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SYNCHROTRON SIDEBAND SPIN RESONANCES IN HIGH ENERGY ELECTRON STORAGE RINGS

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Abstract

The spins of electron and positron beams circulating in high energy storage rings become spontaneously polarized by the emission of synchrotron radiation. The equilibrium degree of polarization is strongly affected by so-called "spin resonances". This paper studies a subset of such resonances called "synchrotron sideband spin resonances". It is shown that the results from several previously published formalisms are equivalent, when the same approximations are made in all of them. In addition, some new calculations are presented, in particular the effects of orbital chromaticity. Some of the results are compared with experimental data, and the agreement is reasonable.

1 Introduction

When electrons and positrons circulate in a high energy storage ring, their spins become spontaneously polarized by the emission of synchrotron radiation. This is called the Sokolov-Ternov effect [1]. They treated a model of particles moving in horizontal circles in a homogenous vertical magnetic field, and found that the equilibrium degree of polarization was $8/(5\sqrt{3}) \simeq 92.4\%$. However, in real storage rings the behavior of the polarization is more complicated. An example of measurements [2] of the equilibrium degree of polarization of the positron beam, taken at the storage ring SPEAR, is shown in fig. 1, where the quantity P_{max} is $8/(5\sqrt{3})$. We see that there are several places where the polarization is much less than $8/(5\sqrt{3})$, at so-called "spin resonances." In this paper, I shall study a subset of the spin resonances, called "synchrotron sideband resonances," to be defined more precisely below. They are generally regarded as the most important family of spin resonances in a high-energy storage ring. The problem has previously been studied by several authors [3,4,5], and more recent results have been given by Buon [6]. In addition, in 1987 I published a formalism [7] for calculating arbitrary higher order spin resonances, subject to the approximation of treating linear orbital dynamics, and coded it into a computer program called SMILE. I shall refer to it below as the "SMILE formalism." Although all of the above formalisms use perturbation theory, the terms are summed in different ways, and the results look different.

However, I shall show below that when one makes the same approximations in all the formalisms, they all yield equivalent results. Any differences obtained in practical applications, by the use of different formalisms, are therefore due to the use of different approximations, or because the formalisms are applied to different storage ring models. In particular, I shall rederive Yokoya's formula [4]. I find that there are some extra terms, which he neglected, but which sometimes turn out to be of comparable magnitude to the terms he retained. I shall show when the neglect of these terms is or is not justified. The above terms are already present in the SMILE formalism [7]. Some of the main points of the work below, such as the above terms, were given in ref. [8], but here I shall give more complete details. I shall also derive some new results to include the effect of orbital chromaticity, and I shall fit the results to some of the data [2] from the polarization measurements at SPEAR. The agreement is reasonable.

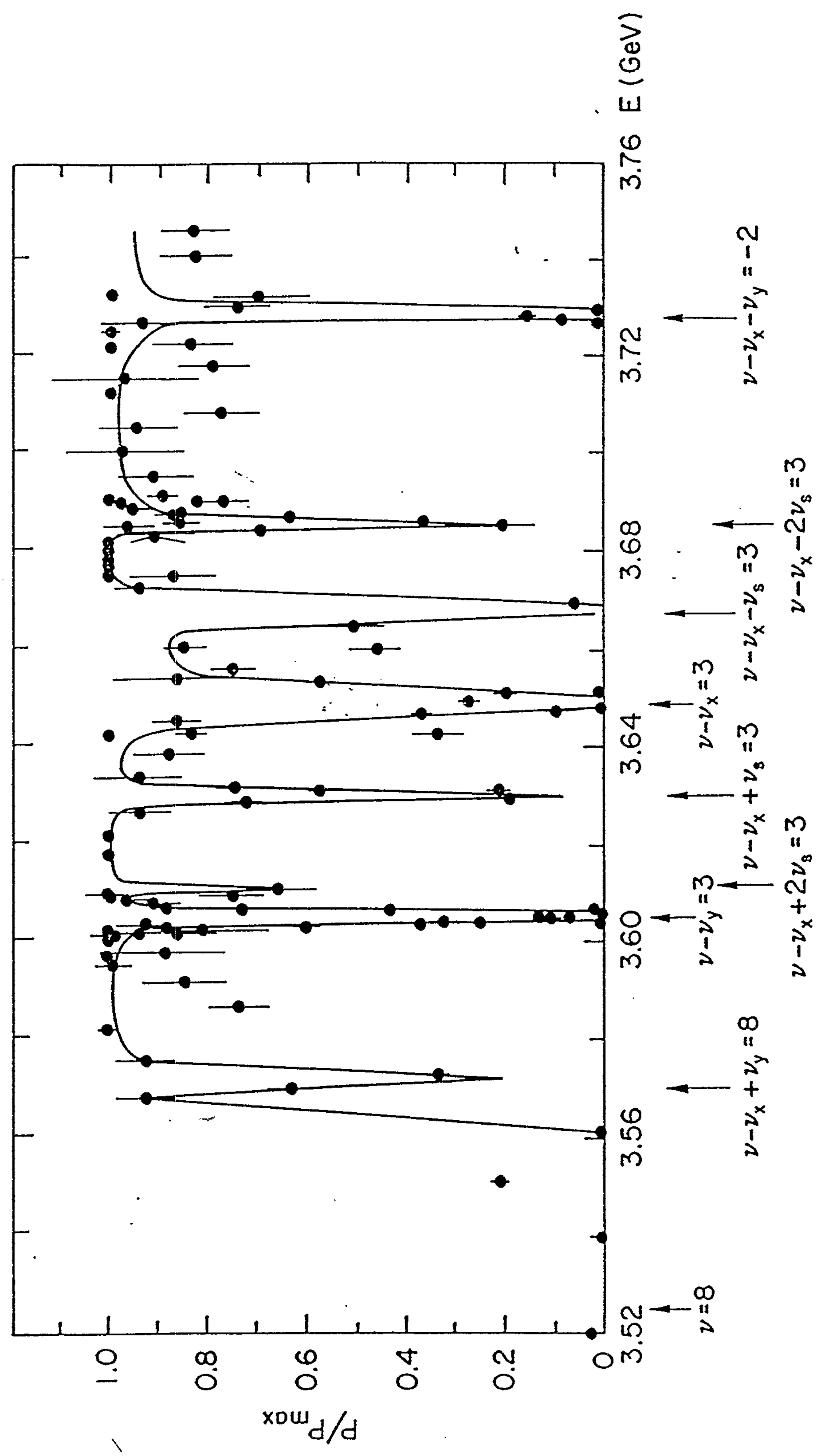

Figure 1. Polarization measurements at SPEAR (from ref. [2]). The quantity P_{max} is $8/(5\sqrt{3}) \simeq 92.4\%$. The curve is a guide for the eye, not a theoretical calculation. Various resonances have been identified in the data. The orbital tunes are called $\nu_{x,y,s}$ instead of $Q_{x,y,s}$. The spin tune is ν. A single beam of positrons was circulated when making measurements. The graph is not a single experiment, but a compilation of many runs.

2 General remarks

Formulas for the equilibrium degree of polarization P_{eq}, and the polarization buildup time τ_{pol}, which include the spin resonances, were derived by Derbenev and Kondratenko [9]

$$
\begin{aligned}
P_{eq} &= \frac{8}{5\sqrt{3}} \frac{\left\langle |\rho|^{-3}\, \hat{b}.\left[\hat{n} - \gamma\dfrac{\partial \hat{n}}{\partial \gamma}\right]\right\rangle}{\left\langle |\rho|^{-3}\left[1 - \dfrac{2}{9}(\hat{n}.\hat{v})^2 + \dfrac{11}{18}\left|\gamma\dfrac{\partial \hat{n}}{\partial \gamma}\right|^2\right]\right\rangle} \\
\tau_{pol}^{-1} &= \left\langle \frac{5\sqrt{3}}{8}\frac{e^2\hbar\gamma^5}{m^2c^2}\frac{1}{|\rho|^3}\left[1 - \frac{2}{9}(\hat{n}.\hat{v})^2 + \frac{11}{18}\left|\gamma\frac{\partial \hat{n}}{\partial \gamma}\right|^2\right]\right\rangle . \qquad (1)
\end{aligned}
$$

Here m and e are the particle mass and charge, respectively, c is the speed of light, $\hbar$ is Planck's constant divided by 2π, γ is the particle energy in units of mc^2, $\vec{v}$ is the particle velocity, $\hat{b} \equiv \vec{v}\times\dot{\vec{v}}/|\vec{v}\times\dot{\vec{v}}|$, ρ is the local radius of curvature of the particle trajectory, $\hat{n}$ is the spin quantization axis on the particle trajectory, and the angular brackets $\langle \dots \rangle$ denote an equilibrium average over the distribution of particle orbits (the action-angle variables of the particles) and the ring azimuth. The quantity of principal, but not exclusive, interest below will be $|\gamma(\partial\hat{n}/\partial\gamma)|^2$. Some authors [6] calculate resonance widths using a so-called "depolarization" formalism, and introduce a depolarization time τ_d, which is related to $|\gamma(\partial\hat{n}/\partial\gamma)|^2$ via

$$
\frac{\tau_0}{\tau_d} = \frac{11}{18}\frac{\langle |\rho|^{-3}\,|\gamma(\partial\hat{n}/\partial\gamma)|^2\rangle}{\langle |\rho|^{-3}\rangle}, \qquad (2)
$$

where

$$
\tau_0^{-1} = \left\langle \frac{5\sqrt{3}}{8}\frac{e^2\hbar\gamma^5}{m^2c^2}\frac{1}{|\rho|^3}\right\rangle, \qquad (3)
$$

is the (inverse) buildup time in the case of a perfectly planar ring. It is known that both formalisms yield the same results for $|\gamma(\partial\hat{n}/\partial\gamma)|^2$ for first order spin resonances. The concept of first order and higher order will be defined more precisely below. I shall show below that both formalisms also yield the same results for synchrotron sideband spin resonances.

3 Tune modulation and chromaticity

3.1 Basic results

In this section, I shall use a tune modulation argument to derive some basic results. I shall also show some new results pertaining to the effects of orbital chromaticity. The notation and formalism below will mainly follow ref. [4]. The ring azimuth is denoted by θ. It will be sufficient to treat only horizontal betatron oscillations and synchrotron oscillations here. Vertical betatron oscillations can be included using a similar derivation. By definition, the vector $\hat{n}$ in eq. (1) satisfies the Thomas-BMT equation [10]

$$\frac{d\hat{n}}{d\theta} = (\vec{\Omega}_0 + \vec{\omega}\,) \times \hat{n}\,. \tag{4}$$

Here $\vec{\Omega}_0$ is the spin precession vector on the closed orbit and $\vec{\omega}$ is the contribution of the orbital oscillations. By definition, $\vec{\omega} = 0$ on the closed orbit. Following Yokoya [4], I express $\hat{n}$ in the form

$$\hat{n} = \hat{n}_0(1 - |\zeta|^2)^{\frac{1}{2}} + \mathrm{Re}(\vec{k}_0^*\zeta)\,, \tag{5}$$

where $\hat{n}_0$ and $\vec{k}_0$ are both solutions of eq. (4) on the closed orbit. The vector $\hat{n}_0$ is the value of $\hat{n}$ on the closed orbit, and $\vec{k}_0$ is orthogonal to $\hat{n}_0$. They have the one-turn periodicities:

$$\begin{aligned} \hat{n}_0(\theta + 2\pi) &= \hat{n}_0(\theta) \\ \vec{k}_0(\theta + 2\pi) &= e^{i2\pi\nu}\vec{k}_0(\theta)\,, \end{aligned} \tag{6}$$

where ν is the spin tune. The equation of motion for ζ can easily be shown to be

$$\frac{d\zeta}{d\theta} = -i\vec{\omega}.\vec{k}_0(1 - |\zeta|^2)^{\frac{1}{2}} + i\vec{\omega}.\hat{n}_0\zeta\,, \tag{7}$$

with $\zeta = 0$ when $\vec{\omega} = 0$.

The above equation is exact. I shall now neglect ζ on the r.h.s., which yields

$$\frac{d\zeta}{d\theta} \simeq -i\vec{\omega}.\vec{k}_0\,. \tag{8}$$

We can decompose $\vec{\omega}$ into components proportional to the various normal modes, i.e.

$$\vec{\omega} = \vec{\omega}_x^{(+)} + \vec{\omega}_x^{(-)} + \vec{\omega}_\epsilon^{(+)} + \vec{\omega}_\epsilon^{(-)} + \ldots \tag{9}$$

The individual components have one-turn phase advances given by their respective orbital tunes, i.e.

$$\vec{\omega}_x^{(\pm)}(\theta + 2\pi) = e^{\pm i2\pi Q_x}\vec{\omega}_x^{(\pm)}(\theta)\,, \tag{10}$$

where Q_x is the horizontal betatron tune, and similarly for other modes. Then we can Fourier decompose

$$\vec{\omega}_x^{(\pm)}.\vec{k}_0 = \sum_{n=-\infty}^{\infty} b_n^{(\pm)} e^{i(n\theta + \nu\theta \pm \psi_x)}\,, \tag{11}$$

where I_x and ψ_x are action-angle variables, and I_x has been absorbed into $b_n^{(\pm)}$. Detailed expressions for the $b_n^{(\pm)}$ are not important here. They can be found, for example, in ref. [4]. Let us now solve for

ζ. In this section, I shall treat only betatron oscillations in $\vec{\omega}.\vec{k}_0$. The effects of treating synchrotron oscillations in $\vec{\omega}.\vec{k}_0$ have been published using a tune modulation formalism in refs. [3] and [5]. Then

$$\frac{d\zeta}{d\theta} = -i \sum_{n=-\infty}^{\infty} b_n^{(+)} e^{i(n\theta+\nu\theta+\psi_x)} + b_n^{(-)} e^{i(n\theta+\nu\theta-\psi_x)} . \tag{12}$$

The solution which satisfies $\zeta \to 0$ when $|\vec{\omega}\,| \to 0$ is

$$\zeta = -\sum_{n=-\infty}^{\infty} b_n^{(+)} \frac{e^{i(n\theta+\nu\theta+\psi_x)}}{n+\nu+Q_x} + b_n^{(-)} \frac{e^{i(n\theta+\nu\theta-\psi_x)}}{n+\nu-Q_x} . \tag{13}$$

As is well known to workers in the field, $\zeta \to \infty$ whenever $\nu = -n \pm Q_x$ (first order horizontal betatron spin resonances). I would also have obtained first order synchrotron resonances, i.e. $\nu = -n \pm Q_s$, had I retained synchrotron oscillations in $\vec{\omega}.\vec{k}_0$. For simplicity, let us pick only a single term in the above expression for $\vec{\omega}.\vec{k}_0$, say

$$\vec{\omega}.\vec{k}_0 \to b_n^{(-)} e^{i(n\theta+\nu\theta-\psi_x)} , \tag{14}$$

so

$$\zeta = -b_n^{(-)} \frac{e^{i(n\theta+\nu\theta-\psi_x)}}{n+\nu-Q_x} . \tag{15}$$

It thus has only a single resonance. It will avoid needless complications below if we deal with only a single resonance to begin with.

The presence of synchrotron oscillations modifies the above solution, including that for a single resonance, because the energy oscillations change the betatron and spin tunes. Following Yokoya [4], I shall write

$$\epsilon = \frac{\delta\gamma}{\gamma_0} = \sqrt{2I_z}\,\cos\psi_z , \tag{16}$$

where I_z and ψ_z are action-angle variables and γ_0 is the average value of γ. If the horizontal chromaticity is ξ_x, then

$$\begin{aligned} Q_x &\to Q_{x0} + \xi_x \frac{\delta\gamma}{\gamma_0} \\ &= Q_{x0} + \xi_x \sqrt{2I_z}\,\cos\psi_z . \end{aligned} \tag{17}$$

The subscript "0" denotes the value of a function in the absence of synchrotron oscillations, in the above equation and also in the ones below. The betatron phase is modified to

$$\begin{aligned} \psi_x &\to \int^{\theta} Q_x\, d\theta' \\ &= \psi_{x0} + \int \xi_x \sqrt{2I_z}\,\cos\psi_z\, d\theta' \end{aligned}$$

$$= \psi_{x0} + \frac{\xi_x\sqrt{2I_z}}{Q_s}\sin\psi_z + \text{const.} \tag{18}$$

where I shall neglect integration constants independent of θ. The above derivation is standard, and is valid in the limit $Q_s \ll 1$ ("quasistatic approximation"), which is therefore an approximation that must be imposed on Q_s in this section. Further, for an approximately planar ring, it is known that $\nu \simeq a\gamma$ (with small corrections due to machine imperfections), where $a = (g-2)/2$, so in the presence of synchrotron oscillations,

$$\begin{aligned}
\nu &\to a\gamma_0(1+\epsilon) \\
&= \nu_0\left(1+\sqrt{2I_z}\,\cos\psi_z\right) \\
\nu\theta &\to \nu_0\theta + \frac{\nu_0\sqrt{2I_z}}{Q_s}\sin\psi_z + \text{const.}
\end{aligned} \tag{19}$$

Thus we find that

$$\begin{aligned}
\vec{\omega}.\vec{k}_0 &\to b_n^{(-)}e^{i(n\theta+\nu_0\theta-\psi_{x0})}e^{i\sqrt{2I_z}\,(\nu_0-\xi_x)Q_s^{-1}\sin\psi_z} \\
&= b_n^{(-)}e^{i(n\theta+\nu_0\theta-\psi_{x0})}\sum_{m=-\infty}^{\infty} e^{im\psi_z}J_m\left(\frac{\sqrt{2I_z}\,(\nu_0-\xi_x)}{Q_s}\right),
\end{aligned} \tag{20}$$

where I have neglected irrelevant phase factors that do not affect the final result. I have also used the Bessel function identity

$$e^{ir\sin\psi} = \sum_{m=-\infty}^{\infty} e^{im\psi}J_m(r)\,. \tag{21}$$

The solution for ζ is now

$$\begin{aligned}
\frac{d\zeta}{d\theta} &= -i\vec{\omega}.\vec{k}_0 \\
&\to -ib_n^{(-)}\sum_m e^{i(n\theta+\nu_0\theta-\psi_{x0}+m\psi_z)}J_m\left(\frac{\sqrt{2I_z}\,(\nu_0-\xi_x)}{Q_s}\right) \\
\zeta &= -b_n^{(-)}\sum_m \frac{e^{i(n\theta+\nu_0\theta-\psi_{x0}+m\psi_z)}}{n+\nu_0-Q_{x0}+mQ_s}J_m\left(\frac{\sqrt{2I_z}\,(\nu_0-\xi_x)}{Q_s}\right).
\end{aligned} \tag{22}$$

Notice that now $\zeta \to \infty$ (a resonance) whenever $\nu_0 = -n + Q_{x0} - mQ_s$, $m = 0, \pm1, \pm2, \ldots$, i.e. instead of a single resonance, we have an infinite family of resonances, equally spaced at intervals of Q_s. These are the synchrotron sideband resonances, with the one at $m = 0$ being the "parent" resonance.

The above solution can be easily generalized to include the full set of parent betatron resonances, by summing over n:

$$\zeta \to -\sum_n b_n^{(-)}\sum_m \frac{e^{i(n\theta+\nu_0\theta-\psi_{x0}+m\psi_z)}}{n+\nu_0-Q_{x0}+mQ_s}J_m\left(\frac{\sqrt{2I_z}\,(\nu_0-\xi_x)}{Q_s}\right). \tag{23}$$

The terms with $b_n^{(+)}$ coefficients can also be easily included, but then one must explicitly include global phase factors which I have neglected up to now, because they are not the same for the $b_n^{(-)}$ and $b_n^{(+)}$ terms. The additional resonances yield no new physics, hence I shall not write them out explicitly. It is also straightforward to include vertical betatron parent resonances — one introduces appropriately defined analogs of $b_n^{(\pm)}$, and obtains a very similar solution to the above.

We see from the above solutions for ζ, eqs. (22) and (23), that the effect of chromaticity is similar to that of the spin tune modulation. In particular, if $\nu_0 = \xi_x$, then all the sideband resonances vanish, and only the $m = 0$ parent resonance is left, because then $\nu\theta - \psi_x = \nu_0\theta - \psi_{x0}$ for arbitrary I_z and ψ_z. The two sources of tune modulation cancel each other [11]. The corresponding condition for resonances of the form $\nu = -n - Q_{x0} - mQ_s$ is $\nu_0 = -\xi_x$. Thus we cannot eliminate both types of synchrotron sidebands at once. The above derivation applies equally well for parent vertical betatron resonances, with $\xi_x \to \xi_y$ obviously. In practice, $\xi_{x,y}$ tends to have a value of a few units, but for very high energy storage rings, e.g. TRISTAN, HERA and LEP, the value of ν_0 is much larger than unity, so that the chromaticity is not likely to affect the synchrotron sideband resonance widths. It may be important in lower energy rings, where $\nu_0 \leq 10$, for example.

I have thus described the subset of spin resonances I wish to treat in this paper, and derived some of their properties using a tune modulation approach. This method has the advantage of being relatively simple, at the expense of several approximations, e.g. $Q_s \ll 1$ and $\nu \simeq \gamma a$, but these are reasonable approximations for most storage rings. In the subsequent sections, I shall use more detailed and comprehensive formalisms to study synchrotron sideband resonances, including sidebands of first order synchrotron resonances. The mathematics will get more complicated, but I shall show that, when I treat parent betatron resonances, and approximate $Q_s \ll 1$ etc., I shall still get the above solution for ζ. In addition, I have not yet calculated $|\gamma(\partial\hat{n}/\partial\gamma)|^2$, or any other function that appears directly in the polarization formula. There is one important caveat however. All of the formalisms I shall study below treat only linear orbital motion, hence they do not include chromaticity. Thus, unless otherwise stated, I shall ignore chromaticity in the rest of this paper.

3.2 Non-planar rings

Before turning to other formalisms, however, let us briefly consider non-planar rings, where the spin tune is not proportional to the particle energy. As an example, let us consider a ring with one or more Siberian Snakes [12], such that $\nu = 1/2$, independent of energy. In that case, we find that the chromaticity still modulates the betatron tune, but the spin tune is not modulated. The above results are still valid, but now the argument of the Bessel function is $-\sqrt{2I_z}\,\xi_x/Q_s$. The condition for a resonance is now

$$Q_x + mQ_s \;=\; n + \frac{1}{2}\,, \qquad m, n = 0, \pm 1, \pm 2, \ldots \tag{24}$$

i.e. it depends on the betatron and synchrotron tunes, and not on energy. The term ν_0 in the argument of the above Bessel functions is really the coefficient of the spin tune modulation. In general, therefore, it is more appropriate to write

$$\nu_0 \to \frac{\partial \nu}{\partial \epsilon} = \gamma_0 \frac{\partial \nu}{\partial \gamma} \equiv \xi_{spin}\,, \tag{25}$$

say. A similar result has been obtained by Yokoya [4]. One should then write

$$J_m\left(\frac{\sqrt{2I_z}\,(\nu_0 - \xi_x)}{Q_s}\right) \to J_m\left(\frac{\sqrt{2I_z}\,(\xi_{spin} - \xi_x)}{Q_s}\right) \tag{26}$$

in eq. (22) or eq. (23). In the above example, Siberian Snakes would destroy the radiative polarization for other reasons anyway, so the tune modulation is not important. The point of this section is to emphasize the distinction between the various ν_0 factors appearing above.

4 Yokoya's formalism

4.1 Solution for $\hat{n}$

The first formalism I shall treat is due to Yokoya [4], and the notation and formalism below will mainly follow his work. It is again adequate to treat only horizontal betatron and synchrotron oscillations. I shall drop the subscript "0" on the tunes ν_0 and Q_{x0}. The horizontal betatron coordinate and energy offset of a particle are

$$\begin{aligned} x_\beta &= \sqrt{2I_x\beta_x}\,\cos(\psi_x + \tilde{\Psi}_x(\theta)) \\ \epsilon &= \sqrt{2I_z}\,\cos\psi_z\,, \end{aligned} \tag{27}$$

where $\{I_j, \psi_j, j = x, z\}$ are action-angle variables, and $\tilde{\Psi}_x(\theta)$ is the periodic part of the betatron phase advance,

$$\tilde{\Psi}_x(\theta) = \int_0^\theta \frac{R\,d\theta'}{\beta_x(\theta')} - Q_x\theta\,, \tag{28}$$

where R is the average ring radius. The horizontal betatron and synchrotron tunes are called Q_x and Q_s, respectively. As stated above, the equation of motion for the Derbenev-Kondratenko $\hat{n}$ axis is

$$\frac{d\hat{n}}{d\theta} = (\vec{\Omega}_0 + \vec{\omega}\,) \times \hat{n}\,, \tag{29}$$

where $\vec{\Omega}_0$ is the spin precession vector on the closed orbit and $\vec{\omega}$ is the contribution of the orbital oscillations ($\vec{\omega} = 0$ on the closed orbit). I shall write $\vec{\omega} = x_\beta \vec{\omega}_x + \epsilon \vec{\omega}_\epsilon$ to denote the couplings to the horizontal betatron and synchrotron oscillations, respectively, and express $\hat{n}$ in the form

$$\hat{n}(I_x, \psi_x, I_z, \psi_z, \theta) = \hat{n}_0(1 - |\zeta|^2)^{\frac{1}{2}} \,+\, \mathrm{Re}(\vec{k}_0^*\zeta)\,. \tag{30}$$

As previously stated, the equation of motion for ζ is

$$\frac{d\zeta}{d\theta} = -i\vec{\omega}.\vec{k}_0(1 - |\zeta|^2)^{\frac{1}{2}} + i\vec{\omega}.\hat{n}_0\zeta\,. \tag{31}$$

Following Yokoya, I approximate $\sqrt{1 - |\zeta|^2} \simeq 1$ and then solve for ζ, which yields

$$\zeta = -ie^{-i\chi(\theta)} \int_{-\infty}^{\theta} e^{i\chi(\theta')}\vec{\omega}.\vec{k}_0(\theta')\, d\theta'\,, \tag{32}$$

where

$$\chi = -\int_{-\infty}^{\theta} \vec{\omega}.\hat{n}_0\, d\theta'\,. \tag{33}$$

The contributions of rapidly oscillating terms in χ are neglected, and only synchrotron oscillations are retained, i.e. $\omega.\hat{n}_0 \simeq \epsilon\vec{\omega}_\epsilon.\hat{n}_0$. In this context, I approximate $Q_s \ll 1$. This is required for making contact with other formalisms, including the tune modulation derivation above. I also define $u_\epsilon = \gamma a/Q_s$. Then, from ref. [4],

$$\chi \simeq \sqrt{2I_z}\, u_\epsilon \sin\psi_z\,. \tag{34}$$

Using the Bessel function identity eq. (21), I obtain Yokoya's result

$$\zeta = e^{-i\chi} \sum_{m=-\infty}^{\infty} \Big[\frac{m}{u_\epsilon} A_m(\theta) + \sum_{\pm} \sqrt{2I_x\beta_x}\, e^{\pm i(\psi_x + \widetilde{\Psi}_x)} B_{m,\pm x}(\theta)\Big] e^{im\psi_z} J_m(\sqrt{2I_z}u_\epsilon)\,, \tag{35}$$

where

$$\begin{aligned}
A_m(\theta) &= \frac{-ie^{-imQ_s\theta}}{e^{i2\pi(\nu+mQ_s)} - 1} \int_{\theta}^{\theta+2\pi} e^{imQ_s\theta'}\vec{\omega}_\epsilon.\vec{k}_0\, d\theta' \\
B_{m,\pm x}(\theta) &= \frac{-(i/2)e^{\mp i(\widetilde{\Psi}_x + Q_x\theta) - imQ_s\theta}}{(e^{i2\pi(\nu\pm Q_x+mQ_s)} - 1)\sqrt{\beta_x}} \int_{\theta}^{\theta+2\pi} e^{\pm i(\widetilde{\Psi}_x + Q_x\theta') + imQ_s\theta'} \sqrt{\beta_x}\vec{\omega}_x.\vec{k}_0\, d\theta'\,.
\end{aligned} \tag{36}$$

4.2 Single betatron parent resonance

The term in A_m describes synchrotron sidebands of the first order synchrotron resonances $\nu + n \pm Q_s = 0$, whereas the terms in $B_{m,\pm x}$ describe synchrotron sidebands of the betatron resonances $\nu + n \pm Q_x = 0$, respectively. I shall focus on sidebands of a single betatron resonance, say $\nu + n - Q_x = 0$, for a given integer n, so I neglect A_m and $B_{m,x}$. I shall study the A_m terms later in this paper. As in the tune modulation approach, I can decompose the solution for ζ into a sum of Fourier harmonics, and I shall keep only the harmonic in $B_{m,-x}$ which diverges at the above resonance, i.e.

$$
\begin{aligned}
B_{m,-x} \equiv B_m &\equiv -\frac{1}{2}\sum_{n'} \frac{b_{n',-x}}{n' + \nu - Q_x + mQ_s} \frac{e^{i\widetilde{\Psi}_x + i(\nu+n')\theta}}{\sqrt{\beta_x}} \\
&\simeq -\frac{1}{2}\frac{b_{n,\pm x}}{n + \nu - Q_x + mQ_s} \frac{e^{i\widetilde{\Psi}_x + i(\nu+n)\theta}}{\sqrt{\beta_x}} .
\end{aligned}
\tag{37}
$$

The solution for ζ, for a single parent betatron resonance, is then

$$
\zeta = -e^{-i\chi} \sum_m \sqrt{I_x/2}\, b_{n,-x} \frac{e^{i(n\theta + \nu\theta - \psi_x + m\psi_z)}}{n + \nu - Q_x + mQ_s} J_m\left(\frac{\sqrt{2I_z}\,\gamma_0 a}{Q_s}\right) . \tag{38}
$$

This is the same as the solution from the tune modulation approach, eq. (22), modulo a global phase factor, and with $\xi_x = 0$. We see that tune modulation gives us the results that are otherwise obtained by retaining the $\epsilon\vec{\omega}_\epsilon.\hat{n}_0$ term in eq. (7), at least in the approximation $Q_s \ll 1$ and $\nu \simeq \gamma a$. We also see that the relationship between $b_n^{(\pm)}$ of the tune modulation derivation and $b_{n,\pm x}$ is

$$
b_n^{(\pm)} = \sqrt{I_x/2}\, b_{n,\pm x} . \tag{39}
$$

4.3 Solution for $\gamma(\partial\hat{n}/\partial\gamma)$

Next, we want $\partial\zeta/\partial\epsilon$, to get $\gamma(\partial\hat{n}/\partial\gamma)$ in eq. (1). Note that [4]

$$
\begin{aligned}
\frac{\partial}{\partial\epsilon}\left[\sqrt{2I_x\beta_x}\, e^{\pm i(\psi_x + \widetilde{\Psi}_x)}\right] &= -\eta_x \pm i(\eta_x'\beta_x + \eta_x\alpha_x) \\
\frac{\partial}{\partial\epsilon}\left[J_m(\sqrt{2I_z}\,u_\epsilon)e^{im\psi_z}\right] &= \frac{u_\epsilon}{2}\left[J_{m-1}(\sqrt{2I_z}\,u_\epsilon)e^{i(m-1)\psi_z} - J_{m+1}(\sqrt{2I_z}\,u_\epsilon)e^{i(m+1)\psi_z}\right],
\end{aligned}
\tag{40}
$$

where a prime denotes a derivative with respect to arc-length, i.e. $R\theta$. Hence

$$
\frac{\partial\zeta}{\partial\epsilon} = e^{-i\chi} \sum_{m=-\infty}^{\infty} B_m(\theta)\left[\frac{\partial}{\partial\epsilon}\left(\sqrt{2I_x\beta_x}\, e^{-i(\psi_x + \widetilde{\Psi}_x)}\right) J_m(\sqrt{2I_z}\,u_\epsilon)e^{im\psi_z}\right.
$$

$$+\sqrt{2I_x\beta_x}\, e^{-i(\psi_x+\widetilde{\Psi}_x)} \frac{\partial}{\partial\epsilon}\Big(J_m(\sqrt{2I_z}\,u_\epsilon)e^{im\psi_z}\Big)\Big]\,. \qquad (41)$$

Yokoya neglected the second term, but I shall retain it. After eq. (3.6) in ref. [4], it is stated that "we have neglected the terms which are proportional to betatron oscillation amplitudes after differentiation." Using eq. (40), and treating only a single betatron parent resonance,

$$\begin{aligned}
\frac{\partial\zeta}{\partial\epsilon} &= e^{-i\chi}\sum_{m=-\infty}^{\infty} B_m(\theta)\Big[\Big(-\eta_x - i(\eta_x'\beta_x + \eta_x\alpha_x)\Big)J_m(\sqrt{2I_z}\,u_\epsilon)e^{im\psi_z} \\
&\qquad + \sqrt{2I_x\beta_x}e^{-i(\psi_x+\widetilde{\Psi}_x)}\frac{u_\epsilon}{2}\Big(J_{m-1}(\sqrt{2I_z}\,u_\epsilon)e^{i(m-1)\psi_z} \\
&\qquad - J_{m+1}(\sqrt{2I_z}\,u_\epsilon)e^{i(m+1)\psi_z}\Big)\Big] \\
&\equiv e^{-i\chi}\sum_{m=-\infty}^{\infty} C_m(\theta)J_m(\sqrt{2I_z}\,u_\epsilon)e^{im\psi_z}\,. \qquad (42)
\end{aligned}$$

4.4 Linear $\gamma(\partial\hat{n}/\partial\gamma)$ term

In this section I shall describe when it is valid to neglect the terms Yokoya ignored in eq. (41). There are two terms in eq. (1) involving $\gamma(\partial\hat{n}/\partial\gamma)$, viz. $\langle\,\hat{b}.\gamma(\partial\hat{n}/\partial\gamma)/|\rho|^3\rangle$ and $\langle|\gamma(\partial\hat{n}/\partial\gamma)|^2/|\rho|^3\rangle$. The terms Yokoya neglected in eq. (41) average to zero for the term linear in $\gamma(\partial\hat{n}/\partial\gamma)$, and so it is valid to neglect them there. The proof is simply to average $\partial\zeta/\partial\epsilon$ in eq. (42) over the betatron action-angle variables. The terms Yokoya neglected are proportional to $e^{-i\psi_x}$. From the fact that $\langle e^{-i\psi_x}\rangle = 0$, we immediately see that these terms average to zero.

However, I shall show below that these same terms do not average to zero when calculating $\langle|\gamma(\partial\hat{n}/\partial\gamma)|^2/|\rho|^3\rangle$ in eq. (1). In fact they yield a result which is comparable in magnitude to the terms Yokoya retained, hence it is not valid to neglect them. Hence the ensemble average over the orbital action-angle variables plays an important role in the calculations in this paper. Note, though, that for first order resonances, $\gamma(\partial\hat{n}/\partial\gamma)$ does not depend on the orbital action-angle variables, hence the ensemble average is trivial, and so $\langle|\gamma(\partial\hat{n}/\partial\gamma)|^2\rangle_{I,\psi}$ is just the absolute square of $\langle\gamma(\partial\hat{n}/\partial\gamma)\rangle_{I,\psi}$ in eq. (1). Hence a term which is negligible in $\langle\,\hat{b}.\gamma(\partial\hat{n}/\partial\gamma)/|\rho|^3\rangle$ is also negligible in $\langle|\gamma(\partial\hat{n}/\partial\gamma)|^2/|\rho|^3\rangle$. However, the above result is not true for higher order resonances.

4.5 Ensemble averages

Averaging over the synchrotron oscillations, assuming a Gaussian distribution,

$$
\begin{aligned}
\left\langle \left| \frac{\partial \zeta}{\partial \epsilon} \right|^2 \right\rangle &= \sum_{m=-\infty}^{\infty} |C_m(\theta)|^2 \int_0^{\infty} \frac{dI_z}{\langle I_z \rangle} e^{-I_z/\langle I_z \rangle} J_m^2(\sqrt{2I_z}\, u_\epsilon) \\
&= \sum_{m=-\infty}^{\infty} |C_m(\theta)|^2 e^{-\alpha} I_m(\alpha)\,, \qquad (43)
\end{aligned}
$$

where $\alpha = \langle I_z \rangle u_\epsilon^2 = \langle I_z \rangle (\gamma a/Q_s)^2$. We also need to average over the betatron orbits in C_m. From eq. (42)

$$
C_m(\theta) = (-\eta_x - i(\eta'_x \beta_x + \eta_x \alpha_x)) B_m + \sqrt{2I_x\beta_x}\, e^{-i(\psi_x + \tilde{\Psi}_x)} \frac{u_\epsilon}{2} (B_{m+1} - B_{m-1})\,. \qquad (44)
$$

For brevity, let us put $\delta = \nu + n - Q_x$. To establish contact with Yokoya's solution, I shall consider the terms proportional $\sqrt{2I_x\beta_x}$ later, but neglect them for now. Then

$$
\begin{aligned}
|C_m(\theta)|^2 &\simeq (\eta_x^2 + (\eta'_x\beta_x + \eta_x\alpha_x)^2)|B_m|^2 \\
&= \frac{1}{4} \frac{\eta_x^2 + (\eta'_x\beta_x + \eta_x\alpha_x)^2}{\beta_x} \frac{|b_{n,-x}|^2}{(\delta + mQ_s)^2}\,, \qquad (45)
\end{aligned}
$$

and so

$$
\left\langle \left| \gamma \frac{\partial \hat{n}}{\partial \gamma} \right|^2 \right\rangle \simeq \frac{1}{4} \frac{\eta_x^2 + (\eta'_x\beta_x + \eta_x\alpha_x)^2}{\beta_x} \sum_{m=-\infty}^{\infty} \frac{|b_{n,-x}|^2}{(\delta + mQ_s)^2} e^{-\alpha} I_m(\alpha)\,, \qquad (46)
$$

which agrees with Yokoya's expression for $\langle |\gamma(\partial \hat{n}/\partial\gamma)|^2 \rangle$ in eq. (3.18) in ref. [4].

4.6 Additional spin integrals

Now let us include the extra terms, which are proportional to $\sqrt{2I_x\beta_x}$ in eq. (44). To do so, note that [13]

$$
\begin{aligned}
\epsilon_\epsilon &\equiv \langle I_z \rangle = \frac{C_q \gamma_0^2 \langle |\rho_0|^{-3} \rangle_\theta}{J_\epsilon \langle |\rho_0|^{-2} \rangle_\theta} \\
\epsilon_{x\beta} &\equiv \langle I_x \rangle = \frac{C_q \gamma_0^2}{J_x \langle |\rho_0|^{-2} \rangle_\theta} \left\langle \frac{1}{|\rho_0|^3} \frac{\eta_x^2 + (\eta'_x\beta_x + \eta_x\alpha_x)^2}{\beta_x} \right\rangle_\theta \equiv \frac{C_q \gamma_0^2}{J_x \langle |\rho_0|^{-2} \rangle_\theta} \left\langle \frac{H}{|\rho_0|^3} \right\rangle_\theta\,, \qquad (47)
\end{aligned}
$$

where $\langle \ldots \rangle_\theta$ denotes an average around the ring circumference, $C_q = 55/(32\sqrt{3})\hbar/(mc) = 3.84 \times 10^{-13}$ m, ρ_0 is the bending radius of the closed orbit, γ_0 is the average electron energy in units of mc^2, and J_ϵ and J_x are the partition numbers of the synchrotron and betatron damping constants.

For brevity, I define $K = C_q\gamma_0^2/\langle|\rho_0|^{-2}\rangle_\theta$. Then, averaging over both the orbital action-angle variables and the ring circumference,

$$\begin{aligned}\left\langle \frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle_{\theta,I,\psi} &\simeq \left\langle \frac{1}{|\rho|^3}\left|\frac{\partial\zeta}{\partial\epsilon}\right|^2\right\rangle_{\theta,I,\psi} \\ &= \sum_m e^{-\alpha}I_m(\alpha)\Big[K^{-1}J_x\epsilon_{x\beta}|\beta_x^{\frac{1}{2}}B_m|^2 + \epsilon_{x\beta}K^{-1}J_\epsilon\epsilon_\epsilon\frac{u_\epsilon^2}{2} \\ &\qquad \times\Big(|\beta_x^{\frac{1}{2}}B_{m+1}|^2 + |\beta_x^{\frac{1}{2}}B_{m-1}|^2 - 2\mathrm{Re}(\beta_x B_{m+1}B_{m-1}^*)\Big)\Big]\,. \qquad (48)\end{aligned}$$

I neglect the cross term $\mathrm{Re}(\beta_x B_{m+1}B_{m-1}^*)$ because it is not as singular as the other terms, and so

$$\begin{aligned}\left\langle \frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle &= \frac{\epsilon_{x\beta}}{K}\sum_{m=-\infty}^{\infty} e^{-\alpha}\Big[J_x|\beta_x^{\frac{1}{2}}B_m|^2 I_m(\alpha) + J_\epsilon\frac{\alpha}{2}|\beta_x^{\frac{1}{2}}B_m|^2(I_{m-1}(\alpha)+I_{m+1}(\alpha))\Big] \\ &= \frac{\epsilon_{x\beta}}{4K}\sum_{m=-\infty}^{\infty}\frac{|b_{n,-x}|^2}{(\delta+mQ_s)^2}e^{-\alpha}\Big[J_xI_m(\alpha) + J_\epsilon\frac{\alpha}{2}(I_{m-1}(\alpha)+I_{m+1}(\alpha))\Big]. \qquad (49)\end{aligned}$$

From the tune modulation derivation, the contribution of chromaticity can be included by modifying $\alpha \to \langle I_z\rangle(\gamma_0 a - \xi_x)^2/Q_s^2$. One of the features of the above result is that because $I_{-m}(\alpha) = I_m(\alpha)$, the terms in m and $-m$ have the same magnitude, provided α is kept constant when measuring the two resonance widths. This can be verified by substituting $m \to -m$ in eq. (49). The inclusion of additional terms, which have been neglected in the above derivation, will change this symmetry. An example of such terms has been given in ref. [8]. Now let us consider the cross term $\mathrm{Re}(\beta_x B_{m+1}B_{m-1}^*)$ which was neglected above. It is easy to show that, for a single parent betatron resonance,

$$\begin{aligned}\mathrm{Re}(\beta_x B_{m+1}B_{m-1}^*) &= \frac{|b_{n,-x}|^2}{4}\frac{e^{-\alpha}I_m(\alpha)}{[\delta+(m-1)Q_s][\delta+(m+1)Q_s]} \\ &= \frac{|b_{n,-x}|^2}{4}\frac{1}{2Q_s}\left[\frac{1}{\delta+(m-1)Q_s} - \frac{1}{\delta+(m+1)Q_s}\right]e^{-\alpha}I_m(\alpha) \qquad (50)\end{aligned}$$

and, after some calculation, eq. (49) is modified to

$$\begin{aligned}\left\langle \frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle &= \frac{\epsilon_{x\beta}}{4K}\sum_{m=-\infty}^{\infty}\left\{\frac{|b_{n,-x}|^2}{(\delta+mQ_s)^2}e^{-\alpha}\Big[J_xI_m(\alpha) + J_\epsilon\frac{\alpha}{2}(I_{m-1}(\alpha)+I_{m+1}(\alpha))\Big]\right. \\ &\qquad \left. + \frac{|b_{n,-x}|^2}{\delta+mQ_s}\frac{mJ_\epsilon}{Q_s}e^{-\alpha}I_m(\alpha)\right\}. \qquad (51)\end{aligned}$$

An equivalent result has been reported by Buon [6], as part of an enhancement factor. Such factors will be derived in the next section. The m and $-m$ sidebands still have equal width, even with the extra term above [14]. However, being linear and not quadratic in the denominator, it will change sign at $\delta + mQ_s = 0$, and so it will skew the lineshape above and below the center of the sideband. Unless otherwise stated, I shall ignore all such less singular terms in the rest of this paper.

4.7 Enhancement factors

4.7.1 Sidebands with $m > 0$

Let us now consider only the terms for which $m \geq 0$ in eq. (49). These are the sidebands $n+\nu = Q_x - Q_s$, $n+\nu = Q_x - 2Q_s$, etc. I shall return to the $m < 0$ terms later. Using the relation

$$\alpha(I_{m-1}(\alpha) - I_{m+1}(\alpha)) = 2mI_m(\alpha) \ , \tag{52}$$

eq. (49) can be written in the form

$$\begin{aligned} \left\langle \frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle &= \frac{\epsilon_{x\beta}}{4K}\sum_m \frac{|b_{n,-x}|^2}{(\delta+mQ_s)^2}e^{-\alpha}\left[J_x I_m(\alpha) + J_\epsilon(mI_m'(\alpha) + \alpha I_{m+1}(\alpha))\right] \\ &= \frac{J_x\epsilon_{x\beta}}{4K}\sum_{m=0}^{\infty}\frac{|b_{n,-x}|^2}{(\delta+mQ_s)^2}e^{-\alpha}\left[\left(1+m\frac{J_\epsilon}{J_x}\right)I_m(\alpha) + \frac{J_\epsilon}{J_x}\alpha I_{m+1}(\alpha)\right] . \end{aligned} \tag{53}$$

We can write the above result as an "enhancement" of the parent resonance

$$\left\langle \frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle = F\left\langle \frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle_{1^{st}\ \mathrm{order}} . \tag{54}$$

The first-order result is

$$\left\langle \frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle_{1^{st}\ \mathrm{order}} = \frac{1}{4}\left\langle\frac{H}{|\rho|^3}\right\rangle_\theta \frac{|b_{n,-x}|^2}{\delta^2} . \tag{55}$$

Hence, for the $m > 0$ sidebands,

$$F_{m\geq 0} = \sum_{m=0}^{\infty} e^{-\alpha}\left[\left(1+m\frac{J_\epsilon}{J_x}\right)I_m(\alpha) + \frac{J_\epsilon}{J_x}\alpha I_{m+1}(\alpha)\right]\frac{\delta^2}{(\delta+mQ_s)^2} . \tag{56}$$

Using the asymptotic relations

$$\begin{aligned} I_m(\alpha) &\simeq \left(\frac{\alpha}{2}\right)^m\frac{1}{m!} \qquad (\alpha \ll m) \ , \\ &\simeq \frac{e^\alpha}{\sqrt{2\pi\alpha}} \qquad (\alpha \gg m) \ , \end{aligned} \tag{57}$$

we see that the term in $\alpha I_{m+1}(\alpha)$ is important only if $\alpha \gg m$.

4.7.2 General case

For the sideband resonances with $m < 0$ in eq. (49), the enhancement factor is

$$F_{m\leq 0} = \sum_{m=-\infty}^{0} e^{-\alpha}\left[\left(1-m\frac{J_\epsilon}{J_x}\right)I_m(\alpha) + \frac{J_\epsilon}{J_x}\alpha I_{m-1}(\alpha)\right]\frac{\delta^2}{(\delta+mQ_s)^2}$$

Table 1: Numerical estimates for $\langle I_z \rangle$ and α

Ring	Relative energy spread $\langle I_z \rangle$	α
SPEAR	8.2×10^{-7}	2.8×10^{-2}
HERA	1.1×10^{-6}	1.42
LEP	6.1×10^{-7}	0.79

$$= \sum_{m=0}^{\infty} e^{-\alpha} \left[\left(1 + m\frac{J_\epsilon}{J_x}\right) I_m(\alpha) + \frac{J_\epsilon}{J_x} \alpha I_{m+1}(\alpha) \right] \frac{\delta^2}{(\delta - mQ_s)^2} . \tag{58}$$

The two enhancement factors are equal, as expected. We can combine the above results into one factor for all sidebands

$$F = \sum_{m=-\infty}^{\infty} e^{-\alpha} \left[\left(1 + |m|\frac{J_\epsilon}{J_x}\right) I_{|m|}(\alpha) + \frac{J_\epsilon}{J_x} \alpha I_{|m|+1}(\alpha) \right] \frac{\delta^2}{(\delta + mQ_s)^2} . \tag{59}$$

It is also convenient to write

$$\left\langle \frac{1}{|\rho|^3} \left| \gamma \frac{\partial \hat{n}}{\partial \gamma} \right|^2 \right\rangle \equiv \sum_{m=-\infty}^{\infty} \frac{W_m^2}{(\delta + mQ_s)^2} . \tag{60}$$

If the resonances are well separated, then it can be shown that the width of the resonance $\delta + mQ_s = 0$ is proportional to W_m. In particular, the ratio of the width of the sideband resonance $\delta + mQ_s = 0$ to that of the parent resonance $\delta = 0$ is equal, not merely proportional, to W_m/W_0. This result is not true, however, if several terms in the above sum contribute significantly to the width of a given resonance, i.e. if the resonances overlap.

4.8 Numerical estimates for α

At this point, let us estimate the value of $\alpha \equiv \epsilon_\epsilon (\gamma_0 a / Q_s)^2$ for various rings. I obtain the value of ϵ_ϵ using eq. (47). For SPEAR at the horizontal betatron resonance $\nu = 3 + Q_x$, $E = 3.65$ GeV, and $\rho_0 \simeq 12$ m, and I put $J_\epsilon = 2$. From the data in ref. [2], $Q_s \simeq 0.045$. For HERA, I use the values $E = 30$ GeV, $\rho_0 = 600$ m, $J_\epsilon = 2$ and $Q_s = 0.06$, and for LEP I use $E = 50$ GeV, $\rho_0 = 3000$ m, $J_\epsilon = 2$ and $Q_s = 0.1$. The values of ϵ_ϵ and α for these three models are given in table 1. We see that $\alpha \ll 1$ for SPEAR, but is of order unity for HERA and LEP.

5 SMILE formalism

5.1 Solution for $\hat{n}$

In this section the SMILE formalism [7] will be used to derive the above results. One need only consider sidebands with $m > 0$. The SMILE formalism uses a power series expansion in powers of the beam emittances, hence we should compare it against the power series expansion of the above Bessel functions. In practice, I shall derive only the leading terms of the Bessel function expansion below. This is sufficient to prove the basic equivalence of the two formalisms. The remaining terms can be obtained with more laborious calculation. In ref. [7], the orbital motion is written as a sum of eigenvectors but here I shall follow Yokoya [4] and write $\vec{\omega} = x_\beta \vec{\omega}_x + \epsilon \vec{\omega}_\epsilon$ instead, and now I put

$$x_\beta = a_{x\beta} \tilde{x}_\beta + \text{c.c.}\,, \qquad \epsilon = a_\epsilon \tilde{\epsilon} + \text{c.c.}\,, \tag{61}$$

where

$$\begin{aligned} a_{x\beta} &= \sqrt{I_x}\, e^{i(\psi_x - Q_x\theta)} & \qquad a_\epsilon &= \sqrt{I_z}\, e^{i(\psi_z - Q_s\theta)} \\ \tilde{x}_\beta &= \sqrt{\beta_x/2}\, e^{i(Q_x\theta + \tilde{\Psi}_x)} & \qquad \tilde{\epsilon} &= \frac{e^{iQ_s\theta}}{\sqrt{2}}\,. \end{aligned} \tag{62}$$

I decompose

$$\hat{n} = n_1 \hat{l}_0 + n_2 \hat{m}_0 + n_3 \hat{n}_0 \tag{63}$$

in terms of a right-handed orthonormal basis $\{\hat{l}_0, \hat{m}_0, \hat{n}_0\}$ of solutions of the Thomas-BMT equation on the closed orbit, and define

$$V_1 = -\frac{n_1 + in_2}{\sqrt{2}}\,, \qquad V_{-1} = \frac{n_1 - in_2}{\sqrt{2}}\,, \qquad V_0 = n_3\,. \tag{64}$$

Then $\vec{k}_0 = \hat{l}_0 + i\hat{m}_0$ and $\zeta = -\sqrt{2}\, V_1$. The solution is given by an azimuth-ordered exponential [7]

$$\begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix} = \mathrm{T}\left\{ \exp\left(i \int_{-\infty}^{\theta} d\theta' \vec{\omega}.\vec{J}^T \right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}\,, \tag{65}$$

where $\vec{J}$ is a vector of spin 1 angular momentum matrices. To obtain a practical solution, I expand the above exponential in a power series and sum the terms one by one.

5.2 Synchrotron sideband resonances

The above exponential contains all combinations of spin integrals, but to get the previous solution for ζ (eq. (32)), I consider only the terms with $x_\beta \vec{\omega}_x.\vec{k}_0$ at first order, followed by powers of $\epsilon\vec{\omega}_\epsilon.\hat{n}_0$, the coupling to the synchrotron oscillations, i.e.

$$\begin{aligned}\zeta = -\sqrt{2}\,V_1 \;\; &\simeq \;\; -i\int_{-\infty}^{\theta} d\theta'\, x_\beta\vec{\omega}_x.\vec{k}_0 \\ &+\int_{-\infty}^{\theta} d\theta'\,\epsilon\vec{\omega}_\epsilon.\hat{n}_0\int_{-\infty}^{\theta'} d\theta''\, x_\beta\vec{\omega}_x.\vec{k}_0 \\ &+i\int_{-\infty}^{\theta} d\theta'\,\epsilon\vec{\omega}_\epsilon.\hat{n}_0\int_{-\infty}^{\theta'} d\theta''\,\epsilon\vec{\omega}_\epsilon.\hat{n}_0\int_{-\infty}^{\theta''} d\theta'''\, x_\beta\vec{\omega}_x.\vec{k}_0 + \ldots \end{aligned} \tag{66}$$

Eq. (65) also contains terms of the form

$$\zeta = \frac{i}{2}\int_{-\infty}^{\theta} d\theta'\,\epsilon\vec{\omega}_\epsilon.\vec{k}_0\int_{-\infty}^{\theta'} d\theta''\,\epsilon\vec{\omega}_\epsilon.\vec{k}_0^*\int_{-\infty}^{\theta''} d\theta'''\, x_\beta\vec{\omega}_x.\vec{k}_0 + \ldots \tag{67}$$

i.e. terms involving $\vec{\omega}_\epsilon.\vec{k}_0$ rather than $\vec{\omega}_\epsilon.\hat{n}_0$, as well as other terms. They come from expanding $\sqrt{1-|\zeta|^2}$ in eq. (7), and do not appear in the tune modulation derivation, or Yokoya's solution, and I shall ignore them below. Let us now evaluate eq. (66) term by term. As before, I approximate

$$e^{-i(Q_x\theta+\widetilde{\Psi}_x)}\sqrt{\beta_x}\,\vec{\omega}_x.\vec{k}_0 \;\simeq\; be^{i(\nu+n-Q_x)\theta} \;\equiv\; be^{i\delta\theta}\,, \tag{68}$$

dropping the subscripts on $b_{n,-x}$. From above, I approximate $\epsilon\vec{\omega}_\epsilon.\hat{n}_0 \simeq -Q_s u_\epsilon\sqrt{2I_z}\cos\psi_z$ because $Q_s \ll 1$. Let us now write $\zeta = \zeta_1+\zeta_2+\zeta_3+\ldots$, where ζ_n is the n^{th} term in eq. (66). Then

$$\begin{aligned}\zeta_1 &= -ia_{x\beta}^*\int_{-\infty}^{\theta}\widetilde{x}_\beta^*\vec{\omega}_x.\vec{k}_0\,d\theta' \\ &= -a_{x\beta}^*\frac{b}{\sqrt{2}}\frac{e^{i\delta\theta}}{\delta}\,. \end{aligned} \tag{69}$$

We also need the results, which can be derived from eq. (40),

$$\begin{aligned}\frac{\partial a'}{\partial\epsilon} &\equiv \frac{\partial}{\partial\epsilon}(a_{x\beta}^* e^{-i(\widetilde{\Psi}_x+Q_x\theta)}) = \frac{-\eta_x - i(\eta'\beta_x+\eta_x\alpha_x)}{\sqrt{2\beta_x}} \\ \frac{\partial a_\epsilon}{\partial\epsilon} &= \frac{e^{-iQ_s\theta}}{\sqrt{2}}\,. \end{aligned} \tag{70}$$

5.3 First order resonance

The derivative of ζ_1 is

$$\frac{\partial\zeta_1}{\partial\epsilon} = -\frac{\partial a'}{\partial\epsilon}\frac{b}{\sqrt{2}\delta}e^{i(\widetilde{\Psi}_x+Q_x\theta)}e^{i\delta\theta} \tag{71}$$

and

$$\left\langle\frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle \;=\; \left\langle\frac{1}{|\rho|^3}\left|\frac{\partial\zeta_1}{\partial\epsilon}\right|^2\right\rangle$$

$$
\begin{aligned}
&= \left\langle \frac{1}{|\rho|^3}\left|\frac{\partial a'}{\partial \epsilon}\right|^2 \right\rangle \frac{|b|^2}{2\delta^2} \\
&= \frac{1}{4}\left\langle \frac{H}{|\rho|^3} \right\rangle \frac{|b|^2}{\delta^2}\,, \\
&= \frac{\epsilon_{x\beta} J_x}{K}\frac{|b|^2}{4\delta^2}
\end{aligned}
\tag{72}
$$

in agreement with the previous calculation.

5.4 $m = 1$ Sideband

The second term in the series for ζ is

$$
\zeta_2 = -a^*_{x\beta}\frac{b}{\sqrt{2}\delta}a_\epsilon i\int_{-\infty}^{\theta}\frac{e^{iQ_s\theta'}}{\sqrt{2}}\vec{\omega}_\epsilon.\hat{n}_0 e^{i\delta\theta'}\,d\theta' + \ldots
\tag{73}
$$

I have neglected the term which gives the resonance $\delta - Q_s = 0$, because I am only considering sidebands $\delta + mQ_s$ with $m > 0$. Using the approximation for $\vec{\omega}_\epsilon.\hat{n}_0$ given above,

$$
\begin{aligned}
\zeta_2 &\simeq -a^*_{x\beta}\frac{b}{\sqrt{2}\delta}a_\epsilon i\int_{-\infty}^{\theta}\frac{-Q_s u_\epsilon}{\sqrt{2}}e^{i(\delta+Q_s)\theta'}\,d\theta' \\
&= \frac{a^*_{x\beta}a_\epsilon}{2}\frac{b}{\delta}Q_s u_\epsilon\frac{e^{i(\delta+Q_s)\theta}}{\delta+Q_s}\,.
\end{aligned}
\tag{74}
$$

Then

$$
\frac{\partial\zeta_2}{\partial\epsilon} = \frac{1}{2}\Big(\frac{\partial a'}{\partial\epsilon}a_\epsilon + a'\frac{\partial a_\epsilon}{\partial\epsilon}\Big)\frac{b}{\delta}\frac{Q_s u_\epsilon}{(\delta+Q_s)}e^{i(\widetilde{\Psi}_x+Q_x\theta)}e^{i(\delta+Q_s)\theta}
\tag{75}
$$

and

$$
\begin{aligned}
\left\langle \frac{1}{|\rho|^3}\left|\frac{\partial\zeta}{\partial\epsilon}\right|^2\right\rangle &\simeq \left\langle \frac{1}{|\rho|^3}\left|\frac{\partial\zeta_1}{\partial\epsilon}\right|^2\right\rangle \\
&\quad + \left\langle \frac{1}{|\rho|^3}\left[\left|\frac{\partial a'}{\partial\epsilon}\right|^2\epsilon_\epsilon + \epsilon_{x\beta}\left|\frac{\partial a_\epsilon}{\partial\epsilon}\right|^2\right]\right\rangle \frac{|b|^2}{4\delta^2}\frac{(Q_s u_\epsilon)^2}{(\delta+Q_s)^2} \\
&= \frac{\epsilon_{x\beta}J_x}{K}\frac{|b|^2}{4\delta^2} + \frac{\epsilon_{x\beta}\epsilon_\epsilon}{2K}(J_x+J_\epsilon)\frac{|b|^2}{4\delta^2}\frac{(Q_s u_\epsilon)^2}{(\delta+Q_s)^2}\,.
\end{aligned}
\tag{76}
$$

Recall the expression for $\langle|\gamma(\partial\hat{n}/\partial\gamma)|^2/|\rho|^3\rangle$ in the previous section could be written in the form

$$
\left\langle \frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle = \sum_{m=-\infty}^{\infty}\frac{W_m^2}{(\delta+mQ_s)^2}\,.
\tag{77}
$$

Unlike eq. (77), the sum in eq. (76) is not automatically separated into terms with distinct resonance denominators $\delta + mQ_s$. The higher order terms also contain lower order resonance denominators. Thus the second term has both δ^2 and $(\delta + Q_s)^2$ in the denominator. This gives a correction to the first order resonance strength of $O(\epsilon_\epsilon u_\epsilon^2) = O(\alpha)$. In the previous calculation, this was given by the nonleading terms in the power series expansions of $e^{-\alpha} I_m(\alpha)$ and $\alpha e^{-\alpha} I_{m+1}(\alpha)$.

We can, however, expand the SMILE solution into partial fractions to separate the various resonance denominators. It is easy to see that

$$\begin{aligned}
\frac{1}{\delta(\delta + Q_s)} &= \frac{1}{Q_s}\left[\frac{1}{\delta} - \frac{1}{\delta + Q_s}\right] \\
\frac{1}{\delta^2(\delta + Q_s)^2} &= \frac{1}{Q_s^2}\left[\frac{1}{\delta^2} + \frac{1}{(\delta + Q_s)^2} - \frac{2}{\delta(\delta + Q_s)}\right] \\
&= \frac{1}{Q_s^2}\left[\frac{1}{\delta^2} + \frac{1}{(\delta + Q_s)^2}\right] - \frac{2}{Q_s^3}\left[\frac{1}{\delta} - \frac{1}{\delta + Q_s}\right] . \qquad (78)
\end{aligned}$$

I shall concentrate on the $(\delta + Q_s)^{-2}$ term, because it yields the leading contribution to $m = 1$ synchrotron sideband. The other terms yield nonleading corrections to various resonances. In fact, as stated above, I shall only treat the leading order contribution in the general case below, i.e. the $(\delta + mQ_s)^{-2}$ term in ζ_{m+1}. To obtain the strength of the first sideband resonance, i.e. W_1, we need the coefficient of the $(\delta + Q_s)^{-2}$ partial fraction, or rather, the ratio of this coefficient to that of the first order resonance. There is a standard mathematical technique to do this: the answer is given by putting $\delta + Q_s = 0$ in the coefficient of $(\delta + Q_s)^{-2}$ in the r.h.s of eq. (76), which yields

$$\frac{W_1^2}{W_0^2} \simeq \frac{\epsilon_{x\beta}}{2K}\frac{|b|^2}{4}(J_x + J_\epsilon)\epsilon_\epsilon u_\epsilon^2 \left[\frac{\epsilon_{x\beta} J_x}{K}\frac{|b|^2}{4}\right]^{-1} = \left(1 + \frac{J_\epsilon}{J_x}\right)\frac{\alpha}{2} \, . \qquad (79)$$

In the general case, we put $\delta + mQ_s = 0$ in the coefficient of $(\delta + mQ_s)^{-2}$ in the expression for ζ_{m+1}. For this power of α, Yokoya's formalism, i.e. eq. (53), yields

$$\begin{aligned}
\frac{W_1^2}{W_0^2} &= \left[\left(1 + \frac{J_\epsilon}{J_x}\right) I_1(\alpha) + \frac{J_\epsilon}{J_x}\alpha I_2(\alpha)\right]\left[I_0(\alpha) + \frac{J_\epsilon}{J_x}\alpha I_1(\alpha)\right]^{-1} \\
&= \left[\left(1 + \frac{J_\epsilon}{J_x}\right)\frac{\alpha}{2} + \ldots + \frac{J_\epsilon}{J_x}\frac{\alpha^3}{8} + \ldots\right]\left[1 + \ldots\right] \\
&= \left(1 + \frac{J_\epsilon}{J_x}\right)\frac{\alpha}{2} + \ldots \qquad (80)
\end{aligned}$$

and so the two formalisms agree.

5.5 General case

Let us now consider the m^{th} sideband $\delta + mQ_s$. By repeatedly multiplying by $\epsilon\vec{\omega}_\epsilon.\hat{n}_0$ and integrating, it can be verified that

$$\zeta_{m+1} \simeq -a^*_{x\beta}\frac{b}{\sqrt{2}\delta}\frac{a^m_\epsilon(-Q_s u_\epsilon)^m}{2^{m/2}(\delta+Q_s)\ldots(\delta+mQ_s)}e^{i(\delta+mQ_s)\theta}, \tag{81}$$

with neglect of $m < 0$ sidebands, etc., and so

$$\begin{aligned}
\left\langle \frac{1}{|\rho|^3}\left|\frac{\partial\zeta_{m+1}}{\partial\epsilon}\right|^2\right\rangle &\simeq \left\langle \frac{1}{|\rho|^3}\left[2\left|\frac{\partial a'}{\partial\epsilon}\right|^2|a_\epsilon|^{2m} + 2|a^*_{x\beta}|^2\left|m\frac{\partial a_\epsilon}{\partial\epsilon}a^{m-1}_\epsilon\right|^2\right]\right\rangle \\
&\quad\times\frac{|b|^2}{4\delta^2}\frac{(Q_s u_\epsilon)^{2m}}{2^m(\delta+Q_s)^2\ldots(\delta+mQ_s)^2} \\
&= \frac{\epsilon_{x\beta}J_x}{K}\frac{|b|^2}{4\delta^2}\left[m!\epsilon^m_\epsilon + m^2(m-1)!\epsilon^m_\epsilon\frac{J_\epsilon}{J_x}\right] \\
&\quad\times\frac{(Q_s u_\epsilon)^{2m}}{2^m(\delta+Q_s)^2\ldots(\delta+mQ_s)^2} \\
&= \frac{\epsilon_{x\beta}J_x}{K}\frac{|b|^2}{4\delta^2}\left(1+m\frac{J_\epsilon}{J_x}\right)\frac{m!\alpha^m Q_s^{2m}}{2^m(\delta+Q_s)^2\ldots(\delta+mQ_s)^2}\,.
\end{aligned} \tag{82}$$

Putting $\delta + mQ_s = 0$ in the coefficient of $(\delta + mQ_s)^{-2}$,

$$\begin{aligned}
\frac{W_m^2}{W_0^2} &\simeq \frac{m!\alpha^m Q_s^{2m}}{2^m\delta^2(\delta+Q_s)^2\ldots(\delta+(m-1)Q_s)^2}\left(1+m\frac{J_\epsilon}{J_x}\right) \\
&= \frac{m!\alpha^m Q_s^2}{2^m m^2(m-1)^2(m-2)^2\ldots1^2}\left(1+m\frac{J_\epsilon}{J_x}\right) \\
&= \frac{m!\alpha^m}{2^m(m!)^2}\left(1+m\frac{J_\epsilon}{J_x}\right) \\
&= \left(1+m\frac{J_\epsilon}{J_x}\right)\frac{\alpha^m}{2^m m!}\,.
\end{aligned} \tag{83}$$

Once again, for the α^m term, Yokoya's formalism, i.e. eq. (53), yields the same result

$$\begin{aligned}
\frac{W_m^2}{W_0^2} &= \left[\left(1+m\frac{J_\epsilon}{J_x}\right)I_m(\alpha) + \frac{J_\epsilon}{J_x}\alpha I_{m+1}(\alpha)\right]\left[I_0(\alpha)+\frac{J_\epsilon}{J_x}\alpha I_1(\alpha)\right]^{-1} \\
&= \left[\left(1+m\frac{J_\epsilon}{J_x}\right)\frac{\alpha^m}{2^m m!} + \ldots + \frac{J_\epsilon}{J_x}\frac{\alpha^{m+2}}{2^{m+1}(m+1)!} + \ldots\right]\left[1+\ldots\right] \\
&= \left(1+m\frac{J_\epsilon}{J_x}\right)\frac{\alpha^m}{2^m m!} + \ldots
\end{aligned} \tag{84}$$

5.6 Nonleading terms

In the above sections I calculated only the leading terms in the power series expansion for the strength of each resonance. In this section I shall briefly consider the nonleading terms for one resonance, viz. $\nu = -n + Q_x$, the parent resonance. I shall only calculate the series expansion for ζ, not $\partial\zeta/\partial\epsilon$, for a single betatron parent resonance.

To avoid needless symbols which are effectively just constants, I shall put $\vec{\omega}.\vec{k}_0 = A\exp[i(n\theta + \nu\theta - \psi_x)]$. From eq. (38), the solution for ζ using Yokoya's formalism [4] is

$$\zeta = -A\,e^{-i\chi}\sum_m \frac{e^{i(n\theta+\nu\theta-\psi_x)}}{n+\nu-Q_x+mQ_s} J_m\left(\frac{\sqrt{2I_z}\,\gamma_0 a}{Q_s}\right) \tag{85}$$

and recall

$$\chi = \frac{\gamma_0 a}{Q_s}\sqrt{2I_z}\,\sin\psi_z\,, \tag{86}$$

so, for the $m = 0$ term,

$$\begin{aligned} \zeta &= -\frac{A}{\delta}e^{i(n\theta+\nu\theta-\psi_x)}\left[1 - i\frac{\gamma_0 a}{Q_s}\sqrt{2I_z}\,\sin\psi_z - \left(\frac{\gamma_0 a}{Q_s}\right)^2 I_z\,\sin^2\psi_z + \ldots\right]\left[1 - \frac{1}{2}\left(\frac{\gamma_0 a}{Q_s}\right)^2 I_z + \ldots\right] \\ &= \frac{A}{\delta}e^{i(n\theta+\nu\theta-\psi_x)}\left[-1 + i\frac{\gamma_0 a}{Q_s}\sqrt{2I_z}\,\sin\psi_z + \left(\frac{\gamma_0 a}{Q_s}\right)^2 I_z\left(\frac{1}{2} + \sin^2\psi_z\right) + \ldots\right]. \end{aligned} \tag{87}$$

I shall derive the above terms using the SMILE formalism. I again expand $\zeta = \zeta_1 + \zeta_2 + \ldots$, where $\zeta_n \propto I_z^{n/2}$. Then

$$\begin{aligned} \frac{d\zeta_1}{d\theta} &= -i\vec{\omega}.\vec{k}_0 = -iAe^{i(n\theta+\nu\theta-\psi_x)} \\ \zeta_1 &= -A\frac{e^{i(n\theta+\nu\theta-\psi_x)}}{n+\nu-Q_x} = -\frac{A}{\delta}e^{i(n\theta+\nu\theta-\psi_x)}\,, \end{aligned} \tag{88}$$

and

$$\begin{aligned} \frac{d\zeta_2}{d\theta} &= i\vec{\omega}.\hat{n}_0\zeta_1 \\ &= i\frac{A}{\delta}\gamma_0 a\,\sqrt{2I_z}\,\cos\psi_z e^{i(n\theta+\nu\theta-\psi_x)} \\ &= i\frac{A}{\delta}\gamma_0 a\,\sqrt{2I_z}\,\frac{1}{2}[e^{i\psi_z} + e^{-i\psi_z}]e^{i(n\theta+\nu\theta-\psi_x)} \\ \zeta_2 &= Ae^{i(n\theta+\nu\theta-\psi_x)}\gamma_0 a\,\sqrt{2I_z}\,\frac{1}{2}\left[\frac{e^{i\psi_z}}{\delta(\delta+Q_s)} + \frac{e^{-i\psi_z}}{\delta(\delta-Q_s)}\right] \\ &= Ae^{i(n\theta+\nu\theta-\psi_x)}\frac{\gamma_0 a}{Q_s}\sqrt{2I_z}\,\frac{1}{2}\left[\frac{e^{i\psi_z}}{\delta} - \frac{e^{i\psi_z}}{\delta+Q_s} - \frac{e^{-i\psi_z}}{\delta} + \frac{e^{-i\psi_z}}{\delta-Q_s}\right] \end{aligned}$$

$$= i\frac{A}{\delta}e^{i(n\theta+\nu\theta-\psi_x)}\frac{\gamma_0 a}{Q_s}\sqrt{2I_z}\,\sin\psi_z+\ldots \tag{89}$$

The third term is given by

$$\begin{aligned}
\frac{d\zeta_3}{d\theta} &= i\vec{\omega}.\hat{n}_0\zeta_2 \\
&= -iA\frac{(\gamma_0 a)^2}{Q_s}I_z\,e^{i(n\theta+\nu\theta-\psi_x)}\frac{1}{2}[e^{i\psi_z}+e^{-i\psi_z}]\left[\frac{e^{i\psi_z}}{\delta}-\frac{e^{-i\psi_z}}{\delta}-\frac{e^{i\psi_z}}{\delta+Q_s}+\frac{e^{-i\psi_z}}{\delta-Q_s}\right] \\
\zeta_3 &= -A\frac{(\gamma_0 a)^2}{Q_s}I_z\,e^{i(n\theta+\nu\theta-\psi_x)}\frac{1}{2}\left[\frac{e^{i2\psi_z}}{\delta(\delta+2Q_s)}+\frac{1}{\delta^2}-\frac{1}{\delta^2}-\frac{e^{-i2\psi_z}}{\delta(\delta-2Q_s)}\right. \\
&\qquad \left.-\frac{e^{i2\psi_z}}{(\delta+Q_s)(\delta+2Q_s)}-\frac{1}{\delta(\delta+Q_s)}+\frac{e^{-2i\psi_z}}{(\delta-Q_s)(\delta-2Q_s)}+\frac{1}{\delta(\delta-Q_s)}\right] \\
&= -Ae^{i(n\theta+\nu\theta-\psi_x)}\left(\frac{\gamma_0 a}{Q_s}\right)^2\frac{I_z}{2}\left[\frac{e^{i2\psi_z}}{2\delta}+\frac{e^{-i2\psi_z}}{2\delta}-\frac{2}{\delta}+\ldots\right] \\
&= -\frac{A}{\delta}e^{i(n\theta+\nu\theta-\psi_x)}\left(\frac{\gamma_0 a}{Q_s}\right)^2\frac{I_z}{2}\left[\cos(2\psi_z)-2\right] \\
&= \frac{A}{\delta}e^{i(n\theta+\nu\theta-\psi_x)}\left(\frac{\gamma_0 a}{Q_s}\right)^2 I_z\left[\frac{1}{2}+\sin^2\psi_z\right].
\end{aligned} \tag{90}$$

Adding the three terms together, we see that $\zeta_1+\zeta_2+\zeta_3$ adds up to the solution for ζ in eq. (87), for the resonance $\nu=-n+Q_x$. Proceeding further, we can obtain higher powers in $\sqrt{I_z}$. The SMILE formalism [7] does not make a distinction between global phase factors etc. for calculating the various terms in $\sqrt{I_z}$.

6 Numerical results

Fig. 1 shows a graph of polarization vs. energy measured at SPEAR [2]. In this section I shall compare the above results with some data from this graph, specifically, the horizontal betatron resonance $\nu=3+Q_x$ at 3.65 GeV and its sideband $\nu=3+Q_x-Q_s$. In this paper, the resonance widths are defined as the interval in which $P/P_0<0.5$, where $P_0=8/(5\sqrt{3})\simeq 92.4\%$. From fig. 1, the widths are approximately 4.1 – 6.4 MeV for the parent $\nu=3+Q_x$ and 1.5 MeV for the sideband $\nu=3+Q_x-Q_s$. The values of the relevant parameters used in the theoretical fit (eq. (56)) are the same as the ones used to calculate α in table 1, and are given in table 2, together with the results for the ratio W_1/W_0 of the resonance widths. Fig. 2 shows a fit to the data using eq. (51) for the sidebands from $m=-2$ to $m=2$. Only the denominators with $-2\le m\le 2$ were retained in the sum, and each Bessel function was expanded up to its first three terms, with

Table 2: Parameters and results of resonance width calculations

Function	Value
ν	8.28
Q_s	0.045
J_ϵ	2
J_x	1
W_1/W_0 (expt.)	0.23 – 0.37
W_1/W_0 (theory)	0.20

$\alpha = 0.028$, e.g. $I_0 \simeq 1 + \alpha^2/4 + \alpha^4/64$. The polarization was calculated using the formula

$$P = \frac{8/(5\sqrt{3})}{1 + (11/18)\langle|\gamma(\partial\hat{n}/\partial\gamma)|^2\rangle} \tag{91}$$

instead of the full formula eq. (1). The overall coefficient of $|\gamma(\partial\hat{n}/\partial\gamma)|^2$ was chosen by fitting to the sideband $\nu = 3 + Q_x - Q_s$, since it is the most precisely determined resonance experimentally. This graph not only yields the resonance widths but also provides information about the resonance lineshapes. The theoretical curve always drops to zero at the center of a resonance. We see that it fits the parent resonance and the sidebands $\nu = 3+Q_x-Q_s$ and $\nu = 3+Q_x-2Q_s$ fairly well. I shall comment on the other sidebands below. From table 2, the theoretical value for W_1/W_0 is roughly in agreement with the data, but slightly smaller. Various nonleading terms are perhaps needed in the theoretical formula. Further, the data were not taken in one experimental run, but in several runs, with slightly different machine tunes, etc. Thus there are points with $P/P_0 \simeq 30\%$ in the center of the resonance $\nu = 3 + Q_x$, for example, hence it is difficult to be very precise about the exact experimental resonance widths, and unfortunately the SPEAR polarimeter has been dismantled, so one cannot easily take more data.

Several other synchrotron sidebands of the same parent resonance are visible in the data, but numerical values for resonance widths are not given here. The reasons are given below. The second sideband resonance $\nu = 3 + Q_x - 2Q_s$ is so narrow that P/P_0 does not drop below 50% in the experimental graph, so one cannot quote an experimental value for W_2/W_0. The width of the $m = -1$ sideband $\nu = 3 + Q_x + Q_s$ could not be determined because of insufficient data. The

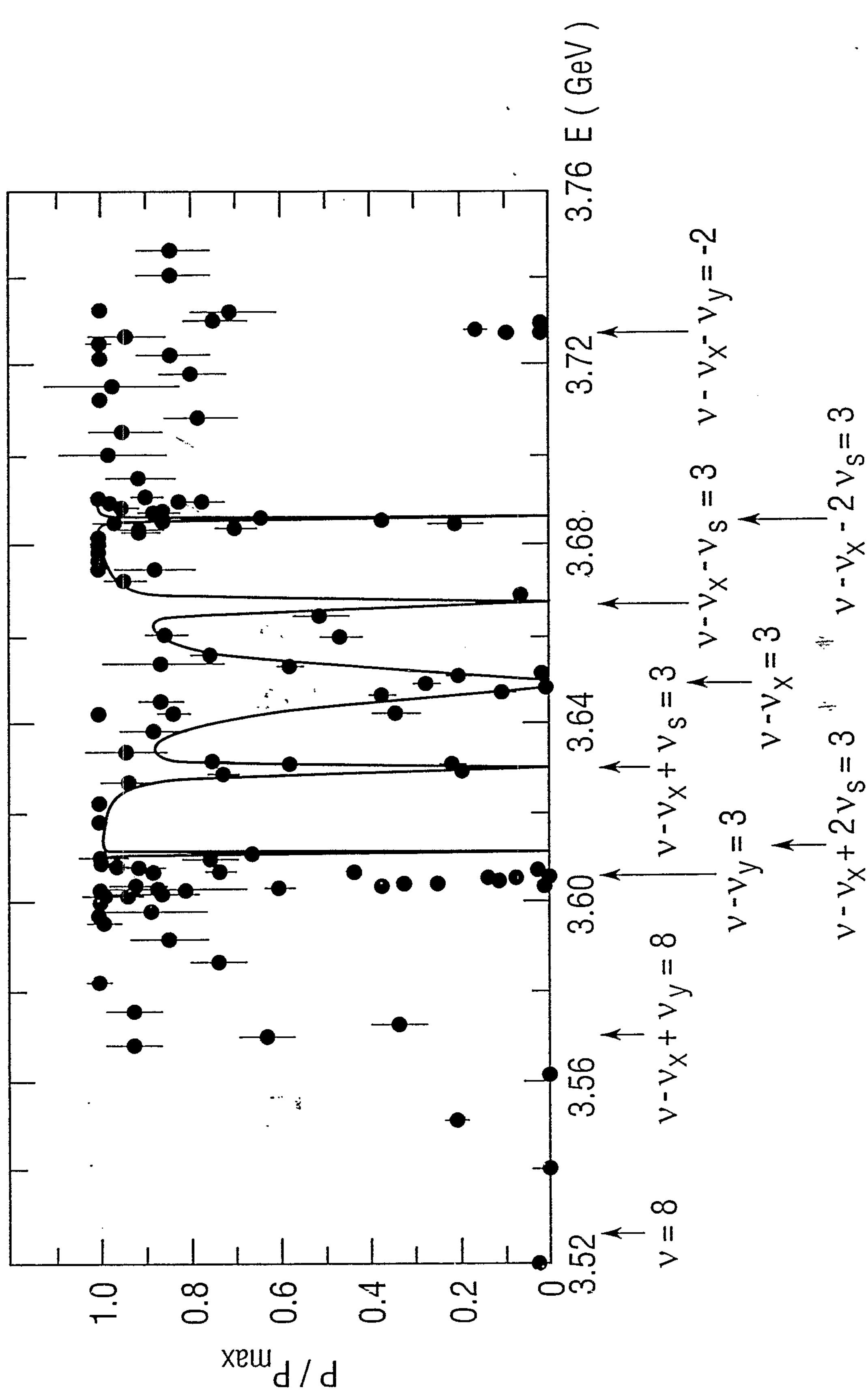

Figure 2. Theoretical fit to the resonances $\nu = 3 + \nu_x - 2\nu_s$ to $\nu = 3 + \nu_x + 2\nu_s$ in fig. 1. As in fig. 1, the orbital tunes are called $\nu_{x,y,s}$ instead of $Q_{x,y,s}$. Once again, P_{max} is $8/(5\sqrt{3}) \simeq 92.4\%$. The theory curve always drops to zero at the center of a resonance. The resonance labelled "$\nu = 3 + \nu_x + 2\nu_s$" in the data is actually an overlap of two resonances, $\nu = 3 + \nu_x + 2\nu_s$ and $\nu = 2\nu_y - 2$, and the latter is not taken into account in this paper.

$m = -2$ sideband $\nu = 3 + Q_x + 2Q_s$ at approximately 3.69 GeV seems to indicate an apparent problem with the above theory because it looks much wider than its $m = 2$ partner $\nu = 3+Q_x-2Q_s$. However, I have shown elsewhere [16] that this resonance actually consists of *two* closely overlapping resonances, viz. $\nu = 3 + Q_x + 2Q_s$ and $\nu = 2Q_y - 2$. When the SMILE computer program [7] is used to fit the data, the theory agrees with the data. Fig. 2 shows the contribution of only the $\nu = 3 + Q_x + 2Q_s$ resonance to the data in this region. The simplified theory above cannot include the contribution of the $\nu = 2Q_y - 2$ resonance.

7 Synchrotron sidebands centered on an integer

The above resonances were centered on a betatron resonance of the form $\nu = Q_x$+integer, and come from the $B_{m,-x}$ terms in eq. (35). Very similar results are obtained for resonances of the form $\nu = -Q_x$+integer, which came from the $B_{m,x}$ terms, and from vertical betatron parent resonances, and so I shall not write them out explicitly. The terms in A_m, however, yield a new set of resonances. In this case the first order parent resonance is itself a synchrotron resonance, viz. $\nu = \pm Q_s$+integer. The technique for calculating $|\gamma(\partial\hat{n}/\partial\gamma)|^2$ and the enhancement factors are, however, similar to those employed above. The results of the SMILE formalism [7] agree with those of Yokoya [4]. There are no extra terms which should be retained, in either derivation, in order to obtain agreement between the two formalisms. I shall quote the results, for sidebands based on a single integer n. Following the notation for $B_{m,-x}$, I write

$$\begin{aligned} A_m &= -\sum_{n'} \frac{a_{n'}}{n' + \nu + mQ_s} e^{i(n'+\nu)\theta} \\ &\simeq -\frac{a_n}{n + \nu + mQ_s} e^{i(n+\nu)\theta} . \end{aligned} \tag{92}$$

Defining $\delta = \nu + n$ now, the results are

$$\begin{aligned} \left\langle \frac{1}{|\rho|^3} \left| \gamma \frac{\partial\hat{n}}{\partial\gamma} \right|^2 \right\rangle &= \left\langle \frac{1}{|\rho|^3} \right\rangle |a_n|^2 \sum_m \left[\frac{\delta}{(\delta + mQ_s)^2 - Q_s^2} \right]^2 e^{-\alpha} I_m(\alpha) \\ F &= (\delta^2 - Q_s^2)^2 \sum_m \frac{e^{-\alpha} I_m(\alpha)}{[(\delta + mQ_s)^2 - Q_s^2]^2} , \end{aligned} \tag{93}$$

for the average of $|\gamma(\partial\hat{n}/\partial\gamma)|^2$ and the enhancement factor F, respectively. Note that the first order resonances consist of a doublet, viz. $\nu = -n \pm Q_s$, as opposed to only one parent resonance. I did not derive the above results in the section on tune modulation, but results equivalent to eq. (93) have been published, using tune modulation, in refs. [3] and [5].

More recently, a new formalism, using correlation functions, has been presented by Buon [6], with details given in ref. [17]. His expression for sidebands of a parent betatron resonance was shown in ref. [6] to be equivalent to eq. (51), but the expression for sidebands of parent synchrotron resonances looks different from eq. (93). I have proved [18] that his solution is equivalent to eq. (93), and give the main points below. Buon's expression for the enhancement factor for synchrotron parent resonances is

$$\begin{aligned} C = \frac{(\delta^2 - Q_s^2)^2}{4\delta^2} \quad \Bigg\{ & \sum_{\eta=\pm1} \frac{e^{-x}}{(\delta + (k+\eta)Q_s)^2} \left\{ I_k \left[k^2 + (k-\eta)\frac{\delta}{Q_s} \right] + \frac{x}{2}(I_{k-1} + I_{k+1}) \right\} \\ & + \frac{e^{-x}}{(\delta + kQ_s)^2} \Bigg\{ (k-1)I_{k-1} \left[k^2 - k + (k-2)\frac{\delta}{Q_s} \right] \\ & \qquad - (k+1)I_{k+1} \left[k^2 + k + (k+2)\frac{\delta}{Q_s} \right] - 2xI_k \Bigg\} \Bigg\}, \end{aligned} \tag{94}$$

The argument α of the Bessel functions has been omitted, and Buon uses C instead of F and x instead of α. He derives C via a depolarization formalism (eq. (2)) and defines [6]

$$C = \frac{[\tau_d]_{1^{st} \text{ order}}}{\tau_d} . \tag{95}$$

The Bessel function recursion relations

$$\begin{aligned} I_\mu(z) &= \frac{z}{2\mu}[I_{\mu-1}(z) - I_{\mu+1}(z)] \\ \frac{dI_\mu}{dz} \equiv I'_\mu(z) &= \frac{1}{2}[I_{\mu-1}(z) + I_{\mu+1}(z)] , \end{aligned} \tag{96}$$

which are valid for real μ and complex z, are useful in the manipulations below. Dropping the prefactor of $(\delta^2 - Q_s^2)^2/(4\delta^2)$, one sees that

$$\begin{aligned} F \quad \propto & \quad \sum_{\eta=\pm1} \frac{e^{-\alpha}}{(\delta + (k+\eta)Q_s)^2} \left\{ I_k \left[k^2 + (k-\eta)\frac{\delta + (k+\eta)Q_s}{Q_s} - (k^2 - \eta^2) \right] + \alpha I'_k \right\} \\ & + \frac{e^{-\alpha}}{(\delta + kQ_s)^2} \left\{ \cdots \right\} \\ = & \quad \sum_{\eta=\pm1} \frac{e^{-\alpha}}{(\delta + (k+\eta)Q_s)^2} \left\{ I_k \left[1 + (k-\eta)\frac{\delta + (k+\eta)Q_s}{Q_s} \right] + \alpha I'_k \right\} \\ & + \frac{e^{-\alpha}}{(\delta + kQ_s)^2} \left\{ \cdots \right\} \\ = & \quad \frac{e^{-\alpha}}{(\delta + kQ_s)^2} \left\{ I_{k-1} \left[1 + (k-2)\frac{\Delta}{Q_s} \right] + \alpha I'_{k-1} + I_{k+1} \left[1 + (k+2)\frac{\Delta}{Q_s} \right] + \alpha I'_{k+1} \right. \\ & \qquad + (k-1)I_{k-1} \left[k^2 - k + (k-2)\frac{\Delta}{Q_s} - k(k-2) \right] \end{aligned}$$

$$
\begin{aligned}
& \qquad -(k+1)I_{k+1}\left[k^2+k+(k+2)\frac{\Delta}{Q_s}-k(k+2)\right]-2\alpha I_k\Bigg\} \\
&= \frac{e^{-\alpha}}{\Delta^2}\Bigg\{I_{k-1}\left[1+(k-2)\frac{\Delta}{Q_s}+k(k-1)+(k-1)(k-2)\frac{\Delta}{Q_s}\right] \\
& \qquad +I_{k-1}\left[1+(k+2)\frac{\Delta}{Q_s}+k(k+1)-(k+1)(k+2)\frac{\Delta}{Q_s}\right] \\
& \qquad +\left[\alpha I_k+(k-1)I_{k-1}\right]+\left[\alpha I_k-(k+1)I_{k+1}\right]-2\alpha I_k\Bigg\} \\
&= \frac{e^{-\alpha}}{\Delta^2}\left\{I_{k-1}\left[k^2+k(k-2)\frac{\Delta}{Q_s}\right]+I_{k+1}\left[k^2-k(k+2)\frac{\Delta}{Q_s}\right]\right\}. \qquad (97)
\end{aligned}
$$

The above terms have been collected so as to have a common resonance denominator $\Delta = \delta + kQ_s$. Other authors [3,4,5] collect terms so as to have a common Bessel function $I_k(\alpha)$, which has also been done in eq. (93). Let us collect terms in this way. I define $\Delta_{\pm 1} = \delta + (k \pm 1)Q_s$, which yields

$$
\begin{aligned}
F &\propto e^{-\alpha}\sum_k \frac{1}{\Delta^2}\left\{I_{k-1}\left[k^2+k(k-2)\frac{\Delta}{Q_s}\right]+I_{k+1}\left[k^2-k(k+2)\frac{\Delta}{Q_s}\right]\right\} \\
&= e^{-\alpha}\sum_k I_k\left\{\frac{1}{\Delta_1^2}\left[(k+1)^2+(k+1)(k-1)\frac{\Delta_1}{Q_s}\right]+\frac{1}{\Delta_{-1}^2}\left[(k-1)^2-(k-1)(k+1)\frac{\Delta_{-1}}{Q_s}\right]\right\} \\
&= e^{-\alpha}\sum_k \frac{I_k}{\Delta_1^2\Delta_{-1}^2}\left\{(k+1)^2\Delta_{-1}^2+(k-1)^2\Delta_1^2+(k^2-1)\left(\frac{\Delta_1\Delta_{-1}^2}{Q_s}-\frac{\Delta_{-1}\Delta_1^2}{Q_s}\right)\right\} \\
&= \ldots\left\{(k+1)^2\Delta_{-1}^2+(k-1)^2\Delta_1^2+(k^2-1)\Delta_1\Delta_{-1}\left(\frac{\Delta_{-1}}{Q_s}-\frac{\Delta_1}{Q_s}\right)\right\} \\
&= \ldots\left\{(k+1)^2\Delta_{-1}^2+(k-1)^2\Delta_1^2+(k^2-1)\Delta_1\Delta_{-1}\left(\frac{\delta}{Q_s}+k-1-\frac{\delta}{Q_s}-(k+1)\right)\right\} \\
&= \ldots\left\{(k+1)^2\Delta_{-1}^2+(k-1)^2\Delta_1^2-2(k-1)(k+1)\Delta_1\Delta_{-1}\right\} \\
&= \ldots\left[(k+1)\Delta_{-1}-(k-1)\Delta_1\right]^2 \\
&= \sum_k e^{-\alpha}I_k\left(\frac{k+1}{\Delta_1}-\frac{k-1}{\Delta_{-1}}\right)^2 \\
&= \sum_k e^{-\alpha}I_k\left(\frac{k+1}{\delta+(k+1)Q_s}-\frac{k-1}{\delta+(k-1)Q_s}\right)^2 \\
&= 4\delta^2\sum_k \frac{e^{-\alpha}I_k}{[(\delta+kQ_s)^2-Q_s^2]^2}, \\
F &= (\delta^2-Q_s^2)^2\sum_k \frac{e^{-\alpha}I_k}{[(\delta+kQ_s)^2-Q_s^2]^2}, \qquad (98)
\end{aligned}
$$

which is exactly eq. (93). In the last step I introduced the previously omitted prefactor of $(\delta^2 - Q_s^2)^2/(4\delta^2)$. Thus this formalism agrees with the others, too.

8 Conclusions

I have studied the synchrotron sideband resonances of both betatron and synchrotron parent resonances, and quoted results from several authors [3] – [7]. I have also used a tune modulation approach to derive results for sidebands of a betatron parent resonance, and I showed how the effects of orbital chromaticity could be included using this approach. I have shown that, for synchrotron sideband spin resonances, one can use any of the above formalisms, and will obtain the same answer (provided enough terms are kept). The various formalisms all use perturbation theory, and they sum the terms in different ways, and the results look different, but in fact they are all equivalent when one makes the same approximations in all of them. In particular, I have assumed that the synchrotron tune is much less than unity ($Q_s \ll 1$), the ring is approximately planar ($\nu \simeq \gamma a$). In practice, $Q_s \simeq 0.1$ for LEP, but $Q_s \ll 1$ for most lower energy storage rings, and so it is a good approximation for most storage rings. It is also true that most storage rings are planar, and so the approximation $\nu \simeq \gamma a$ is also valid in general. Throughout most of this paper I have treated only a single parent betatron resonance ($\nu = -n + Q_{x,y}$) or only one doublet parent synchrotron resonance ($\nu = -n \pm Q_s$). This approximation for the parent resonances is actually not required by any of the above formalisms, but was made for convenience, to avoid obscuring the mathematics with too many terms. Note that for the solutions involving Bessel functions, it is not necessary for the argument of the Bessel functions to be much less than unity. The solutions are valid for arbitrary values of the argument. In terms of a power series expansion [7], larger arguments mean more terms are required to obtain a satisfactorily convergent answer, but the series will still eventually converge to the Bessel function solution.

The reader should realize that the individual formalisms do not all require the synchrotron tune to be small, or that the ring should be approximately planar, or that the resonances should be isolated. It is only necessary to assume $Q_s \ll 1$ in those formalisms which use tune modulation, i.e. refs. [3], [5], and the tune modulation derivation above. The calculations in refs. [4], [6], [7] and [17] sometimes approximate $Q_s \ll 1$ to derive explicit analytical results, but the *formalisms* may be applicable to more general models. Unfortunately, explicit analytical results become much more difficult to write down if we do not assume the synchrotron tune is much less than unity. The transverse orbital normal modes do not have to be horizontal and vertical betatron oscillations; linear transverse coupling is permissible. In fact, the results for linear transverse coupling would look formally the same as those in this paper, by reinterpreting Q_x and Q_y as the tunes of the transverse normal modes. The detailed expressions for the integrands in the calculations above, in terms of the lattice functions of the storage ring, would of course become more complicated.

For sidebands of a betatron parent resonance, I have shown that Yokoya's formula [4] for synchrotron sidebands of a first order betatron spin resonance, together with some terms he neglected (eq. (53)), Buon's formula [6], and the SMILE formalism [7] (eqs. (72), (80) and (84)), are equivalent. Strictly speaking, only the leading contribution (in powers of the beam emittances) to each resonance was retained. A few nonleading terms were calculated to prove that they were also in agreement, and to show how to derive the remaining terms. The theory was used to fit certain data from SPEAR polarization measurements [2], specifically the ratio of the widths of the resonances $\nu = 3 + Q_x$ and $\nu = 3 + Q_x - Q_s$, and a graph through the sidebands from $\nu = 3 + Q_x - 2Q_s$ to $\nu = 3 + Q_x + 2Q_s$. The agreement was reasonable. For sidebands of a parent synchrotron doublet, there are more formulas [3] – [6], but they all agree. Unfortunately, there are no experimental results to test this aspect of the theory.

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