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Quadrupole end-field effects: going beyond the simple hard-edge approximation

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1. Introduction

In an accelerator or beam line, quadrupole magnets are used to focus the beam. When the accelerator is to be modelled on a computer, the quadrupoles are represented in the simplest possible approximation by what is called the hard-edge model. In this model, the beam particles are treated as travelling in a field-free drift region till they reach the edge of a quadrupole. At this point they enter the quadrupole field region which is characterised by a transverse field gradient that is uniform along the axis of the magnet. At the other end the field terminates equally abruptly.

Real magnets do not have such sharply terminating fields. They have regions near the ends where field lines fringe, and the transverse field gradient decreases with distance from the end of the magnet in a manner determined by the geometry of the magnet. To accurately determine the optical characteristics of such a magnet, an axial map of the field is required. The path of a charged particle through this field can then be determined by numerical integration. The phase-space transfer matrices for the two transverse directions can thus be obtained. One then has the choice of either using these matrices as they are, or deriving from them two equivalent hard-edge models, one for each plane. The equivalent model has a hard-edge quadrupole surrounded by two drift spaces of equal length such that the total length of the combination is the same as that of the original magnet between the points at which the field gradient is essentially zero. The hard-edge part of the equivalent model has different lengths and strengths in the focussing and defocussing planes. Usually one just uses the average of the two lengths as the length of the magnet and the average of the strengths as its effective strength. This procedure introduces some errors which are acceptable in most cases.

A detailed field map is often not available. In such a situation one takes recourse to an even greater simplification. The magnetic field is measured by rotating a long coil in its bore. If the coil is chosen to be longer than the field region, then the measurement yields the field gradient integrated over the axial direction. The value of the field gradient near the centre of the magnet is measured separately; this is taken to be the effective strength of the hard-edge magnet. Its effective length is obtained by dividing the integrated strength by the length.

2. Idealised quadrupole with linear end-fields

The lens strength resulting from the field of an idealised quadrupole magnet is shown in Figure 1. (This figure is based on Fig. 1-21 in [**1**] , and we have used the same symbol definitions to facilitate comparisons.) The magnet consists of an inner region Q with the familiar two-dimensional quadrupole field, leading to constant lens strength. This is surrounded by two end field regions P and R in which the strength changes linearly with distance from the end of the magnet.

To obtain expressions for transformation matrices through this quadrupole, we consider the differential equation (D.E.) for particle motion in each region. The solution in region P must have the boundary conditions specified at the point 0. The solution in region Q must join smoothly at points 1 and 2 to the solutions in regions P and R. This means that the solutions and their derivatives with respect to the longitudinal coordinate must be continuous at these points.

3. Notation

Let ψ be a column vector (the "solution vector") formed by grouping together the general solution of the D.E. and its derivative. A superscript on ψ identifies the region in which the solution holds. Thus, ψ^Q is the solution vector in region Q.

$$\psi^Q \equiv \begin{pmatrix} z \\ \dfrac{dz}{ds} \end{pmatrix}^Q$$

The general solution is a linear combination, with arbitrary coefficients, of the linearly independent solutions of the D.E. . In the region Q, the D.E. in the focussing plane is

$$\frac{d^2z}{ds^2} + k_m\, z = 0$$

This D.E. has the general solution

$$z = E \cos \kappa s + F \sin \kappa s$$

where $\kappa = k_m^{1/2}$, and E and F are arbitrary constants. The derivative of this solution is

$$\frac{dz}{ds} = -\kappa E \sin \kappa s + \kappa F \cos \kappa s$$

From these two, we can write the vector ψ^Q as a product of a matrix and a column vector of the coefficients E and F.

$$\psi^Q = \begin{pmatrix} \cos \kappa s & \sin \kappa s \\ -\kappa \sin \kappa s & \kappa \cos \kappa s \end{pmatrix} \cdot \begin{pmatrix} E \\ F \end{pmatrix}$$

We use $\mathbf{M}^Q$ as a shorthand for the matrix in the above equation. Similar matrices in the other regions are denoted by appropriate superscripts. They consist of the two independent solutions in the top row and their derivatives with respect to s in the bottom row.

An Arabic numeral subscript on, say, ψ^Q or matrix $\mathbf{M}^Q$ indicates its value at the point in question. Thus,

$$\mathbf{M}_1^Q = \mathbf{M}^Q \,|_{s=s_1}$$

4. Formal expression for transformation matrices

With the notation established above, we can now write the solution vectors in each region as following:--

$$\psi^P = \mathbf{M}^P \cdot \begin{pmatrix} C \\ D \end{pmatrix}$$

$$\psi^Q = \mathbf{M}^Q \cdot \begin{pmatrix} E \\ F \end{pmatrix}$$

$$\psi^R = \mathbf{M}^R \cdot \begin{pmatrix} G \\ H \end{pmatrix}$$

By considering the boundary conditions and solution matching between different regions, we can write a formal expression for the transformation matrix from point 0 to point 3.

The boundary conditions at $s = s_0$ are

$$\mathbf{M}_0^P \cdot \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} z_0 \\ z_0' \end{pmatrix}$$

This equation can be solved for the vector of coefficients C and D.

$$\begin{pmatrix} C \\ D \end{pmatrix} = \left(\mathbf{M}_0^P\right)^{-1} \cdot \begin{pmatrix} z_0 \\ z_0' \end{pmatrix} \tag{1}$$

The match at point 1 is

$$\mathbf{M}_1^Q \cdot \begin{pmatrix} E \\ F \end{pmatrix} = \mathbf{M}_1^P \cdot \begin{pmatrix} C \\ D \end{pmatrix}$$

From this, we get

$$\begin{pmatrix} E \\ F \end{pmatrix} = \left(\mathbf{M}_1^Q\right)^{-1} \cdot \mathbf{M}_1^P \cdot \begin{pmatrix} C \\ D \end{pmatrix} \tag{2}$$

The match at point 2 is

$$\mathbf{M}_2^R \cdot \begin{pmatrix} G \\ H \end{pmatrix} = \mathbf{M}_2^Q \cdot \begin{pmatrix} E \\ F \end{pmatrix}$$

From this, we get

$$\begin{pmatrix} G \\ H \end{pmatrix} = \left(\mathbf{M}_2^R\right)^{-1} \cdot \mathbf{M}_2^Q \cdot \begin{pmatrix} E \\ F \end{pmatrix} \tag{3}$$

At the point 3, we have

$$\begin{pmatrix} z_3 \\ z_3' \end{pmatrix} = \mathbf{M}_3^R \cdot \begin{pmatrix} G \\ H \end{pmatrix} \tag{4}$$

Combining Eqs. (1) - (4), we get

$$\begin{pmatrix} z_3 \\ z_3' \end{pmatrix} = \mathbf{M}_3^R \cdot \left(\mathbf{M}_2^R\right)^{-1} \cdot \mathbf{M}_2^Q \cdot \left(\mathbf{M}_1^Q\right)^{-1} \cdot \mathbf{M}_1^P \cdot \left(\mathbf{M}_0^P\right)^{-1} \cdot \begin{pmatrix} z_0 \\ z_0' \end{pmatrix} \tag{5}$$

The required transformation matrix is just the matrix product on the right hand side of Eq. (5).

5. Explicit expressions

We obtain explicit expressions for the various matrices in Eq. (5) by considering the solutions in the three regions. The D.E. has the general form:--

$$\frac{d^2z}{ds^2} + k(s)\,z = 0$$

In region P $(0 \le s \le b)$, this becomes

$$\frac{d^2z}{ds^2} + \left(\frac{k_m}{b}\right) sz = 0$$

We introduce a new independent variable u through,

$$u = \alpha s$$

where

$$\alpha \equiv \left(\frac{k_m}{b}\right)^{1/3} \tag{6}$$

This transforms the D.E. to

$$\frac{d^2z}{du^2} + uz = 0 \tag{7}$$

One pair of linearly independent solutions of Eq. (7) are the Airy functions $\mathrm{Ai}(-u)$ and $\mathrm{Bi}(-u)$ [2]. The derivative with respect to s is

$$\frac{d}{ds}\mathrm{Ai}(-u) = \frac{du}{ds}\frac{d}{du}\mathrm{Ai}(-u) = -\alpha\,\mathrm{Ai}'(-u)$$

From these, we obtain

$$\mathbf{M}^P = \begin{pmatrix} \mathrm{Ai}(-u) & \mathrm{Bi}(-u) \\ -\alpha\,\mathrm{Ai}'(-u) & -\alpha\,\mathrm{Bi}'(-u) \end{pmatrix}$$

At the point 0, $u = 0$. Therefore,

$$\mathbf{M}_0^P = \begin{pmatrix} \mathrm{Ai}(0) & \mathrm{Bi}(0) \\ -\alpha\,\mathrm{Ai}'(0) & -\alpha\,\mathrm{Bi}'(0) \end{pmatrix}$$

The determinant of this matrix is $-\alpha$ times the Wronskian of the Airy functions, which is just π^{-1}. Then,

$$\left(\mathbf{M}_0^P\right)^{-1} = \frac{\pi}{\alpha}\begin{pmatrix} \alpha\,\mathrm{Bi}'(0) & \mathrm{Bi}(0) \\ -\alpha\,\mathrm{Ai}'(0) & -\mathrm{Ai}(0) \end{pmatrix} \tag{8}$$

At the point 1, $u = \alpha b$. Then,

$$\mathbf{M}_1^P = \begin{pmatrix} \mathrm{Ai}(-\alpha b) & \mathrm{Bi}(-\alpha b) \\ -\alpha\,\mathrm{Ai}'(-\alpha b) & -\alpha\,\mathrm{Bi}'(-\alpha b) \end{pmatrix} \tag{9}$$

The D.E. and its solutions in region Q have been considered in Section 2 above, whence we have:--

$$\mathbf{M}^Q = \begin{pmatrix} \cos\kappa s & \sin\kappa s \\ -\kappa\sin\kappa s & \kappa\cos\kappa s \end{pmatrix}$$

At the point 1, $s = b$, while at the point 2, $s = a + b$. We get expressions for $\mathbf{M}_1^Q$ and $\mathbf{M}_2^Q$ by using these values in the above equation, and obtain the following result for the matrix product pertaining to the region Q that occurs in Eq. (5) :--

$$\mathbf{M}_2^Q \cdot \left(\mathbf{M}_1^Q\right)^{-1} = \begin{pmatrix} \cos\kappa a & \frac{1}{\kappa}\sin\kappa a \\ -\kappa\sin\kappa a & \cos\kappa a \end{pmatrix} \tag{10}$$

In the region R, the D.E. can be transformed to the form

$$\frac{d^2z}{dv^2} + vz = 0$$

by changing the independent variable to v through

$$v = \alpha\,(a + 2b - s)$$

We construct the following matrix from the solutions:--

$$\mathbf{M}^R = \begin{pmatrix} \mathrm{Ai}(-v) & \mathrm{Bi}(-v) \\ \alpha\,\mathrm{Ai}'(-v) & \alpha\,\mathrm{Bi}'(-v) \end{pmatrix}$$

At the point 2, $v = \alpha b$. This gives

$$\mathbf{M}_2^R = \begin{pmatrix} \mathrm{Ai}(-\alpha b) & \mathrm{Bi}(-\alpha b) \\ \alpha\,\mathrm{Ai}'(-\alpha b) & \alpha\,\mathrm{Bi}'(-\alpha b) \end{pmatrix}$$

The inverse of the above matrix is

$$\left(\mathbf{M}_2^R\right)^{-1} = \frac{\pi}{\alpha} \begin{pmatrix} \alpha\,\mathrm{Bi}'(-\alpha b) & -\mathrm{Bi}(-\alpha b) \\ -\alpha\,\mathrm{Ai}'(-\alpha b) & \mathrm{Ai}(-\alpha b) \end{pmatrix} \tag{11}$$

At the point 3, $v = 0$. This gives

$$\mathbf{M}_3^R = \begin{pmatrix} \mathrm{Ai}(0) & \mathrm{Bi}(0) \\ \alpha\,\mathrm{Ai}'(0) & \alpha\,\mathrm{Bi}'(0) \end{pmatrix} \tag{12}$$

The expressions for the various matrices in Eqs. (8) - (12) can be substituted into Eq. (5) to give the transformation matrix for the focussing plane.

$$\mathbf{M}_{foc} = \frac{\pi^2}{\alpha^2} \begin{pmatrix} \mathrm{Ai}(0) & \mathrm{Bi}(0) \\ \alpha\,\mathrm{Ai}'(0) & \alpha\,\mathrm{Bi}'(0) \end{pmatrix} \cdot \begin{pmatrix} \alpha\,\mathrm{Bi}'(-\alpha b) & -\mathrm{Bi}(-\alpha b) \\ -\alpha\,\mathrm{Ai}'(-\alpha b) & \mathrm{Ai}(-\alpha b) \end{pmatrix} \cdot \begin{pmatrix} \cos\kappa a & \frac{1}{\kappa}\sin\kappa a \\ -\kappa\sin\kappa a & \cos\kappa a \end{pmatrix} \cdot$$

$$\begin{pmatrix} \mathrm{Ai}(-\alpha b) & \mathrm{Bi}(-\alpha b) \\ -\alpha\,\mathrm{Ai}'(-\alpha b) & -\alpha\,\mathrm{Bi}'(-\alpha b) \end{pmatrix} \cdot \begin{pmatrix} \alpha\,\mathrm{Bi}'(0) & \mathrm{Bi}(0) \\ -\alpha\,\mathrm{Ai}'(0) & -\mathrm{Ai}(0) \end{pmatrix} \quad (13)$$

Analysis for the defocussing plane proceeds on similar lines, and results in the following expression for the transformation matrix :--

$$\mathbf{M}_{def} = \frac{\pi^2}{\alpha^2} \begin{pmatrix} \mathrm{Ai}(0) & \mathrm{Bi}(0) \\ -\alpha\,\mathrm{Ai}'(0) & -\alpha\,\mathrm{Bi}'(0) \end{pmatrix} \cdot \begin{pmatrix} \alpha\,\mathrm{Bi}'(\alpha b) & \mathrm{Bi}(\alpha b) \\ -\alpha\,\mathrm{Ai}'(\alpha b) & -\mathrm{Ai}(\alpha b) \end{pmatrix} \cdot \begin{pmatrix} \cosh \kappa a & \frac{1}{\kappa} \sinh \kappa a \\ \kappa \sinh \kappa a & \cosh \kappa a \end{pmatrix} \cdot$$

$$\begin{pmatrix} \mathrm{Ai}(\alpha b) & \mathrm{Bi}(\alpha b) \\ \alpha\,\mathrm{Ai}'(\alpha b) & \alpha\,\mathrm{Bi}'(\alpha b) \end{pmatrix} \cdot \begin{pmatrix} \alpha\,\mathrm{Bi}'(0) & -\mathrm{Bi}(0) \\ -\alpha\,\mathrm{Ai}'(0) & \mathrm{Ai}(0) \end{pmatrix} \quad (14)$$

It is instructive to consider the limiting case of Eqs. (13) and (14) as $b \rightarrow 0$. Then, although α goes to infinity, the product $\alpha b \rightarrow 0$. The leftmost matrix multiplied by the limiting value of its neighbour on the right yields a unit matrix times $\alpha\,\pi^{-1}$, as does the rightmost matrix multiplied by its neighbour on the left. The total transformation matrix then reduces to the middle matrix, which is recognised as the usual expression for the hard-edge quadrupole. For small αb, the exact transformation matrix will deviate slightly from the hard-edge matrix.

6. Numerical values

The transformation matrices of Eqs. (13) and (14) were evaluated for two types of magnets discussed in [1]. Type I is a long magnet with short fringe field regions. Its dimensions are:--

a = 0.92216 m

b = 0.12092 m

Type II is a short magnet with long fringe field regions. It has the dimensions:--

a = 0.394 m

b = 0.266 m

The transformation coefficients were calculated at the same values of k_m as in [1] , namely, 0.25, 1.0 and 4.0 m^{-2}. All calculations were performed to a fractional accuracy of order 10^{-13} with the SMP 1.6.0 system on node BNLCL2 of the AMD VAX Cluster. The results are shown in columns labelled EXACT in Tables 1 and 2. These agree with the results in Table 1-3 of [1].*

7. Discussion and conclusions

The columns labelled SIMPLE in Tables 1 and 2 contain the linear transformation coefficients for a hard-edge quadrupole whose parameters are obtained by the usual approach described in Section 3 above. For the field geometry under consideration, this leads to an effective strength k_m, and an effective length L given by

$$L = a + b$$

* There is no indication in [1] about how the numbers in its Table 1-3 were obtained; their source is listed as a private communication.

Differences between coefficients calculated by the two methods are obvious. For Type I magnet, the largest relative deviations occur in the r_{12} coefficient (the "lever arm"); their sign beahaves differently in the two planes, but they have the greatest magnitude at the largest value of k_m. The r_{21} coefficient has the smallest (almost negligible) relative deviations, while those of the other two (principal diagonal) coefficients fall between these extremes. The last also show an increase with k_m, albeit a slightly smaller one in the defocussing plane than in the focussing one.

The same remarks generally apply to Type II magnet also. However, the relative deviation in the r_{12} element decreases with increasing k_m in the focussing plane, but increases in the defocussing one. In both planes these deviations are much larger than for Type I. The relative deviations in other coefficients are comparable in magnitude to those of Type I magnet at low k_m but become almost twice as large at $k_m = 4.0\ \mathrm{m}^{-2}$. The most dramatic difference is seen in the focussing plane at this value of k_m. The exact calculation gives negative numbers for the pricipal diagonal elements, making this magnet much more strongly focussing than predicted by the simple model.

The large deviations for $k_m = 4.0\ \mathrm{m}^{-2}$ run counter to the intuitive expectation that the hard-edge model would be a good approximation when the slope of the strength in the fringe regions is large. But, closer examination reveals that the argument of the Airy functions in Eqs.(13) and (14), αb, grows with k_m.

$$\alpha b = (k_m b^2)^{1/3}$$

Then, since αb is large, the total transformation must deviate more from the hard-edge approximate values.

It is to be emphasised that the above analysis does not take into account the component of the fringe field parallel to the magnet axis. If the magnet is shaped carefully to have a linearised fringe field of the type considered above, its axial field component is given by

$$B_s = g_0' \, zx$$

This component would cause a coupling between the two transverse planes, an effect that can be analysed by considering the higher-order aberration coefficients, as has been done in [1].

Even in the absence of a full treatment of the axial component, a linear fringe field is a better approximation to the true field than a crude hard-edge model, whether the magnet has been designed to have linear end fields or not. It models the real situation a little more faithfully than the alternative so extensively used.

References

[1] Klaus G. Steffen, *High Energy Beam Optics,* (Wiley, New York, 1965), Sec. 1.14.

[2] H. A. Antosiewicz, in *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (U.S. National Bureau of Standards, Washington, D.C., 1970), Sec.10.4.

TABLE 1

Transformation coefficients for idealised quadrupole magnet Type I for different values of k_m. Magnet dimensions are: a = 0.92216 m and b = 0.12092 m. The coefficients in the columns labelled EXACT are calculated from Eqs. 13 and 14. Those labelled SIMPLE are for a hard-edge model derived in the usual way.

	$k_m = 0.25\ m^{-2}$		$k_m = 1.0\ m^{-2}$		$k_m = 4.0\ m^{-2}$	
	EXACT	SIMPLE	EXACT	SIMPLE	EXACT	SIMPLE
			Focussing plane			
r_{11}	0.851995	0.867053	0.451377	0.503562	-0.597643	-0.492851
r_{12}	1.100669	0.996432	0.922747	0.863959	0.371242	0.435057
r_{21}	-0.249034	-0.249108	-0.862922	-0.863959	-1.731547	-1.740227
r_{22}	0.851995	0.867053	0.451377	0.503562	-0.597643	-0.492851
			Defocussing plane			
r_{11}	1.155607	1.139113	1.670366	1.595156	4.570241	4.089046
r_{12}	1.229421	1.091014	1.438707	1.242788	2.496301	1.982441
r_{21}	0.272835	0.272754	1.244257	1.242788	7.966628	7.929766
r_{22}	1.155607	1.139113	1.670366	1.595156	4.570241	4.089046

TABLE 2

Transformation coefficients for idealised quadrupole magnet Type II for different values of k_m. Magnet dimensions are: a = 0.394 m and b = 0.266 m. The coefficients in the columns labelled EXACT are calculated from Eqs. 13 and 14. Those labelled SIMPLE are for a hard-edge model derived in the usual way.

	$k_m = 0.25\ m^{-2}$		$k_m = 1.0\ m^{-2}$		$k_m = 4.0\ m^{-2}$	
	EXACT	SIMPLE	EXACT	SIMPLE	EXACT	SIMPLE
			Focussing plane			
r_{11}	0.924516	0.946042	0.708797	0.789992	-0.004911	0.248175
r_{12}	0.897825	0.648086	0.816062	0.613117	0.528026	0.484358
r_{21}	-0.161801	-0.162022	-0.609765	-0.613117	-1.893799	-1.937430
r_{22}	0.924516	0.946042	0.708797	0.789992	-0.004911	0.248175
			Defocussing plane			
r_{11}	1.077315	1.054946	1.320511	1.225822	2.475179	2.005278
r_{12}	0.954644	0.672044	1.043442	0.708971	1.444274	0.869072
r_{21}	0.168238	0.168011	0.712784	0.708971	3.549544	3.476286
r_{22}	1.077315	1.054946	1.320511	1.225822	2.475179	2.005278

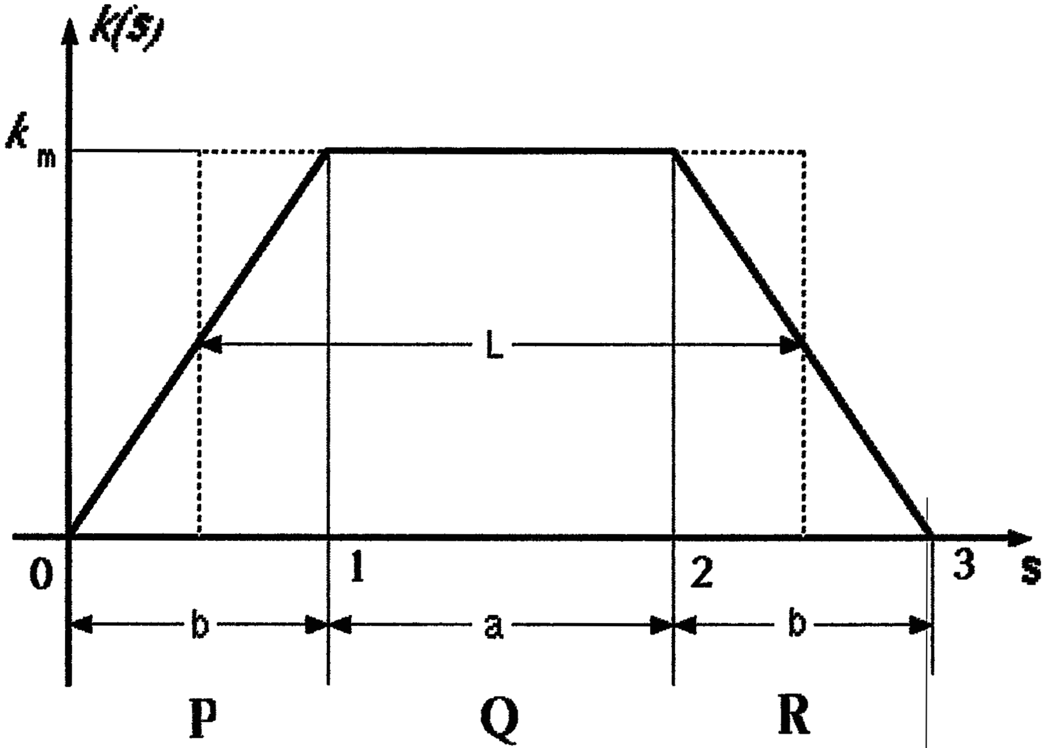

Figure 1 - Lens strength of an idealised quadrupole magnet.
The effective length of a simple hard-edge model equals L.