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Some Calculations of Dispersion in the Booster

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1 Introduction

Recent modeling [1, 2] of the BTA transport line suggests that one may achieve better transport efficiency if the periodic dispersion and its derivative are made to be zero (or near zero) at the point of extraction in the F6 straight section of the Booster. In principle this can be accomplished by appropriately adjusting the currents in some subset of the booster quadrupoles, and recent searches [1, 2] using the MAD code have yielded some possible solutions. The most promising of these requires adjusting the current in a single quadrupole—the horizontal focusing quadrupole at F2. Following are some algebraic calculations meant to supplement the results obtained with the MAD code. We consider first the case of a single thin quadrupole and calculate its effect on the periodic dispersion. The case of four thin quadrupoles arranged in pairs such as those employed in the gamma-transition-jump scheme of the AGS is then considered. The resulting formulae are applied to the manipulation of the periodic dispersion at F6.

2 Notation

We use $\mathbf{T}_i$ to denote the unperturbed (i.e. without perturbing quadrupoles) transfer matrix for one turn around a ring starting at some point, s_i, on the design orbit. The perturbed transfer matrix for one turn starting at s_0 will be denoted by $\mathbf{T}$. For motion in the horizontal plane, these are three-by-three matricies which we partition as follows:

$$\mathbf{T}_i = \begin{pmatrix} \mathbf{M}_i & \mathbf{d}_i \\ \mathbf{0} & 1 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \mathbf{M} & \mathbf{d} \\ \mathbf{0} & 1 \end{pmatrix} \tag{1}$$

where

$$\mathbf{M}_i = \mathbf{I}\cos 2\pi\nu_0 + \mathbf{J}_i \sin 2\pi\nu_0, \quad \mathbf{M} = \mathbf{I}\cos 2\pi\nu + \mathbf{J}\sin 2\pi\nu, \tag{2}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J}_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\gamma_i & -\alpha_i \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \tag{3}$$

and

$$\mathbf{d}_i = \begin{pmatrix} d_i \\ d_i' \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} d \\ d' \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \end{pmatrix}. \tag{4}$$

Here α_i, β_i, γ_i are the unperturbed Courant-Snyder parameters at s_i, and ν_0 is the unperturbed horizontal tune. α, β, γ are the corresponding perturbed parameters at s_0 and ν is the perturbed horizontal tune. The unperturbed and perturbed periodic dispersion vectors at s_0 are denoted by

$$\mathbf{D}_0 = \begin{pmatrix} D_0 \\ D_0' \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} D \\ D' \end{pmatrix}. \tag{5}$$

These must satisfy

$$\mathbf{M}_0\mathbf{D}_0 + \mathbf{d}_0 = \mathbf{D}_0, \quad \mathbf{M}\mathbf{D} + \mathbf{d} = \mathbf{D} \tag{6}$$

and therefore

$$\mathbf{D}_0 = (\mathbf{I} - \mathbf{M}_0)^{-1}\mathbf{d}_0, \quad \mathbf{D} = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{d}. \tag{7}$$

The unperturbed transfer matrix from point s_i to point s_j is given by

$$\mathbf{U}_{ji} = \begin{pmatrix} \mathbf{N}_{ji} & \mathbf{n}_{ji} \\ \mathbf{0} & 1 \end{pmatrix}, \quad \mathbf{N}_{ji} = \begin{pmatrix} A_{ji} & B_{ji} \\ E_{ji} & F_{ji} \end{pmatrix} \tag{8}$$

where

$$\mathbf{n}_{ji} = \begin{pmatrix} n_{ji} \\ n_{ji}' \end{pmatrix} = \mathbf{D}_j - \mathbf{N}_{ji}\mathbf{D}_i, \tag{9}$$

$\mathbf{D}_i$ is the (unperturbed) periodic dispersion vector at s_i, and

$$A_{ji} = \sqrt{\beta_j/\beta_i}\,(\cos\Delta\psi + \alpha_i \sin\Delta\psi), \quad B_{ji} = \sqrt{\beta_j\beta_i}\sin\Delta\psi \tag{10}$$

$$E_{ji} = -\left(\frac{\alpha_j - \alpha_i}{\sqrt{\beta_j\beta_i}}\right)\cos\Delta\psi - \left(\frac{1 + \alpha_j\alpha_i}{\sqrt{\beta_j\beta_i}}\right)\sin\Delta\psi \tag{11}$$

$$F_{ji} = \sqrt{\beta_i/\beta_j}\,(\cos\Delta\psi - \alpha_j \sin\Delta\psi). \tag{12}$$

Here $\Delta\psi = \psi_j - \psi_i$ and ψ_i is the unperturbed betatron phase advance from s_0 to s_i. For the case $i = j - 1$ we define

$$\mathbf{U}_j = \mathbf{U}_{ji}, \quad A_j = A_{ji}, \quad B_j = B_{ji}, \quad E_j = E_{ji}, \quad F_j = F_{ji}. \tag{13}$$

The transfer matrix for a thin quadrupole at s_i is given by

$$\mathbf{R}_i = \begin{pmatrix} \mathbf{Q}_i & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \quad \mathbf{Q}_i = \mathbf{I} + \mathbf{G}_i, \quad \mathbf{G}_i = \begin{pmatrix} 0 & 0 \\ G_i & 0 \end{pmatrix} \tag{14}$$

where G_i is the strength of the quadrupole.

Following Courant and Snyder we also define the symplectic conjugate of a two-by-two matrix $\mathbf{A}$ to be

$$\overline{\mathbf{A}} = -\mathbf{S}\mathbf{A}^{\dagger}\mathbf{S} \tag{15}$$

where

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{16}$$

and a dagger denotes the transpose of the matrix. Thus we have

$$\overline{\mathbf{A}} = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}, \tag{17}$$

and it follows that

$$\mathbf{A}\overline{\mathbf{A}} = \overline{\mathbf{A}}\mathbf{A} = (A_{11}A_{12} - A_{11}A_{22})\mathbf{I} = \mathbf{I}\,|\mathbf{A}|, \tag{18}$$

$$\mathbf{A} + \overline{\mathbf{A}} = (A_{11} + A_{22})\mathbf{I} = \mathbf{I}\,\mathrm{Tr}\mathbf{A}. \tag{19}$$

If $|\mathbf{A}| = 1$ then (18) implies $\overline{\mathbf{A}} = \mathbf{A}^{-1}$.

3 Single Quadrupole Perturbation

Let us now consider a single thin quadrupole placed at a point, s_1, on the design orbit such that

$$s_0 < s_1 < s_2, \quad s_2 = s_0 + C \tag{20}$$

where C is the ring circumference. We wish to calculate the effect of this quadrupole on the periodic dispersion at s_0. In terms of the notation introduced in the previous section we have

$$\mathbf{T}_0 = \mathbf{U}_2\mathbf{U}_1, \quad \mathbf{T}_1 = \mathbf{U}_1\mathbf{U}_2, \quad \mathbf{T} = \mathbf{U}_2\mathbf{R}_1\mathbf{U}_1 \tag{21}$$

and therefore

$$\mathbf{M}_0 = \mathbf{N}_2\mathbf{N}_1, \quad \mathbf{M}_1 = \mathbf{N}_1\mathbf{N}_2, \quad \mathbf{M} = \mathbf{N}_2\mathbf{Q}_1\mathbf{N}_1, \tag{22}$$

$$\mathbf{d}_0 = \mathbf{N}_2\mathbf{n}_1 + \mathbf{n}_2, \quad \mathbf{d}_1 = \mathbf{N}_1\mathbf{n}_2 + \mathbf{n}_1, \quad \mathbf{d} = \mathbf{N}_2\mathbf{Q}_1\mathbf{n}_1 + \mathbf{n}_2, \tag{23}$$

where

$$\mathbf{n}_1 = \mathbf{D}_1 - \mathbf{N}_1\mathbf{D}_0, \quad \mathbf{n}_2 = \mathbf{D}_2 - \mathbf{N}_2\mathbf{D}_1 = \mathbf{D}_0 - \mathbf{N}_2\mathbf{D}_1. \tag{24}$$

(Note that since $s_2 = s_0 + C$ we have $\mathbf{D}_2 = \mathbf{D}_0$.) Using (24) in the last of equations (23) we then have

$$\mathbf{d} = \mathbf{N}_2\mathbf{Q}_1(\mathbf{D}_1 - \mathbf{N}_1\mathbf{D}_0) + \mathbf{D}_0 - \mathbf{N}_2\mathbf{D}_1 \tag{25}$$

and using $\mathbf{Q}_1 = \mathbf{I} + \mathbf{G}_1$ we have

$$\mathbf{d} = \mathbf{N}_2\mathbf{G}_1\mathbf{D}_1 + (\mathbf{I} - \mathbf{M})\mathbf{D}_0 \tag{26}$$

where

$$\mathbf{M} = \mathbf{N}_2\mathbf{Q}_1\mathbf{N}_1 = \mathbf{M}_0 + \mathbf{N}_2\mathbf{G}_1\mathbf{N}_1. \tag{27}$$

The perturbed periodic dispersion vector at s_0 is then

$$\mathbf{D} = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{d} = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{N}_2\mathbf{G}_1\mathbf{D}_1 + \mathbf{D}_0 \tag{28}$$

and therefore

$$\mathbf{D} - \mathbf{D}_0 = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{N}_2\mathbf{G}_1\mathbf{D}_1. \tag{29}$$

Thus (29) and (27) give the difference between the perturbed and unperturbed periodic dispersion vectors at s_0 in terms of the strength of the thin quadrupole at s_1 and the unperturbed lattice parameters. Equation (29) has the same form as the expression for the Periodic Closed Orbit Distortion at s_0 due to a dipole perturbation at s_1. With a little more algebra it can be simplified a bit further. Using (2), (18) and (19) we have

$$(\mathbf{I} - \mathbf{M})(\mathbf{I} - \overline{\mathbf{M}}) = 2\mathbf{I} - (\mathbf{M} + \overline{\mathbf{M}}) = W\mathbf{I}, \tag{30}$$

where

$$W = 2 - \mathrm{Tr}\mathbf{M} = 2(1 - \cos 2\pi\nu). \tag{31}$$

Thus

$$(\mathbf{I} - \mathbf{M})^{-1} = \frac{1}{W}(\mathbf{I} - \overline{\mathbf{M}}) \tag{32}$$

where

$$\overline{\mathbf{M}} = \overline{\mathbf{M}}_0 - \overline{\mathbf{N}}_1\mathbf{G}_1\overline{\mathbf{N}}_2. \tag{33}$$

Now, multiplying (33) by $\mathbf{N}_2\mathbf{G}_1$ and using $\overline{\mathbf{N}}_2\mathbf{N}_2 = \mathbf{I}$ and $\mathbf{G}_1^2 = \mathbf{0}$, we have

$$\overline{\mathbf{M}}\mathbf{N}_2\mathbf{G}_1 = \overline{\mathbf{M}}_0\mathbf{N}_2\mathbf{G}_1 - \overline{\mathbf{N}}_1\mathbf{G}_1^2 = \overline{\mathbf{M}}_0\mathbf{N}_2\mathbf{G}_1 \tag{34}$$

and therefore

$$(\mathbf{I} - \mathbf{M})^{-1}\mathbf{N}_2\mathbf{G}_1 = \frac{1}{W}(\mathbf{I} - \overline{\mathbf{M}})\mathbf{N}_2\mathbf{G}_1 = \frac{1}{W}(\mathbf{I} - \overline{\mathbf{M}}_0)\mathbf{N}_2\mathbf{G}_1. \tag{35}$$

Thus (29) becomes

$$\mathbf{D} - \mathbf{D}_0 = \frac{1}{W}(\mathbf{I} - \overline{\mathbf{M}}_0)\mathbf{N}_2\mathbf{G}_1\mathbf{D}_1, \tag{36}$$

and since

$$\overline{\mathbf{M}}_0\mathbf{N}_2 = \mathbf{N}_2\overline{\mathbf{M}}_1, \tag{37}$$

which follows from (22), we also have

$$\mathbf{D} - \mathbf{D}_0 = \frac{1}{W}\mathbf{N}_2(\mathbf{I} - \overline{\mathbf{M}}_1)\mathbf{G}_1\mathbf{D}_1. \tag{38}$$

To obtain W in terms of the unperturbed parameters we take the trace of (27). Thus

$$\mathrm{Tr}\mathbf{M} = \mathrm{Tr}\mathbf{M}_0 + \mathrm{Tr}(\mathbf{N}_2\mathbf{G}_1\mathbf{N}_1), \tag{39}$$

where

$$\mathrm{Tr}(\mathbf{N}_2\mathbf{G}_1\mathbf{N}_1) = \mathrm{Tr}(\mathbf{G}_1\mathbf{N}_1\mathbf{N}_2) = \mathrm{Tr}(\mathbf{G}_1\mathbf{M}_1), \tag{40}$$

$$\mathbf{G}_1\mathbf{M}_1 = \mathbf{G}_1\cos\phi + \mathbf{G}_1\mathbf{J}_1\sin\phi, \quad \phi = 2\pi\nu_0, \tag{41}$$

and therefore

$$\cos 2\pi\nu = \cos\phi + \frac{1}{2}G_1\beta_1\sin\phi \tag{42}$$

and

$$W = 2(1 - \cos 2\pi\nu) = 2(1 - \cos\phi) - G_1\beta_1\sin\phi. \tag{43}$$

Now, carrying out the matrix multiplication in (38) we obtain the components of $\mathbf{D} - \mathbf{D}_0$. Thus

$$D - D_0 = \frac{G_1 D_1}{W}\{B_2(1 - \cos\phi) + (\beta_1 A_2 - \alpha_1 B_2)\sin\phi\}, \tag{44}$$

and

$$D' - D'_0 = \frac{G_1 D_1}{W}\{F_2(1 - \cos\phi) + (\beta_1 E_2 - \alpha_1 F_2)\sin\phi\} \tag{45}$$

where A_2, B_2, E_2, F_2 are given by (10–13) and $\phi = 2\pi\nu_0$. Collecting terms we find

$$\beta_1 A_2 - \alpha_1 B_2 = \sqrt{\beta_1\beta_2}\cos\psi, \tag{46}$$

$$\beta_1 E_2 - \alpha_1 F_2 = -\sqrt{\beta_1/\beta_2}(\alpha_2\cos\psi + \sin\psi) \tag{47}$$

where

$$\psi = \psi_2 - \psi_1, \quad \psi_2 = 2\pi\nu_0 = \phi, \quad \beta_2 = \beta_0, \quad \alpha_2 = \alpha_0 \tag{48}$$

and equations (44–45) become

$$D - D_0 = \frac{G_1 D_1}{W}\sqrt{\beta_0\beta_1}\{\sin\psi + \sin\psi_1\} \tag{49}$$

$$D' - D'_0 = \frac{G_1 D_1}{W}\sqrt{\beta_1/\beta_0}\{\cos\psi - \cos\psi_1 - \alpha_0(\sin\psi + \sin\psi_1)\}. \tag{50}$$

4 Four-Quadrupole Perturbation

Independent manipulation of the periodic dispersion and its derivative requires at least two perturbing quadrupoles and these will in general alter the machine tunes and other lattice parameters. In this section we show that by adding two more quadrupoles to a minimal scheme of two, we can manipulate the periodic dispersion vector at s_0 without affecting the horizontal lattice parameters there. The scheme is based on the pairing of quadrupoles employed in the gamma-transition-jump schemes of the AGS and other machines [3–6].

We consider four thin quadrupoles placed at points s_1, s_2, s_3, and s_4 such that

$$s_0 < s_1 < s_2 < s_3 < s_4 < s_5, \quad s_5 = s_0 + C \tag{51}$$

where C is the ring circumference. We then have

$$\mathbf{T}_0 = \mathbf{U}_5\mathbf{U}_4\mathbf{U}_3\mathbf{U}_2\mathbf{U}_1, \quad \mathbf{T}_2 = \mathbf{U}_2\mathbf{U}_1\mathbf{U}_5\mathbf{U}_4\mathbf{U}_3, \tag{52}$$

$$\mathbf{T}_4 = \mathbf{U}_4\mathbf{U}_3\mathbf{U}_2\mathbf{U}_1\mathbf{U}_5, \quad \mathbf{T} = \mathbf{U}_5\mathbf{R}_4\mathbf{U}_4\mathbf{R}_3\mathbf{U}_3\mathbf{R}_2\mathbf{U}_2\mathbf{R}_1\mathbf{U}_1 \tag{53}$$

$$\mathbf{M}_0 = \mathbf{N}_5\mathbf{N}_4\mathbf{N}_3\mathbf{N}_2\mathbf{N}_1, \quad \mathbf{M}_2 = \mathbf{N}_2\mathbf{N}_1\mathbf{N}_5\mathbf{N}_4\mathbf{N}_3, \tag{54}$$

$$\mathbf{M}_4 = \mathbf{N}_4\mathbf{N}_3\mathbf{N}_2\mathbf{N}_1\mathbf{N}_5, \quad \mathbf{M} = \mathbf{N}_5\mathbf{Q}_4\mathbf{N}_4\mathbf{Q}_3\mathbf{N}_3\mathbf{Q}_2\mathbf{N}_2\mathbf{Q}_1\mathbf{N}_1, \tag{55}$$

$$\mathbf{d}_0 = \mathbf{N}_5\mathbf{N}_4\mathbf{N}_3\mathbf{N}_2\mathbf{n}_1 + \mathbf{N}_5\mathbf{N}_4\mathbf{N}_3\mathbf{n}_2 + \mathbf{N}_5\mathbf{N}_4\mathbf{n}_3 + \mathbf{N}_5\mathbf{n}_4 + \mathbf{n}_5, \tag{56}$$

and

$$\begin{aligned} \mathbf{d} &= \mathbf{N}_5\mathbf{Q}_4\mathbf{N}_4\mathbf{Q}_3\mathbf{N}_3\mathbf{Q}_2\mathbf{N}_2\mathbf{Q}_1\mathbf{n}_1 + \mathbf{N}_5\mathbf{Q}_4\mathbf{N}_4\mathbf{Q}_3\mathbf{N}_3\mathbf{Q}_2\mathbf{n}_2 + \\ &+ \mathbf{N}_5\mathbf{Q}_4\mathbf{N}_4\mathbf{Q}_3\mathbf{n}_3 + \mathbf{N}_5\mathbf{Q}_4\mathbf{n}_4 + \mathbf{n}_5. \end{aligned} \tag{57}$$

Now suppose the quadrupoles are arranged in pairs such that

$$\sin(\psi_2 - \psi_1) = 0, \quad \sin(\psi_4 - \psi_3) = 0 \tag{58}$$

and

$$\beta_2 G_2 + \beta_1 G_1 = 0, \quad \beta_4 G_4 + \beta_3 G_3 = 0. \tag{59}$$

It then follows from (10–13) that

$$B_2 = B_4 = 0, \quad A_2 G_2 + F_2 G_1 = 0, \quad A_4 G_4 + F_4 G_3 = 0 \tag{60}$$

and therefore

$$\mathbf{Q}_4\mathbf{N}_4\mathbf{Q}_3 = \mathbf{N}_4, \quad \mathbf{Q}_2\mathbf{N}_2\mathbf{Q}_1 = \mathbf{N}_2. \tag{61}$$

The second of equations (55) then becomes

$$\mathbf{M} = \mathbf{N}_5\mathbf{N}_4\mathbf{N}_3\mathbf{N}_2\mathbf{N}_1 = \mathbf{M}_0 \tag{62}$$

and it follows that the horizontal tune and the horizontal lattice parameters at s_0 are left unchanged by the four quadrupoles. Using (61) in (57) we also have

$$\mathbf{d} = \mathbf{N}_5\mathbf{N}_4\mathbf{N}_3\mathbf{N}_2\mathbf{n}_1 + \mathbf{N}_5\mathbf{N}_4\mathbf{N}_3\mathbf{Q}_2\mathbf{n}_2 + \mathbf{N}_5\mathbf{N}_4\mathbf{n}_3 + \mathbf{N}_5\mathbf{Q}_4\mathbf{n}_4 + \mathbf{n}_5 \tag{63}$$

and subtracting (56) from (63) we have

$$\mathbf{d} - \mathbf{d}_0 = \mathbf{N}_5\mathbf{N}_4\mathbf{N}_3\mathbf{G}_2\mathbf{n}_2 + \mathbf{N}_5\mathbf{G}_4\mathbf{n}_4. \tag{64}$$

Then since $\mathbf{M} = \mathbf{M}_0$ it follows from (7) and (32) that

$$\mathbf{D} - \mathbf{D}_0 = (\mathbf{I} - \mathbf{M}_0)^{-1}(\mathbf{d} - \mathbf{d}_0) = \frac{1}{W_0}(\mathbf{I} - \overline{\mathbf{M}}_0)(\mathbf{d} - \mathbf{d}_0) \tag{65}$$

where

$$W_0 = 2 - \mathrm{Tr}\mathbf{M}_0 = 2(1 - \cos 2\pi\nu_0). \tag{66}$$

Thus

$$\mathbf{D} - \mathbf{D}_0 = \frac{1}{W_0}(\mathbf{I} - \overline{\mathbf{M}}_0)\{\mathbf{N}_5\mathbf{N}_4\mathbf{N}_3\mathbf{G}_2\mathbf{n}_2 + \mathbf{N}_5\mathbf{G}_4\mathbf{n}_4\} \tag{67}$$

and since

$$\overline{\mathbf{M}}_0\mathbf{N}_5\mathbf{N}_4\mathbf{N}_3 = \mathbf{N}_5\mathbf{N}_4\mathbf{N}_3\overline{\mathbf{M}}_2, \quad \overline{\mathbf{M}}_0\mathbf{N}_5 = \mathbf{N}_5\overline{\mathbf{M}}_4 \tag{68}$$

we have

$$\mathbf{D} - \mathbf{D}_0 = \frac{1}{W_0}\mathbf{N}_5\mathbf{N}_4\mathbf{N}_3(\mathbf{I} - \overline{\mathbf{M}}_2)\mathbf{G}_2\mathbf{n}_2 + \frac{1}{W_0}\mathbf{N}_5(\mathbf{I} - \overline{\mathbf{M}}_4)\mathbf{G}_4\mathbf{n}_4 \tag{69}$$

where

$$\mathbf{n}_2 = \mathbf{D}_2 - \mathbf{N}_2\mathbf{D}_1, \quad \mathbf{n}_4 = \mathbf{D}_4 - \mathbf{N}_4\mathbf{D}_3. \tag{70}$$

Carrying out the matrix multiplication in (69–70) we obtain the components of $\mathbf{D} - \mathbf{D}_0$. Collecting terms as for the case of a single quadrupole perturbation we find

$$D - D_0 = \frac{G_2 n_2}{W_0}\sqrt{\beta_0\beta_2}\{\sin\phi_2 + \sin\psi_2\} + \frac{G_4 n_4}{W_0}\sqrt{\beta_0\beta_4}\{\sin\phi_4 + \sin\psi_4\} \tag{71}$$

$$\begin{aligned} D' - D'_0 &= \frac{G_2 n_2}{W_0}\sqrt{\beta_2/\beta_0}\{\cos\phi_2 - \cos\psi_2 - \alpha_0(\sin\phi_2 + \sin\psi_2)\} \\ &+ \frac{G_4 n_4}{W_0}\sqrt{\beta_4/\beta_0}\{\cos\phi_4 - \cos\psi_4 - \alpha_0(\sin\phi_4 + \sin\psi_4)\} \end{aligned} \tag{72}$$

where

$$\phi_2 = 2\pi\nu_0 - \psi_2, \quad \phi_4 = 2\pi\nu_0 - \psi_4 \tag{73}$$

and

$$n_2 = D_2 - A_2 D_1 - B_2 D'_1, \quad n_4 = D_4 - A_4 D_3 - B_4 D'_3. \tag{74}$$

5 Application to Booster

Let us now apply the formulae of the previous sections to the manipulation of dispersion in the F6 straight section. We take s_0 to be the center of the F6 quadrupole and use the formulae to calculate $\mathbf{D} - \mathbf{D}_0$ there. For the case of a single quadrupole perturbation we take s_1 to be the center of the F2 quadrupole and imagine that a thin quadrupole with strength G_1 is placed there. We shall assume that alpha is zero at the centers of the F2 and F6 quadrupoles and that the horizontal beta function at both points has the value B. We assume further that the derivative of the periodic dispersion is zero at both points. Thus we have

$$\alpha_2 = \alpha_1 = \alpha_0 = 0, \quad \beta_2 = \beta_1 = \beta_0 = B \tag{75}$$

and

$$D_2' = D_1' = D_0' = 0. \tag{76}$$

Equations (49–50) then become

$$D - D_0 = \frac{BG_1D_1}{W}(\sin\psi + \sin\psi_1), \tag{77}$$

$$D' - D_0' = \frac{G_1D_1}{W}(\cos\psi - \cos\psi_1), \tag{78}$$

where

$$W = 2(1-\cos\phi) - BG_1\sin\phi, \quad \psi = \psi_2 - \psi_1, \quad \psi_2 = 2\pi\nu_0 = \phi. \tag{79}$$

These equations give the change in the periodic dispersion at F6 due to the thin quad at F2. Using (79) in (77), and solving for BG_1 we obtain

$$BG_1 = \frac{2(D-D_0)(1-\cos\phi)}{D_1(\sin\psi + \sin\psi_1) + (D-D_0)\sin\phi} \tag{80}$$

which gives the quadrupole strength required to change the dispersion at F6 by $D - D_0$. If we want the perturbed dispersion D to be zero, (80) becomes

$$BG_1 = \frac{2D_0(\cos\phi - 1)}{D_1(\sin\psi + \sin\psi_1) - D_0\sin\phi}. \tag{81}$$

Now, if we take the unperturbed horizontal tune to be $\nu_0 = 4.8 = 24/5$, then the betatron phase advance per superperiod is $2\pi\nu_0/6 = 8\pi/5$ and the betatron phase advance between adjacent quadrupoles is approximately $\pi/5$. The betatron phase advance between the centers of the F2 and F6 quadrupoles is then approximately $4\pi/5$. Thus we have

$$\phi = 48\pi/5, \quad \psi = 4\pi/5, \quad \psi_1 = 44\pi/5 \tag{82}$$

and (81) becomes

$$BG_1 = \frac{(-1.3820)D_0}{(1.1756)D_1 + (.9511)D_0}. \tag{83}$$

Then taking $D_0 = 2.9$ meters and $D_1 = 1.4$ meters we have $G_1B = -0.91$, and taking $B = 13.6$ meters we find that the required focal length for the thin quad at F2 is $G_1^{-1} = -14.9$ meters. This is to be compared with the focal length of approximately -3.6 meters for the horizontally focusing lattice quadrupoles. Thus, in order to make the periodic dispersion at F6 equal to zero, the strength of the F2 quadrupole must be increased by about 24 per cent. This is in agreement with the result obtained with the MAD code.

Let us now consider the four-quadrupole perturbation. We shall assume, as with the single quadrupole case, that

$$\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0, \tag{84}$$

$$\beta_0 = \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = B, \tag{85}$$

$$D_0' = D_1' = D_2' = D_3' = D_4' = D_5' = 0. \tag{86}$$

Equations (71–74) then become

$$D - D_0 = \frac{BG_2n_2}{W_0}\{\sin\phi_2 + \sin\psi_2\} + \frac{BG_4n_4}{W_0}\{\sin\phi_4 + \sin\psi_4\} \tag{87}$$

$$D' - D_0' = \frac{G_2n_2}{W_0}\{\cos\phi_2 - \cos\psi_2\} + \frac{G_4n_4}{W_0}\{\cos\phi_4 - \cos\psi_4\} \tag{88}$$

where

$$\phi_2 = 2\pi\nu_0 - \psi_2, \quad \phi_4 = 2\pi\nu_0 - \psi_4 \tag{89}$$

and

$$n_2 = D_2 - D_1\cos(\psi_2 - \psi_1), \quad n_4 = D_4 - D_3\cos(\psi_4 - \psi_3). \tag{90}$$

Taking $D = D' = 0$ and solving (87–88) for G_2n_2 and G_4n_4 we find

$$G_2n_2 = \frac{D_0(\cos\psi_4 - \cos\phi_4)}{B\sin(\psi_4 - \psi_2)}, \quad G_4n_4 = \frac{-D_0(\cos\psi_2 - \cos\phi_2)}{B\sin(\psi_4 - \psi_2)} \tag{91}$$

which give the quadrupole strengths required to make the periodic dispersion and its derivative equal to zero at the center of the F6 quadrupole. Now, as before, we take the unperturbed horizontal tune to be

$\nu_0 = 4.8 = 24/5$, so that the betatron phase advance between adjacent quadrupoles is approximately $\pi/5$. Then in order to satisfy equations (58) and (85) we must take $\psi_4 - \psi_3$ and $\psi_2 - \psi_1$ to be even multiples of π. (If they are odd multiples, then one quadrupole in each pair will be at a horizontal beta maximum and the other will be at a horizontal beta minimum.) As one possible arrangement we take s_1, s_2, s_3, s_4 to be the centers of quadrupoles C6, D8, E2, F4 respectively. We then have

$$\psi_1 = 24\pi/5, \quad \psi_2 = 34\pi/5, \quad \psi_3 = 36\pi/5, \quad \psi_4 = 46\pi/5, \tag{92}$$

$$\phi_2 = 14\pi/5, \quad \phi_4 = 2\pi/5, \quad \psi_4 - \psi_2 = 12\pi/5 \tag{93}$$

and equations (91) become

$$G_2(D_2 - D_1) = (-1.176)D_0/B, \quad G_4(D_4 - D_3) = 0. \tag{94}$$

(Note that $G_4(D_4 - D_3)$ happens to be zero in this case because $\cos\phi_2 = \cos\psi_2$.) Taking $D_0 = D_1 = D_4 = 2.9$ meters, $D_2 = D_3 = 1.4$ meters, and $B = 13.6$ meters we then have $G_2^{-1} = 6.0$ meters, and according to (59) we must also have $G_1^{-1} = -6.0$ meters. Thus, increasing the strength of the C6 quadrupole by 60 per cent and decreasing that of the D8 quadrupole by the same amount will make the periodic dispersion and its derivative equal to zero in the F6 quadrupole.

6 References

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