\gamma_1 \sin\psi_1 = -A_{21}, \quad \gamma_2 \sin\psi_2 = -B_{21}, \tag{19}$$

where ψ_1 and ψ_2 are real, $\sin\psi_1$ and $\sin\psi_2$ are nonzero, and β_1 and β_2 are positive. Our assumptions also imply $\cos\psi_1 \neq \cos\psi_2$ and therefore $\mathrm{Tr}(\mathbf{A}) \neq \mathrm{Tr}(\mathbf{B})$. Thus $\mathbf{A}$ and $\mathbf{B}$ are of the form

$$\mathbf{A} = \mathbf{I}\cos\psi_1 + \mathbf{J}\sin\psi_1, \quad \mathbf{B} = \mathbf{I}\cos\psi_2 + \mathbf{K}\sin\psi_2 \tag{20}$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} \tag{21}$$

and since $|\mathbf{A}| = |\mathbf{B}| = 1$ we have

$$\beta_1\gamma_1 - \alpha_1^2 = 1, \quad \beta_2\gamma_2 - \alpha_2^2 = 1, \quad \mathbf{J}^2 = -\mathbf{I}, \quad \mathbf{K}^2 = -\mathbf{I}. \tag{22}$$

In addition to the matricies $\mathbf{J}$ and $\mathbf{K}$ it will be useful to define the matricies

$$\mathbf{F} = -\mathbf{JS} = \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix}, \quad \mathbf{G} = -\mathbf{KS} = \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix}, \tag{23}$$

which have only positive eigenvalues and are therefore positive-definite.

Now, since the matrix $\mathbf{U}$ contains only six independent parameters, the remaining four parameters needed to completely specify $\mathbf{T}$ are contained in the matrix $\mathbf{R}$. In the treatments of linear coupled motion given by Edwards and Teng [5, 6], and by Roser [7], $\mathbf{R}$ is expressed explicitly in terms of four independent parameters which, in turn, are expressed in terms of the matrix elements of $\mathbf{T}$. This proceedure could be followed here; however, we want to show that the envelope parameters do not depend on any particular form chosen for $\mathbf{R}$. Thus we write $\mathbf{R}$ in the general form

$$\mathbf{R} = \begin{pmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{V} & \mathbf{W} \end{pmatrix}, \tag{24}$$

where $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{V}$, and $\mathbf{W}$ are two-by-two matricies. Since $\mathbf{R}$ is symplectic, we have

$$|\mathbf{P}| + |\mathbf{V}| = 1, \quad |\mathbf{W}| + |\mathbf{Q}| = 1, \quad \overline{\mathbf{P}}\mathbf{Q} + \overline{\mathbf{V}}\mathbf{W} = \mathbf{0} \tag{25}$$

$$|\mathbf{P}| + |\mathbf{Q}| = 1, \quad |\mathbf{W}| + |\mathbf{V}| = 1, \quad \mathbf{P}\overline{\mathbf{V}} + \mathbf{Q}\overline{\mathbf{W}} = \mathbf{0}, \tag{26}$$

and defining $D = |\mathbf{P}|$, we also have

$$|\mathbf{W}| = |\mathbf{P}| = D, \quad |\mathbf{V}| = |\mathbf{Q}| = 1 - D. \tag{27}$$

We now carry out some algebraic manipulations which yield expressions for the four matricies, $\mathbf{PA\overline{P}}$, $\mathbf{WB\overline{W}}$, $\mathbf{QB\overline{Q}}$, and $\mathbf{VA\overline{V}}$. In the following section we show that these matricies contain the desired envelope parameters. Writing

$$\mathbf{T} = \begin{pmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{pmatrix} = \mathbf{RU\overline{R}} = \begin{pmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{V} & \mathbf{W} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \overline{\mathbf{P}} & \overline{\mathbf{V}} \\ \overline{\mathbf{Q}} & \overline{\mathbf{W}} \end{pmatrix} \tag{28}$$

and carrying out the matrix multiplications we find

$$\mathbf{M} = \mathbf{PA\overline{P}} + \mathbf{QB\overline{Q}}, \quad \mathbf{N} = \mathbf{VA\overline{V}} + \mathbf{WB\overline{W}}, \tag{29}$$

$$\mathbf{m} = \mathbf{V}\mathbf{A}\overline{\mathbf{P}} + \mathbf{W}\mathbf{B}\overline{\mathbf{Q}}, \quad \mathbf{n} = \mathbf{P}\mathbf{A}\overline{\mathbf{V}} + \mathbf{Q}\mathbf{B}\overline{\mathbf{W}}. \tag{30}$$

Let us now define

$$A = \mathrm{Tr}(\mathbf{A}), \quad B = \mathrm{Tr}(\mathbf{B}), \quad M = \mathrm{Tr}(\mathbf{M}), \quad N = \mathrm{Tr}(\mathbf{N}) \tag{31}$$

and

$$T = \mathrm{Tr}(\mathbf{M} - \mathbf{N}) = M - N, \quad U = \mathrm{Tr}(\mathbf{A} - \mathbf{B}) = A - B. \tag{32}$$

Then taking the trace of equations (29), and using (27), we have

$$M = DA + (1 - D)B, \quad N = (1 - D)A + DB. \tag{33}$$

Adding and subtracting these equations we obtain

$$M + N = A + B, \quad M - N = T = (2D - 1)U. \tag{34}$$

Thus, using the first of equations (34) in (16), we have

$$2\cos\psi_1 = \frac{1}{2}(M + N + U), \quad 2\cos\psi_2 = \frac{1}{2}(M + N - U), \tag{35}$$

and solving the last of equations (34) for D we have

$$D = \frac{U + T}{2U}, \quad D(1 - D) = \frac{U^2 - T^2}{4U^2}. \tag{36}$$

(Note that $\mathrm{Tr}(\mathbf{A}) \neq \mathrm{Tr}(\mathbf{B})$ and therefore $U \neq 0$.) Now adding $\mathbf{m}$ and $\overline{\mathbf{n}}$ we have

$$\mathbf{m} + \overline{\mathbf{n}} = \mathbf{V}(\mathbf{A} + \overline{\mathbf{A}})\overline{\mathbf{P}} + \mathbf{W}(\mathbf{B} + \overline{\mathbf{B}})\overline{\mathbf{Q}} = A\mathbf{V}\overline{\mathbf{P}} + B\mathbf{W}\overline{\mathbf{Q}}, \tag{37}$$

and using (26) we obtain

$$\mathbf{m} + \overline{\mathbf{n}} = U\mathbf{V}\overline{\mathbf{P}} = -U\mathbf{W}\overline{\mathbf{Q}}. \tag{38}$$

Taking the determinant of this equation, and using (27), we have

$$|\mathbf{m} + \overline{\mathbf{n}}| = U^2|\mathbf{V}||\mathbf{P}| = U^2 D(1 - D). \tag{39}$$

Then using (36) we obtain

$$U^2 = T^2 + 4|\mathbf{m} + \overline{\mathbf{n}}|. \tag{40}$$

Thus we have an expression for U which contains only the matrix elements of $\mathbf{T}$. This can be used in (35) to obtain $\cos\psi_1$ and $\cos\psi_2$, and in (36) to obtain D.

Now multiplying the second of equations (30) by (38), and using (27) and (29), we obtain

$$\begin{aligned} \mathbf{n}(\mathbf{m}+\overline{\mathbf{n}}) &= U\mathbf{PA}\overline{\mathbf{V}}\mathbf{V}\overline{\mathbf{P}} - U\mathbf{QB}\overline{\mathbf{W}}\mathbf{W}\overline{\mathbf{Q}} \\ &= U(1-D)\mathbf{PA}\overline{\mathbf{P}} - UD\mathbf{QB}\overline{\mathbf{Q}} \\ &= U\mathbf{PA}\overline{\mathbf{P}} - UD\mathbf{M} \end{aligned} \tag{41}$$

and

$$\begin{aligned} (\mathbf{m}+\overline{\mathbf{n}})\mathbf{n} &= U\mathbf{V}\overline{\mathbf{P}}\mathbf{PA}\overline{\mathbf{V}} - U\mathbf{W}\overline{\mathbf{Q}}\mathbf{QB}\overline{\mathbf{W}} \\ &= UD\mathbf{VA}\overline{\mathbf{V}} - U(1-D)\mathbf{WB}\overline{\mathbf{W}} \\ &= UD\mathbf{N} - U\mathbf{WB}\overline{\mathbf{W}}. \end{aligned} \tag{42}$$

Solving these equations for $\mathbf{PA}\overline{\mathbf{P}}$ and $\mathbf{WB}\overline{\mathbf{W}}$ we find

$$\mathbf{PA}\overline{\mathbf{P}} = D\mathbf{M} + \frac{1}{U}\mathbf{n}(\mathbf{m}+\overline{\mathbf{n}}), \tag{43}$$

$$\mathbf{WB}\overline{\mathbf{W}} = D\mathbf{N} - \frac{1}{U}(\mathbf{m}+\overline{\mathbf{n}})\mathbf{n}, \tag{44}$$

and using (29) we also have

$$\mathbf{QB}\overline{\mathbf{Q}} = \mathbf{M} - \mathbf{PA}\overline{\mathbf{P}} = (1-D)\mathbf{M} - \frac{1}{U}\mathbf{n}(\mathbf{m}+\overline{\mathbf{n}}), \tag{45}$$

$$\mathbf{VA}\overline{\mathbf{V}} = \mathbf{N} - \mathbf{WB}\overline{\mathbf{W}} = (1-D)\mathbf{N} + \frac{1}{U}(\mathbf{m}+\overline{\mathbf{n}})\mathbf{n}. \tag{46}$$

These equations, together with (36) and (40), show that the four matricies, $\mathbf{PA}\overline{\mathbf{P}}$, $\mathbf{WB}\overline{\mathbf{W}}$, $\mathbf{QB}\overline{\mathbf{Q}}$, and $\mathbf{VA}\overline{\mathbf{V}}$, depend only on the matrix elements of $\mathbf{T}$ and are independent of the particular form chosen for the matrix $\mathbf{R}$.

4 Envelope Parameters

Let us now define matricies $\mathbf{E}_\lambda$ such that

$$\mathbf{E}_{x1} = \mathbf{PFP}^\dagger, \quad \mathbf{E}_{x2} = \mathbf{QGQ}^\dagger, \quad \mathbf{E}_{y1} = \mathbf{VFV}^\dagger, \quad \mathbf{E}_{y2} = \mathbf{WGW}^\dagger. \tag{47}$$

Then it follows from (20), (23), and (27) that

$$\mathbf{PA}\overline{\mathbf{P}} = D\mathbf{I}\cos\psi_1 + \mathbf{E}_{x1}\mathbf{S}\sin\psi_1, \tag{48}$$

$$\mathbf{QB\overline{Q}} = (1-D)\mathbf{I}\cos\psi_2 + \mathbf{E}_{x2}\mathbf{S}\sin\psi_2, \tag{49}$$

$$\mathbf{VA\overline{V}} = (1-D)\mathbf{I}\cos\psi_1 + \mathbf{E}_{y1}\mathbf{S}\sin\psi_1, \tag{50}$$

$$\mathbf{WB\overline{W}} = D\mathbf{I}\cos\psi_2 + \mathbf{E}_{y2}\mathbf{S}\sin\psi_2, \tag{51}$$

and, using (43–46) and (35) in these equations, we see that the matricies $\mathbf{E}_\lambda$ can be expressed entirely in terms of the matrix elements of $\mathbf{T}$ and are independent of the particular form chosen for $\mathbf{R}$. In this section we show that these matricies contain the desired envelope parameters.

Let $\mathbf{T}_0$ and $\mathbf{T}$ be the transfer matricies for one turn around the machine starting at s_0 and s respectively, and let $\mathcal{M}$ be the transfer matrix from s_0 to s. Then we have

$$\mathbf{T} = \mathcal{M}\mathbf{T}_0\mathcal{M}^{-1}, \tag{52}$$

and it follows that $\mathbf{T}_0$ and $\mathbf{T}$ have the same eigenvalues which we have assumed are all distinct and lie on the unit circle in the complex plane. Thus we can write

$$\mathbf{T}_0 = \mathbf{R}_0\mathbf{U}_0\mathbf{R}_0^{-1}, \quad \mathbf{T} = \mathbf{RUR}^{-1} \tag{53}$$

where $\mathbf{R}_0$, $\mathbf{R}$, $\mathbf{U}_0$, and $\mathbf{U}$ are symplectic and

$$\mathbf{U}_0 = \begin{pmatrix} \mathbf{A}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_0 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \tag{54}$$

$$\mathbf{A}_0 = \mathbf{I}\cos\psi_1 + \mathbf{J}_0\sin\psi_1, \quad \mathbf{B}_0 = \mathbf{I}\cos\psi_2 + \mathbf{K}_0\sin\psi_2 \tag{55}$$

$$\mathbf{A} = \mathbf{I}\cos\psi_1 + \mathbf{J}\sin\psi_1, \quad \mathbf{B} = \mathbf{I}\cos\psi_2 + \mathbf{K}\sin\psi_2. \tag{56}$$

As in the previous section we define matricies

$$\mathbf{F}_0 = -\mathbf{J}_0\,\mathbf{S}, \quad \mathbf{G}_0 = -\mathbf{K}_0\,\mathbf{S}, \quad \mathbf{F} = -\mathbf{J}\,\mathbf{S}, \quad \mathbf{G} = -\mathbf{K}\,\mathbf{S} \tag{57}$$

which are positive-definite and have unit determinant. Now using (53) in (52) we have

$$\mathbf{RUR}^{-1} = \mathcal{M}\mathbf{R}_0\mathbf{U}_0\mathbf{R}_0^{-1}\mathcal{M}^{-1} \tag{58}$$

and therefore

$$\mathbf{U} = \mathbf{R}^{-1}\mathcal{M}\mathbf{R}_0\mathbf{U}_0\mathbf{R}_0^{-1}\mathcal{M}^{-1}\mathbf{R} = \mathbf{LU}_0\mathbf{L}^{-1} \tag{59}$$

where

$$\mathbf{L} = \mathbf{R}^{-1}\mathcal{M}\mathbf{R}_0 = \begin{pmatrix} \mathbf{L}_1 & \mathbf{l}_2 \\ \mathbf{l}_1 & \mathbf{L}_2 \end{pmatrix}. \tag{60}$$

Thus $\mathbf{L}$ produces a similarity transformation which transforms $\mathbf{U}_0$ into $\mathbf{U}$. We now show that, under our assumptions, $\mathbf{L}$ must be block-diagonal. Writing (59) as $\mathbf{UL} = \mathbf{LU}_0$ we find

$$\mathbf{AL}_1 = \mathbf{L}_1\mathbf{A}_0, \quad \mathbf{BL}_2 = \mathbf{L}_2\mathbf{B}_0, \quad \mathbf{Al}_2 = \mathbf{l}_2\mathbf{B}_0, \quad \mathbf{Bl}_1 = \mathbf{l}_1\mathbf{A}_0, \tag{61}$$

and therefore

$$\mathbf{A}|\mathbf{L}_1| = \mathbf{L}_1\mathbf{A}_0\overline{\mathbf{L}}_1, \quad \mathbf{B}|\mathbf{L}_2| = \mathbf{L}_2\mathbf{B}_0\overline{\mathbf{L}}_2 \tag{62}$$

and

$$\mathbf{A}|\mathbf{l}_2| = \mathbf{l}_2\mathbf{B}_0\bar{\mathbf{l}}_2, \quad \mathbf{B}|\mathbf{l}_1| = \mathbf{l}_1\mathbf{A}_0\bar{\mathbf{l}}_1. \tag{63}$$

Now, if $|\mathbf{l}_1| \neq 0$ or if $|\mathbf{l}_2| \neq 0$, then either $\mathbf{B}$ and $\mathbf{A}_0$, or $\mathbf{A}$ and $\mathbf{B}_0$ are related by a similarity transformation. It then follows that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues, which contradicts our assumption that the eigenvalues of $\mathbf{T}$ are all distinct. Thus we must have

$$|\mathbf{l}_1| = |\mathbf{l}_2| = 0 \tag{64}$$

and since $\mathbf{L}$ is symplectic we then have

$$|\mathbf{L}_1| = |\mathbf{L}_2| = 1. \tag{65}$$

Thus (62) and (63) become

$$\mathbf{A} = \mathbf{L}_1\mathbf{A}_0\overline{\mathbf{L}}_1, \quad \mathbf{B} = \mathbf{L}_2\mathbf{B}_0\overline{\mathbf{L}}_2, \quad \mathbf{l}_2\mathbf{B}_0\bar{\mathbf{l}}_2 = \mathbf{0}, \quad \mathbf{l}_1\mathbf{A}_0\bar{\mathbf{l}}_1 = \mathbf{0} \tag{66}$$

and using (55–57) in (66) we have

$$\mathbf{F} = \mathbf{L}_1\mathbf{F}_0\,\mathbf{L}_1^\dagger, \quad \mathbf{G} = \mathbf{L}_2\mathbf{G}_0\,\mathbf{L}_2^\dagger \tag{67}$$

and

$$\mathbf{l}_2\mathbf{G}_0\,\mathbf{l}_2^\dagger = \mathbf{0}, \quad \mathbf{l}_1\mathbf{F}_0\,\mathbf{l}_1^\dagger = \mathbf{0}. \tag{68}$$

Now since $\mathbf{F}_0$ and $\mathbf{G}_0$ are real, symmetric, and positive definite one can show, by going to the representations in which $\mathbf{F}_0$ and $\mathbf{G}_0$ are diagonal, that (68) implies

$$\mathbf{l}_1 = \mathbf{l}_2 = \mathbf{0}. \tag{69}$$

Thus $\mathbf{L}$ is block-diagonal and we can write

$$\mathbf{L} = \mathbf{R}^{-1}\mathcal{M}\mathbf{R}_0 = \begin{pmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 \end{pmatrix}. \tag{70}$$

Now let x_0, x_0', y_0, and y_0' be the horizontal and vertical positions and angles (or cononical momenta) of a beam particle at s_0. Then the positions and angles at s are given by

$$\mathbf{Z} = \mathcal{M}\mathbf{Z}_0, \tag{71}$$

where

$$\mathbf{Z}_0 = \begin{pmatrix} \mathbf{X}_0 \\ \mathbf{Y}_0 \end{pmatrix}, \quad \mathbf{X}_0 = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}, \quad \mathbf{Y}_0 = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}, \tag{72}$$

$$\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} y \\ y' \end{pmatrix}. \tag{73}$$

Defining new coordinates

$$\widehat{\mathbf{Z}} = \mathbf{R}^{-1}\mathbf{Z}, \quad \widehat{\mathbf{Z}}_0 = \mathbf{R}_0^{-1}\mathbf{Z}_0 \tag{74}$$

we then have

$$\mathbf{Z} = \mathbf{R}\widehat{\mathbf{Z}}, \quad \mathbf{Z}_0 = \mathbf{R}_0\widehat{\mathbf{Z}}_0 \tag{75}$$

and

$$\widehat{\mathbf{Z}} = \mathbf{R}^{-1}\mathbf{Z} = \mathbf{R}^{-1}\mathcal{M}\mathbf{Z}_0 = \mathbf{R}^{-1}\mathcal{M}\mathbf{R}_0\widehat{\mathbf{Z}}_0. \tag{76}$$

Thus

$$\widehat{\mathbf{Z}} = \mathbf{L}\widehat{\mathbf{Z}}_0 \tag{77}$$

and we see that, in terms of the new coordinates, $\mathbf{L}$ is the transfer matrix from s_0 to s. Writing

$$\widehat{\mathbf{Z}} = \begin{pmatrix} \widehat{\mathbf{X}} \\ \widehat{\mathbf{Y}} \end{pmatrix}, \quad \widehat{\mathbf{Z}}_0 = \begin{pmatrix} \widehat{\mathbf{X}}_0 \\ \widehat{\mathbf{Y}}_0 \end{pmatrix}, \tag{78}$$

where $\widehat{\mathbf{X}}$, $\widehat{\mathbf{Y}}$, $\widehat{\mathbf{X}}_0$, $\widehat{\mathbf{Y}}_0$ are two-dimensional vectors, we then have

$$\widehat{\mathbf{X}} = \mathbf{L}_1\widehat{\mathbf{X}}_0, \quad \widehat{\mathbf{Y}} = \mathbf{L}_2\widehat{\mathbf{Y}}_0. \tag{79}$$

Thus, in terms of the new coordinates, the motion is decoupled and it follows from (67) that

$$\widehat{\mathbf{X}}^\dagger\mathbf{F}^{-1}\widehat{\mathbf{X}} = \widehat{\mathbf{X}}_0^\dagger\mathbf{F}_0^{-1}\widehat{\mathbf{X}}_0 = \epsilon_1 \tag{80}$$

and

$$\widehat{\mathbf{Y}}^\dagger\mathbf{G}^{-1}\widehat{\mathbf{Y}} = \widehat{\mathbf{Y}}_0^\dagger\mathbf{G}_0^{-1}\widehat{\mathbf{Y}}_0 = \epsilon_2. \tag{81}$$

Here we see that, since $\mathbf{F}_0$, $\mathbf{G}_0$, $\mathbf{F}$, and $\mathbf{G}$ are positive definite, $\widehat{\mathbf{X}}_0$, $\widehat{\mathbf{Y}}_0$, $\widehat{\mathbf{X}}$, and $\widehat{\mathbf{Y}}$ are constrained to lie on ellipses, and ϵ_1 and ϵ_2 are the Courant-Snyder invariants of the motion.

Now using (24) in first of equations (75) we have

$$\mathbf{X} = \mathbf{P}\widehat{\mathbf{X}} + \mathbf{Q}\widehat{\mathbf{Y}}, \quad \mathbf{Y} = \mathbf{V}\widehat{\mathbf{X}} + \mathbf{W}\widehat{\mathbf{Y}} \tag{82}$$

and therefore

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2, \quad \mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 \tag{83}$$

where

$$\mathbf{X}_1 = \mathbf{P}\widehat{\mathbf{X}}, \quad \mathbf{X}_2 = \mathbf{Q}\widehat{\mathbf{Y}}, \quad \mathbf{Y}_1 = \mathbf{V}\widehat{\mathbf{X}}, \quad \mathbf{Y}_2 = \mathbf{W}\widehat{\mathbf{Y}}. \tag{84}$$

Thus, if $|\mathbf{P}| \neq 0$ and $|\mathbf{Q}| \neq 0$, the matricies $\mathbf{PFP}^\dagger$, $\mathbf{QGQ}^\dagger$, $\mathbf{VFV}^\dagger$, $\mathbf{WGW}^\dagger$ all have inverses and we have

$$\mathbf{X}_1^\dagger(\mathbf{PFP}^\dagger)^{-1}\mathbf{X}_1 = \widehat{\mathbf{X}}^\dagger\mathbf{F}^{-1}\widehat{\mathbf{X}} = \epsilon_1, \tag{85}$$

$$\mathbf{X}_2^\dagger(\mathbf{QGQ}^\dagger)^{-1}\mathbf{X}_2 = \widehat{\mathbf{Y}}^\dagger\mathbf{G}^{-1}\widehat{\mathbf{Y}} = \epsilon_2, \tag{86}$$

$$\mathbf{Y}_1^\dagger(\mathbf{VFV}^\dagger)^{-1}\mathbf{Y}_1 = \widehat{\mathbf{X}}^\dagger\mathbf{F}^{-1}\widehat{\mathbf{X}} = \epsilon_1, \tag{87}$$

$$\mathbf{Y}_2^\dagger(\mathbf{WGW}^\dagger)^{-1}\mathbf{Y}_2 = \widehat{\mathbf{Y}}^\dagger\mathbf{G}^{-1}\widehat{\mathbf{Y}} = \epsilon_2. \tag{88}$$

Using (47) we can then write

$$\mathbf{X}_1^\dagger\mathbf{E}_{x1}^{-1}\mathbf{X}_1 = \epsilon_1, \quad \mathbf{X}_2^\dagger\mathbf{E}_{x2}^{-1}\mathbf{X}_2 = \epsilon_2 \tag{89}$$

$$\mathbf{Y}_1^\dagger\mathbf{E}_{y1}^{-1}\mathbf{Y}_1 = \epsilon_1, \quad \mathbf{Y}_2^\dagger\mathbf{E}_{y2}^{-1}\mathbf{Y}_2 = \epsilon_2. \tag{90}$$

Then, since $\mathbf{F}$ and $\mathbf{G}$ are positive-definite, it follows that the matricies $\mathbf{E}_\lambda$ are positive-definite and equations (89–90) therefore describe ellipses. Thus $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ are each constrained to lie on an ellipse, and equations (83–90) show that the positions and angles x, x', y, y' are given by the superposition of two modes of oscillation which we shall label 1 and 2. In mode 1 we have $\epsilon_1 \neq 0$ and $\epsilon_2 = 0$, and it follows that $\mathbf{X}_2 = \mathbf{Y}_2 = 0$ and therefore $\mathbf{X} = \mathbf{X}_1$ and $\mathbf{Y} = \mathbf{Y}_1$. Similarly, in mode 2 we have $\epsilon_2 \neq 0$ and $\epsilon_1 = 0$, and it follows that $\mathbf{X} = \mathbf{X}_2$ and $\mathbf{Y} = \mathbf{Y}_2$. Thus, for each single mode of oscillation, the motion in each plane is constrained to lie on a single ellipse. If ϵ_1 and ϵ_2 are both nonzero, then both modes of oscillation are present and the motion in each plane is characterized by the superposition of two ellipses. This characterization of the motion in terms of two ellipses was first derived by Ripken [8, 9].

Now, since the matricies $\mathbf{E}_\lambda$ are symmetric, we can write

$$\mathbf{E}_\lambda = \begin{pmatrix} \beta_\lambda & -\alpha_\lambda \\ -\alpha_\lambda & \gamma_\lambda \end{pmatrix}, \tag{91}$$

where

$$\beta_{x1}\gamma_{x1} - \alpha_{x1}^2 = D^2, \quad \beta_{x2}\gamma_{x2} - \alpha_{x2}^2 = (1-D)^2 \tag{92}$$

$$\beta_{y1}\gamma_{y1} - \alpha_{y1}^2 = (1-D)^2, \quad \beta_{y2}\gamma_{y2} - \alpha_{y2}^2 = D^2. \tag{93}$$

Then writing

$$\mathbf{X}_1 = \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} x_2 \\ x_2' \end{pmatrix}, \quad \mathbf{Y}_1 = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}, \quad \mathbf{Y}_2 = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} \tag{94}$$

equations (89–90) become

$$\gamma_{x1}x_1^2 + 2\alpha_{x1}x_1x_1' + \beta_{x1}x_1'^2 = \epsilon_1 D^2, \tag{95}$$

$$\gamma_{x2}x_2^2 + 2\alpha_{x2}x_2x_2' + \beta_{x2}x_2'^2 = \epsilon_2 (1-D)^2, \tag{96}$$

$$\gamma_{y1}y_1^2 + 2\alpha_{y1}y_1y_1' + \beta_{y1}y_1'^2 = \epsilon_1 (1-D)^2, \tag{97}$$

$$\gamma_{y2}y_2^2 + 2\alpha_{y2}y_2y_2' + \beta_{y2}y_2'^2 = \epsilon_2 D^2. \tag{98}$$

The maximum possible values of x, x', y, and y' are then given by

$$|x| \le \sqrt{\beta_{x1}\epsilon_1} + \sqrt{\beta_{x2}\epsilon_2}, \quad |x'| \le \sqrt{\gamma_{x1}\epsilon_1} + \sqrt{\gamma_{x2}\epsilon_2} \tag{99}$$

$$|y| \le \sqrt{\beta_{y1}\epsilon_1} + \sqrt{\beta_{y2}\epsilon_2}, \quad |y'| \le \sqrt{\gamma_{y1}\epsilon_1} + \sqrt{\gamma_{y2}\epsilon_2} \tag{100}$$

and we see that β_λ and γ_λ are the desired envelope parameters. These are analogous to the Courant-Snyder parameters for uncoupled motion, but their normalization is given by equations (92–93) rather than $\beta_\lambda\gamma_\lambda - \alpha_\lambda^2 = 1$.

Equations (85–100) are valid only if $|\mathbf{P}| \neq 0$ and $|\mathbf{Q}| \neq 0$. For the case in which either $|\mathbf{P}| = D = 0$ or $|\mathbf{Q}| = 1 - D = 0$ the corresponding ellipses degenerate into line segments.

5 Summary

We summarize our results with a brief recipe for calculating the envelope parameters at s:

1) The first of two ingredients is the transfer matrix, $\mathbf{T}_0$, for one turn around a machine starting at some point, s_0, on the design orbit. The second ingredient is the transfer matrix, $\mathcal{M}$, from s_0 to some other point, s, on the design orbit.

2) Then the transfer matrix for one turn starting at s is given by $\mathbf{T} = \mathcal{M}\mathbf{T}_0\mathcal{M}^{-1}$.

3) Using equations (40) and (36) we obtain the parameters U and D in terms of the matrix elements of $\mathbf{T}$. Equations (35) give $\cos\psi_1$ and $\cos\psi_2$.

4) The matricies $\mathbf{PA\overline{P}}$, $\mathbf{WB\overline{W}}$, $\mathbf{QB\overline{Q}}$, and $\mathbf{VA\overline{V}}$ are then calculated using equations (43–46).

5) Finally, the matricies $\mathbf{E}_\lambda$ are calculated from equations (48–51). (The signs of $\sin\psi_1$ and $\sin\psi_2$ are determined by the requirement that the parameters β_λ be positive.)

Thus, the envelope parameters are given entirely in terms of the matrix elements of $\mathbf{T}$, and are independent of the form chosen for $\mathbf{R}$. The reduction of $\mathbf{T}$ to block-diagonal form therefore serves only as a scaffold for constructing the envelope parameters. It is worth noting here that $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, and $\mathbf{Y}_2$ are also independent of the form chosen for $\mathbf{R}$. This can be seen by substituting the first of equations (74) into (84). Using (27) and (38) we then have

$$\mathbf{X}_1 = \mathbf{P}(\overline{\mathbf{P}}\mathbf{X} + \overline{\mathbf{V}}\mathbf{Y}) = D\mathbf{X} + \frac{1}{U}(\overline{\mathbf{m}} + \mathbf{n})\mathbf{Y}, \tag{101}$$

$$\mathbf{X}_2 = \mathbf{Q}(\overline{\mathbf{Q}}\mathbf{X} + \overline{\mathbf{W}}\mathbf{Y}) = (1-D)\mathbf{X} - \frac{1}{U}(\overline{\mathbf{m}} + \mathbf{n})\mathbf{Y}, \tag{102}$$

$$\mathbf{Y}_1 = \mathbf{V}(\overline{\mathbf{P}}\mathbf{X} + \overline{\mathbf{V}}\mathbf{Y}) = (1-D)\mathbf{Y} + \frac{1}{U}(\overline{\mathbf{n}} + \mathbf{m})\mathbf{X}, \tag{103}$$

$$\mathbf{Y}_2 = \mathbf{W}(\overline{\mathbf{Q}}\mathbf{X} + \overline{\mathbf{W}}\mathbf{Y}) = D\mathbf{Y} - \frac{1}{U}(\overline{\mathbf{n}} + \mathbf{m})\mathbf{X}. \tag{104}$$

6 References

1. E.D. Courant and H.S. Snyder, Annals of Physics 3, 1–48 (1958).

2. K.L. Brown and R.V. Servranckx, Particle Accelerators 36, 121–139 (1991).

3. A.J. Dragt, AIP Conference Proceedings No. 87, pp. 147–313 (1982).

4. M. Berz, AIP Conference Proceedings No. 249, pp. 476–479 (1989).

5. L.C. Teng, NAL Report FN-229, May 3, 1971.

6. D.A. Edwards and L.C. Teng, IEEE Trans. on Nucl. Sc. 20, 885 (1973).

7. T. Roser, University of Michigan Report UM-HE 88-8 (1988).

8. G. Ripken, DESY Report R1-70/04 (1970).

9. F. Willeke and G. Ripken, AIP Conference Proceedings No. 184, pp. 758–819 (1989).