# The Vector Potential In Accelerator Magnets 

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October 1989

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 Brookhaven National Laboratory
## U.S. Department of Energy <br> USDOE Office of Science (SC)

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Upton, New York 11973
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Technical Note
AGS/AD/Tech. Note No. 331

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October 11, 1989

## 1 Introduction

The standard treatment of the static magnetic field in an accelerator involves expanding the field about the reference orbit and expressing it in terms of normal and skew multipoles, which arise in regions where the field is transverse to the reference orbit, and in terms of other non-multipole terms which arise in regions where the field has longitudinal as well as transverse components or where the curvature of the reference orbit is nonzero. The coefficients in the expansion depend only on the values of the field and its derivatives on the reference orbit, and are essentially what we measure and use to characterize the various magnetic elements in an accelerator.

The magnetic vector potential, which is of interest by virtue of its explicit appearance in the hamiltonian for particle motion in the accelerator, may also be expanded about the reference orbit. The coefficients in this expansion will depend on the coefficients in the magnetic field expansion but are not completely determined by them owing to the fact that one can always add the gradient of an arbitrary scalar function to the vector potential without changing the magnetic field. The general expansion of the vector potential about the reference orbit will therefore contain terms which do not contribute to the magnetic field and which, although they appear explicitly in the hamiltonian, do not contribute to the equations of motion. These unnecessary terms can be eliminated by making an appropriate gauge transformation. The coefficients in the expansion of the vector potential will then be completely determined by the coefficients in the magnetic field expansion.
In regions where the magnetic field is transverse to the reference orbit-typically far inside the accelerator magnets away from the magnet
ends-the choice of an appropriate gauge transformation is straight forward. Here one can find a gauge transformation such that the vector potential has no transverse components. In this case the vector potential is completely determined except for the addition of the gradient of an arbitrary function of the position along the reference orbit which may be taken to be zerc. The coefficients in the expansion of the vector potential about the reference orbit will then be completely determined by the coefficients in the expansion of the magnetic field.

Near the ends of the magnets the magnetic field has longitudinal components and the vector potential must therefore have transverse components. The choice of an appropriate gauge transformation in these and other regions where the magnetic field has longitudinal components is the subject of this report. It will be shown that a gauge transformation can be found such that the vector potential has the required transverse components near the magnet ends, has no transverse components in regions where the magnetic field is transverse to the reference orbit, and has expansion coefficients which are completely determined by the coefficients in the expansion of the magnetic field.
The resulting expression for the vector potential may be of use in small accelerators or storage rings with large acceptances where it has been shown that the effects of the magnet end-fields can become important. These effects are discussed in references [2-5].

## 2 Expansion of the Magnetic Field about the Reference Orbit

Here we follow the treatment of the magnetic field given by K. L. Brown and R. V. Servranckx in reference [1].
Inside the aperture of an accelerator the curl of the static magnetic field, $\mathbf{B}$, is zero which implies that $\mathbf{B}$ can be expressed as the gradient of a scalar potential, $\phi$. Thus

$$
\begin{equation*}
\mathbf{B}=\nabla \phi, \tag{1}
\end{equation*}
$$

and since the divergence of $\mathbf{B}$ must also vanish we have

$$
\begin{equation*}
\nabla^{2} \phi=0 . \tag{2}
\end{equation*}
$$

The desired expansion of the magnetic field can therefore be obtained by
first expanding $\phi$ about the reference orbit and then inserting this result into equation (1).
We shall assume that the reference orbit lies in a plane and shall employ the right-handed curvilinear coordinate system ( $x, y, s$ ) introduced in reference [1]. At each point along the reference orbit the three mutually perpendicular unit vectors $\mathbf{x}, \mathrm{y}$, and s are defined as follows: s is the unit vector tangent to the reference orbit; $x$ is the unit vector which lies in the plane of the orbit, is perpendicular to $s$, and points outward from the reference orbit; $y$ is the unit vector perpendicular to the plane of the orbit and pointing upward. The coordinate $s$ is the distance along the reference orbit measured from some reference point, and the coordinates $x$ and $y$ are the distances from the reference orbit in the $\mathbf{x}$ and $\mathbf{y}$ directions.
In terms of these coordinates we have

$$
\begin{equation*}
\nabla \phi=\mathbf{x} \frac{\partial \phi}{\partial x}+\mathbf{y} \frac{\partial \phi}{\partial y}+\mathbf{s}\left(\frac{1}{1+h x}\right) \frac{\partial \phi}{\partial s} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\nabla^{2} \phi=\frac{1}{1+h x} \frac{\partial}{\partial x}\left[(1+h x) \frac{\partial \phi}{\partial x}\right]+\frac{\partial^{2} \phi}{\partial y^{2}}+\frac{1}{1+h x} \frac{\partial}{\partial s}\left[\frac{1}{1+h x} \frac{\partial \phi}{\partial s}\right] \tag{4}
\end{equation*}
$$

where $h=h(s)$ is the curvature of the reference orbit.
Now for each $s$ along the reference orbit we can expand the potential, $\phi$, as follows:

$$
\begin{equation*}
\phi(x, y, s)=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m n} \frac{x^{n}}{n!} \frac{y^{m}}{m!} \tag{5}
\end{equation*}
$$

where the coefficients, $C_{m n}$, are functions of s. Then putting this expansion into equation (4) we find the following recursion relation for the coefficients $C_{m n}$ :

$$
\begin{align*}
-C_{m+2, n} & =C_{m n}^{\prime \prime}+n h C_{m, n-1}^{\prime \prime}-n h^{\prime} C_{m, n-1}^{\prime}+ \\
& +C_{m, n+2}+(3 n+1) h C_{m, n+1}+n(3 n-1) h^{2} C_{m n}+ \\
& +n(n-1)^{2} h^{3} C_{m, n-1}+3 n h C_{m+2, n-1}+ \\
& +3 n(n-1) h^{2} C_{m+2, n-2}+n(n-1)(n-2) h^{3} C_{m+2, n-3} \tag{6}
\end{align*}
$$

where a prime denotes differentiation with respect to $s$ and all coefficients with one or more negative subscripts are, by definition, zero. Putting (5)
into equations (3) and (1) we find that the components of $\mathbf{B}$ in the $\mathbf{x}, \mathbf{y}$, and s directions are respectively

$$
\begin{align*}
B_{x} & =\frac{\partial \phi}{\partial x}=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m, n+1} \frac{x^{n}}{n!} \frac{y^{m}}{m!} \\
B_{y} & =\frac{\partial \phi}{\partial y}=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m+1, n} \frac{x^{n}}{n!} \frac{y^{m}}{m!} \\
B_{s} & =\frac{1}{1+h x} \frac{\partial \phi}{\partial s}=\frac{1}{1+h x} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m, n}^{\prime} \frac{x^{n}}{n!} \frac{y^{m}}{m!} \tag{7}
\end{align*}
$$

This is the desired expansion of the magnetic field about the reference orbit.
Differentiating $B_{y}$ and $B_{x}$ with respect to $x$ we find

$$
\begin{equation*}
C_{1 n}=\left(\frac{\partial^{n} B_{y}}{\partial x^{n}}\right)_{x=0, y=0}, \quad C_{10}=B_{y}(0,0, s) \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{0, n+1}=\left(\frac{\partial^{n} B_{x}}{\partial x^{n}}\right)_{x=0, y=0}, \quad C_{01}=B_{x}(0,0, s) \tag{9}
\end{equation*}
$$

which are the normal and skew multipole strengths, respectively, evaluated on the reference orbit. The longitudinal component of the magnetic field on the reference orbit is given by

$$
\begin{equation*}
C_{00}^{\prime}=B_{s}(0,0, s) \tag{10}
\end{equation*}
$$

It follows from the recursion relation (6) that all of the coefficients in the magnetic field expansion can be expressed in terms of these multipole and longitudinal field strengths and their derivatives with respect to $s$.
Equations (7) then show that the field at any point for which the expansion is valid is completely determined by the multipole and longitudinal field strengths on the reference orbit. Defining

$$
\begin{equation*}
B_{n}=C_{1 n}, \quad A_{n}=C_{0, n+1}, \quad C_{0}=C_{00}^{\prime} \tag{11}
\end{equation*}
$$

we find that the coefficients required to expand the field to third order are given by (8-11) and

$$
C_{20}=-A_{1}-h A_{0}-C_{0}^{\prime}
$$

$$
\begin{align*}
C_{21}= & -A_{2}-h A_{1}+h^{2} A_{0}-A_{0}^{\prime \prime}+2 h C_{0}^{\prime}+h^{\prime} C_{0} \\
C_{22}= & -A_{3}-h A_{2}+2 h^{2} A_{1}-2 h^{3} A_{0} \\
& -A_{1}^{\prime \prime}+4 h A_{0}^{\prime \prime}+2 h^{\prime} A_{0}^{\prime}-6 h^{2} C_{0}^{\prime}-6 h h^{\prime} C_{0} \\
C_{30}= & -B_{2}-h B_{1}-B_{0}^{\prime \prime} \\
C_{31}= & -B_{3}-h B_{2}+h^{2} B_{1}-B_{1}^{\prime \prime}+2 h B_{0}^{\prime \prime}+h^{\prime} B_{0}^{\prime} \\
C_{40}= & -C_{22}-h C_{21}-C_{20}^{\prime \prime} . \tag{12}
\end{align*}
$$

## 3 Expansion of the Vector Potential about the Reference Orbit

Since its divergence vanishes, the magnetic field can be expressed as the curl of a vector potential, A. Thus

$$
\begin{equation*}
\mathbf{B}=\nabla \times \mathbf{A} \tag{13}
\end{equation*}
$$

which, in terms of the curvilinear coordinate system introduced in the previous section, becomes

$$
\begin{align*}
B_{x} & =\frac{\partial A_{s}}{\partial y}-\frac{1}{1+h x} \frac{\partial A_{y}}{\partial s} \\
B_{y} & =\frac{1}{1+h x}\left(\frac{\partial A_{x}}{\partial s}-h A_{s}\right)-\frac{\partial A_{s}}{\partial x} \\
B_{s} & =\frac{\partial A_{y}}{\partial x}-\frac{\partial A_{x}}{\partial y} \tag{14}
\end{align*}
$$

Here, as before, the subscripts $x, y$, and $s$ denote the components of the vectors in the $\mathbf{x}, \mathbf{y}$, and $\mathbf{s}$ directions.

Expanding $A_{x}, A_{y}$, and $A_{s}$ about the reference orbit we have

$$
\begin{align*}
& A_{x}=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m n} \frac{x^{n}}{n!} \frac{y^{m}}{m!} \\
& A_{y}=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m n} \frac{x^{n}}{n!} \frac{y^{m}}{m!} \\
& A_{s}=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{m n} \frac{x^{n}}{n!} \frac{y^{m}}{m!} . \tag{15}
\end{align*}
$$

where $a_{m n}, b_{m n}$, and $d_{m n}$ are functions of $s$.
Inserting (15) into (14) and comparing with (7) we find the following relations between the magnetic field and vector potential expansion coefficients:

$$
\begin{gather*}
C_{m, n+1}+n h C_{m n}=d_{m+1, n}-b_{m n}^{\prime}+n h d_{m+1, n-1} \\
C_{m+1, n}+n h C_{m+1, n-1}=a_{m n}^{\prime}-d_{m, n+1}-(1+n) h d_{m n} \\
C_{m n}^{\prime}=b_{m, n+1}-a_{m+1, n}+n h b_{m n}-n h a_{m+1, n-1} \tag{16}
\end{gather*}
$$

These relations alone are not enough to completely determine the coefficients $a_{m n}, b_{m n}$, and $d_{m n}$. We can remedy this situation by making an appropriate gauge transformation.

## 4 Choosing a Gauge Transformation

Since we can add the gradient of a scalar function, $f$, to the vector potential, $\mathbf{A}$, without changing the magnetic field, the general expansion of A about the reference orbit is of the form

$$
\begin{equation*}
\mathbf{A}=\mathbf{V}+\nabla f \tag{17}
\end{equation*}
$$

where

$$
\begin{align*}
& V_{x}=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{m n} \frac{x^{n}}{n!} \frac{y^{m}}{m!} \\
& V_{y}=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} v_{m n} \frac{x^{n}}{n!} \frac{y^{m}}{m!} \\
& V_{s}=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} w_{m n} \frac{x^{n}}{n!} \frac{y^{m}}{m!} \tag{18}
\end{align*}
$$

and

$$
\begin{equation*}
f=\sum_{m=0}^{\infty} \sum_{m=0}^{\infty} f_{m n} \frac{x^{n}}{n!} \frac{y^{m}}{m!} \tag{19}
\end{equation*}
$$

Thus, using (18) and (19) in (17) and comparing with (15) we find

$$
a_{m n}=u_{m n}+f_{m, n+1}
$$

$$
\begin{gather*}
b_{m n}=v_{m n}+f_{m+1, n} \\
d_{m n}+n h d_{m, n-1}=w_{m n}+n h w_{m, n-1}+f_{m n}^{\prime} . \tag{20}
\end{gather*}
$$

Now, since $f$ is arbitrary we may choose

$$
\begin{equation*}
f_{m, n+1}=-u_{m n}, \quad f_{m+1,0}=-v_{m 0}, \quad f_{00}^{\prime}=-w_{00} \tag{21}
\end{equation*}
$$

in which case we have

$$
\begin{equation*}
a_{m n}=0, \quad b_{m 0}=0, \quad d_{00}=0, \tag{22}
\end{equation*}
$$

and equations (16) become

$$
\begin{gather*}
C_{m, n+1}+n h C_{m n}=d_{m+1, n}-b_{m n}^{\prime}+n h d_{m+1, n-1} \\
C_{m+1, n}+n h C_{m+1, n-1}=-d_{m, n+1}-(1+n) h d_{m n} \\
C_{m n}^{\prime}=b_{m, n+1}+n h b_{m n} \tag{23}
\end{gather*}
$$

These relations completely determine the nonzero coefficients $b_{m n}$ and $d_{m n}$. Solving (23) one obtains these coefficients entirely in terms of the coefficients, $C_{m n}$, which in turn can be expressed in terms of the multipole and longitudinal field strengths defined by equations (8-11). Using (11-12) and (22) in (23) we find

$$
\begin{align*}
& d_{01}=-B_{0} \\
& d_{02}=-B_{1}+h B_{0} \\
& d_{03}=-B_{2}+h B_{1}-3 h^{2} B_{0} \\
& d_{04}=-B_{3}+h B_{2}-4 h^{2} B_{1}+12 h^{3} B_{0} \\
& d_{10}=A_{0} \\
& d_{11}=A_{1}+C_{0}^{\prime} \\
& d_{12}=A_{2}+A_{0}^{\prime \prime}-3 h C_{0}^{\prime}-h^{\prime} C_{0} \\
& d_{13}=A_{3}+A_{1}^{\prime \prime}-5 h A_{0}^{\prime \prime}-2 h^{\prime} A_{0}^{\prime}+11 h^{2} C_{0}^{\prime}+7 h h^{\prime} C_{0} \\
& d_{20}=B_{1} \\
& d_{21}=B_{2}+B_{0}^{\prime \prime} \\
& d_{22}=B_{3}+B_{1}^{\prime \prime}-3 h B_{0}^{\prime \prime}-h^{\prime} B_{0}^{\prime} \\
& d_{30}=-A_{2}-h A_{1}+h^{2} A_{0}-A_{0}^{\prime \prime}+2 h C_{0}^{\prime}+h^{\prime} C_{0} \\
& d_{31}=-A_{3}-h A_{2}+2 h^{2} A_{1}-2 h^{3} A_{0} \\
&-2 A_{1}^{\prime \prime}+3 h A_{0}^{\prime \prime}-h^{\prime \prime} A_{0}-C_{0}^{\prime \prime \prime}-6 h^{2} C_{0}^{\prime}-6 h h^{\prime} C_{0} \\
& d_{40}=-B_{3}-h B_{2}+h^{2} B_{1}-B_{1}^{\prime \prime}+2 h B_{0}^{\prime \prime}+h^{\prime} B_{0}^{\prime}, \tag{24}
\end{align*}
$$

and

$$
\begin{align*}
& b_{01}=C_{0} \\
& b_{02}=A_{0}^{\prime}-h C_{0} \\
& b_{03}=A_{1}^{\prime}-2 h A_{0}^{\prime}+2 h^{2} C_{0} \\
& b_{04}=A_{2}^{\prime}-3 h A_{1}^{\prime}+6 h^{2} A_{0}^{\prime}-6 h^{3} C_{0} \\
& b_{11}=B_{0}^{\prime} \\
& b_{12}=B_{1}^{\prime}-h B_{0}^{\prime} \\
& b_{13}=B_{2}^{\prime}-2 h B_{1}^{\prime}+2 h^{2} B_{0}^{\prime} \\
& b_{21}=-A_{1}^{\prime}-h^{\prime} A_{0}-h A_{0}^{\prime}-C_{0}^{\prime \prime} \\
& b_{22}=-A_{2}^{\prime}-h^{\prime} A_{1}+3 h h^{\prime} A_{0}+2 h^{2} A_{0}^{\prime}-A_{0}^{\prime \prime \prime}+3 h^{\prime} C_{0}^{\prime}+3 h C_{0}^{\prime \prime}+h^{\prime \prime} C_{0} \\
& b_{31}=-B_{2}^{\prime}-h^{\prime} B_{1}-h B_{1}^{\prime}-B_{0}^{\prime \prime \prime} . \tag{25}
\end{align*}
$$

Continuing in this way one obtains all of the coefficients $b_{m n}$ and $d_{m n}$ in terms of the multipole and longitudinal field strengths and their derivatives with respect to $s$. For the case in which the magnetic field is transverse to the reference orbit, the coefficients $C_{m n}^{\prime}$ are zero and it follows from (22) and the last of equations (23) that the coefficients $b_{m n}$ are all zero. The vector potential then has only the longitudinal component $A_{s}$. In regions where the coefficients $C_{m n}^{\prime}$ are nonzero the magnetic field has a longitudinal component, the coefficients $b_{m n}$ are nonzero, and the vector potential has a transverse component $A_{y}$.

The components of the vector potential to fourth order can be obtained by inserting (22) and (24-25) into (15). Collecting terms we find

$$
\begin{equation*}
A_{s}=A_{s}^{a}+A_{s}^{b}, \tag{26}
\end{equation*}
$$

where (to fourth order)

$$
\begin{aligned}
A_{s}^{a}= & -B_{0}\left[x-\frac{h}{2} x^{2}+\frac{h^{2}}{2} x^{3}-\frac{h^{3}}{2} x^{4}\right] \\
& -B_{1}\left[\frac{1}{2}\left(x^{2}-y^{2}\right)-\frac{h}{6} x^{3}+\frac{h^{2}}{24}\left(4 x^{4}-y^{4}\right)\right] \\
& -B_{2}\left[\frac{1}{6}\left(x^{3}-3 x y^{2}\right)-\frac{h}{24}\left(x^{4}-y^{4}\right)\right] \\
& -B_{3}\left[\frac{1}{24}\left(x^{4}-6 x^{2} y^{2}+y^{4}\right)\right]+
\end{aligned}
$$

$$
\begin{align*}
& -A_{0}\left[-y-\frac{h^{2}}{6} y^{3}+\frac{h^{3}}{3} x y^{3}\right] \\
& -A_{1}\left[-x y+\frac{h}{6} y^{3}-\frac{h^{2}}{3} x y^{3}\right] \\
& -A_{2}\left[\frac{1}{6}\left(y^{3}-3 x^{2} y\right)+\frac{h}{6} x y^{3}\right] \\
& -A_{3}\left[\frac{1}{6}\left(x y^{3}-x^{3} y\right)\right], \tag{27}
\end{align*}
$$

and

$$
\begin{align*}
A_{s}^{b}= & -B_{0}^{\prime \prime}\left[-\frac{1}{2} x y^{2}+\frac{h}{12}\left(9 x^{2} y^{2}-y^{4}\right)\right]-B_{0}^{\prime} h^{\prime}\left[\frac{1}{24}\left(6 x^{2} y^{2}-y^{4}\right)\right] \\
& -B_{1}^{\prime \prime}\left[\frac{1}{24}\left(y^{4}-6 x^{2} y^{2}\right)\right] \\
& -A_{0}^{\prime \prime}\left[\frac{1}{6}\left(y^{3}-3 x^{2} y\right)+\frac{h}{6}\left(5 x^{3} y-3 x y^{3}\right)\right] \\
& -A_{0}^{\prime} h^{\prime}\left[\frac{1}{3} x^{3} y\right]-A_{0} h^{\prime \prime}\left[\frac{1}{6} x y^{3}\right] \\
& -A_{1}^{\prime \prime}\left[\frac{1}{6}\left(2 x y^{3}-x^{3} y\right)\right] \\
& -C_{0}^{\prime}\left[-x y-\frac{h}{6}\left(2 y^{3}-9 x^{2} y\right)+\frac{h^{2}}{6}\left(6 x y^{3}-11 x^{3} y\right)\right] \\
& -C_{0} h^{\prime}\left[\frac{1}{6}\left(3 x^{2} y-y^{3}\right)+\frac{h}{6}\left(6 x y^{3}-7 x^{3} y\right)\right]-C_{0}^{\prime \prime \prime}\left[\frac{1}{6} x y^{3}\right] \tag{28}
\end{align*}
$$

and

$$
\begin{aligned}
A_{y}= & h^{\prime} A_{0}\left[-\frac{x y^{2}}{2}+3 h \frac{x^{2} y^{2}}{4}\right]-A_{0}^{\prime \prime \prime}\left[\frac{x^{2} y^{2}}{4}\right] \\
& +A_{0}^{\prime}\left[\frac{x^{2}}{2}-\frac{h}{6}\left(2 x^{3}+3 x y^{2}\right)+\frac{h^{2}}{4}\left(x^{4}+2 x^{2} y^{2}\right)\right] \\
& +A_{1}^{\prime}\left[\frac{1}{6}\left(x^{3}-3 x y^{2}\right)-\frac{h}{8} x^{4}\right]+h^{\prime} A_{1}\left[-\frac{x^{2} y^{2}}{4}\right] \\
& +A_{2}^{\prime}\left[\frac{1}{24}\left(x^{4}-6 x^{2} y^{2}\right)\right]+
\end{aligned}
$$

$$
\begin{align*}
& +B_{0}^{\prime}\left[x y-h \frac{x^{2} y}{2}+2 h^{2} \frac{x^{3} y}{6}\right]-B_{0}^{\prime \prime \prime}\left[\frac{x y^{3}}{6}\right] \\
& +B_{1}^{\prime}\left[\frac{x^{2} y}{2}-\frac{h}{6}\left(2 x^{3} y+x y^{3}\right)\right]-h^{\prime} B_{1}\left[\frac{x y^{3}}{6}\right] \\
& +B_{2}^{\prime}\left[\frac{1}{6}\left(x^{3} y-x y^{3}\right)\right] \\
& +C_{0}^{\prime}\left[3 h^{\prime} \frac{x^{2} y^{2}}{4}\right]+C_{0}^{\prime \prime}\left[-\frac{x y^{2}}{2}+3 h \frac{x^{2} y^{2}}{4}\right] \\
& +C_{0}\left[x-h \frac{x^{2}}{2}+2 h^{2} \frac{x^{3}}{6}-6 h^{3} \frac{x^{4}}{24}+h^{\prime \prime} \frac{x^{2} y^{2}}{4}\right] . \tag{29}
\end{align*}
$$

Equations (27-29) show explicitly that in the regions where $C_{0}=0$ and the multipole and longitudinal field strengths and curvature are independent of $s$, the vector potential has only a longitudinal component and is given by $A_{s}^{a}$. Outside these regions, and in particular near the magnet ends, the vector potential has the additional terms given by $A_{s}^{b}$ and. $A_{y}$. In regions where the curvature, $h$, is zero $A_{s}^{a}$ contains only pure multipole terms.

## 5 A Quadrupole with End-Fields

Here we illustrate the use of the formulae developed in the previous sections for the case of a quadrupole of finite length. We shall assume that the reference orbit coincides with the longitudinal axis of the quadrupole so that the curvature, $h$, is zero. As discussed in reference [2], the scalar potential for a pure quadrupole is an odd function of both $x$ and $y$. The scalar and vector potentials to fourth order are therefore

$$
\begin{align*}
\phi & =B_{1} x y+B_{3} \frac{x^{3} y}{6}-\left(B_{3}+B_{1}^{\prime \prime}\right) \frac{x y^{3}}{6} \\
A_{s} & =-\frac{1}{2} B_{1}\left(x^{2}-y^{2}\right)-\frac{1}{24} B_{3}\left(x^{4}-6 x^{2} y^{2}+y^{4}\right)-\frac{1}{24} B_{1}^{\prime \prime}\left(y^{4}-6 x^{2} y^{2}\right) \\
A_{y} & =B_{1}^{\prime} \frac{x^{2} y}{2} \tag{30}
\end{align*}
$$

with the constraint that

$$
\begin{equation*}
B_{3}=-\frac{1}{2} B_{1}^{\prime \prime}, \tag{31}
\end{equation*}
$$

which follows from the additional requirement that $\phi(x, y, s)=\phi(y, x, s)$ for a pure quadrupole.
The components of the magnetic field, obtained from equations (1) and (3) or from equations (14), are then

$$
\begin{align*}
B_{x} & =B_{1} y-\frac{1}{12} B_{1}^{\prime \prime}\left(3 x^{2} y+y^{3}\right) \\
B_{y} & =B_{1} x-\frac{1}{12} B_{1}^{\prime \prime}\left(x^{3}+3 x y^{2}\right) \\
B_{s} & =B_{1}^{\prime} x y \tag{32}
\end{align*}
$$

Here we see that the longitudinal variation of $B_{1}$ near the ends of the quadrupole produces a third-order transverse field (which is not an octupole field) and a second-order longitudinal field. If the transverse coordinates of a particle do not change significantly as it passes through the quadrupole, then the net effects of the terms involving $B_{1}^{\prime}$ and $B_{1}^{\prime \prime}$ on the particle trajectory are small. This is typically the case for the quadrupoles (and other magnetic elements) in high-energy particle accelerators. In smaller accelerators at lower energies, the transverse coordinates can change significantly as the particle passes through the end-field regions and the net effects of the terms involving the longitudinal variation of $B_{1}$ can become important. These effects and the conditions under which they become important are discussed in references [2-5].
The longitudinal variation of $B_{1}$ near the ends of the quadrupole also contributes fourth-order terms to the longitudinal component of the vector potential and a third-order term to the transverse component, as seen in equations (30). These terms can become important in the excitation of transverse resonances in an accelerator if the betatron functions, $\beta_{x}$ and $\beta_{y}$, or the betatron phase advance times the order of the resonance, vary significantly along the reference trajectory as it passes through the quadrupole.

For the case of high-energy particle accelerators, the effects of the terms of the vector potential involving the variation of the multipole and longitudinal field strengths with respect to $s$ are generally unimportant.

However, in smaller accelerators at lower energies, these terms can become important in the excitation of transverse resonances. It is for these cases that the expansion of the vector potential developed in the previous section becomes useful.

## 6 References

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