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RESONANCE EXCITATION IN THE AGS BOOSTER DUE TO CLOSED ORBIT DISTORTION IN MULTIPOLE MAGNETS

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1 Introduction

A number of studies [1,2] were carried out in 1993 to determine the corrections required to reduce or eliminate beam loss due to excitation of transverse resonances in the AGS Booster. During the course of these studies it was found that the quadrupole corrections could not completely eliminate the beam loss observed as the $2Q_x = 9$ and $2Q_y = 9$ resonances were crossed. It was thought initially that an 18th harmonic octupole field might be responsible for the residual loss, but then it was found that the required corrections varied linearly with orbit radius which strongly suggested that quadrupole fields arising from the displacement of the closed orbit in sextupole fields were exciting the resonances. Beam particles whose momentum differs by δp from the central momentum oscillate obout orbits which are displaced in the sextupoles and therefore see a quadrupole field which is proportional to δp. If the sextupole field possesses a ninth harmonic, or if some combination of orbit harmonics and sextupole harmonics produce a ninth harmonic, then the quadrupole-driven resonances can be excited and this can account for the residual loss. It was found that by introducing a ninth harmonic with available sextupoles, the dependence of the required quadrupole correction on radius could be eliminated and the residual loss could be reduced substantially.

Further studies and careful analysis by Y. Shoji [2,3] showed that the skew quadrupole and sextupole corrections required to eliminate beam loss as the $Q_x + Q_y = 9$ and $3Q_x = 14$ resonances are crossed also depend linearly on the orbit radius. This again suggested that the displacement of the closed orbit in higher order multipole fields—in this case skew sextupole and octupole fields—gives rise to lower order multipoles (skew quadrupole and sextupole) which excite the lower order resonances. The purpose of this note is to work out the details of how the displacement of the closed orbit in higher order multipoles gives rise to lower order multipoles which excite lower order resonances. The treatment is an extension of that given in Ref.[4] for the case of closed orbit distortions in sextupoles. We shall assume that the reference orbit lies in a plane and shall employ the right-handed curvilinear coordinate system (**x**, **y**, **s**) introduced in Ref.[5].

2 The Multipole Vector Potential

The hamiltonian treatment given in subsequent sections requires expressions for the vector potential of a multipole magnet. Here we develop these and other formulae for use later on.

Inside a multipole magnet, far from the magnet ends, the magnetic field is transverse to the reference orbit and one can choose a gauge such that the vector potential has no transverse components. The x and y components of the magnetic field are then

$$B_x = \frac{\partial A_s}{\partial y}, \quad B_y = -\frac{\partial A_s}{\partial x} \tag{1}$$

where A_s is the longitudinal component of the vector potential. Since the curl of the magnetic field is zero we then have

$$\frac{\partial^2 A_s}{\partial x^2} + \frac{\partial^2 A_s}{\partial y^2} = 0 \tag{2}$$

which has solutions

$$A_n(z) = \frac{z^n}{n!} = \frac{1}{n!}(x+iy)^n = U_n(x,y) + iV_n(x,y) \tag{3}$$

where U_n and V_n are real functions of x and y. Differentiation of (3) with respect to x and y yields the Cauchy-Riemann equations

$$\frac{\partial U_n}{\partial x} = \frac{\partial V_n}{\partial y}, \quad \frac{\partial U_n}{\partial y} = -\frac{\partial V_n}{\partial x} \tag{4}$$

from which it follows that

$$\frac{\partial^2 U_n}{\partial x^2} + \frac{\partial^2 U_n}{\partial y^2} = 0, \quad \frac{\partial^2 V_n}{\partial x^2} + \frac{\partial^2 V_n}{\partial y^2} = 0. \tag{5}$$

Thus both $U_n(x,y)$ and $V_n(x,y)$ are real solutions of (2) and we call them respectively the normal and skew $2n$-pole vector potentials. Expanding $(x+iy)^n$ in (3) we find the following normal and skew vector potentials:

Dipole:

$$U_1 = x, \quad V_1 = y \tag{6}$$

Quadrupole:

$$U_2 = \frac{1}{2}(x^2 - y^2), \quad V_2 = xy \tag{7}$$

Sextupole:

$$U_3 = \frac{1}{6}(x^3 - 3xy^2), \quad V_3 = -\frac{1}{6}(y^3 - 3x^2y) \tag{8}$$

Octupole:

$$U_4 = \frac{1}{24}(x^4 - 6x^2y^2 + y^4), \quad V_4 = \frac{1}{24}(4x^3y - 4xy^3) \tag{9}$$

Decapole:

$$U_5 = \frac{1}{120}(x^5 - 10x^3y^2 + 5xy^4), \quad V_5 = \frac{1}{120}(5x^4y - 10x^2y^3 + y^5). \tag{10}$$

Differentiation of (3) with respect to x and y also yields the relations

$$\frac{\partial U_n}{\partial x} = U_{n-1}, \quad \frac{\partial V_n}{\partial x} = V_{n-1}, \quad \frac{\partial U_n}{\partial y} = -V_{n-1}, \quad \frac{\partial V_n}{\partial y} = U_{n-1}. \tag{11}$$

Expanding $(z+d)^n$, where $d = a + ib$, we obtain

$$\frac{(z+d)^n}{n!} = \frac{z^n}{n!} + \frac{dz^{n-1}}{(n-1)!} + \frac{d^2z^{n-2}}{2!(n-2)!} + \cdots + \frac{d^{n-1}z}{(n-1)!} + \frac{d^n}{n!}, \tag{12}$$

which may be written as

$$\begin{aligned} A_n(z+d) &= A_n(z) + dA_{n-1}(z) + \frac{1}{2!}d^2A_{n-2}(z) + \cdots \\ &+ \frac{1}{2!}z^2A_{n-2}(d) + zA_{n-1}(d) + A_n(d). \end{aligned} \tag{13}$$

Expressing each A_m as $U_m + iV_m$ we then have

$$\begin{aligned} U_n(x+a, y+b) &= U_n(x,y) + aU_{n-1}(x,y) - bV_{n-1}(x,y) \\ &+ \frac{1}{2}(a^2 - b^2)U_{n-2}(x,y) - abV_{n-2}(x,y) + \cdots \\ &+ \frac{1}{2}(x^2 - y^2)U_{n-2}(a,b) - xyV_{n-2}(a,b) \\ &+ xU_{n-1}(a,b) - yV_{n-1}(a;b) + U_n(a,b) \qquad (14) \end{aligned}$$

and

$$\begin{aligned} V_n(x+a, y+b) &= V_n(x,y) + aV_{n-1}(x,y) + bU_{n-1}(x,y) \\ &+ \frac{1}{2}(a^2 - b^2)V_{n-2}(x,y) + abU_{n-2}(x,y) + \cdots \\ &+ \frac{1}{2}(x^2 - y^2)V_{n-2}(a,b) + xyU_{n-2}(a,b) \\ &+ xV_{n-1}(a,b) + yU_{n-1}(a,b) + V_n(a,b). \qquad (15) \end{aligned}$$

3 Hamiltonian for Oscillations about the Reference Trajectory

The hamiltonian for oscillations about the reference trajectory is [6]

$$H = -(1+hx)\{1 - (P_x - U)^2 - (P_y - V)^2\}^{1/2} - (1+hx)W \qquad (16)$$

where

$$U = \frac{eA_x}{cp}, \quad V = \frac{eA_y}{cp}, \quad W = \frac{eA_s}{cp}, \quad P_x = \frac{p_x}{p}, \quad P_y = \frac{p_y}{p}, \qquad (17)$$

A_x, A_y, A_s are the components of the vector potential, p_x and p_y are the components of the momentum along **x** and **y**, and p is the particle momentum. The momentum of the reference particle is p_0. We shall assume that the effects of any longitudinal magnetic fields (typically near the ends of magnets) can be neglected, in which case the transverse components of the vector potential may be set to zero. The longitudinal component of the vector potential is then

$$A_s = Ay - Bx + Bhx^2/2 - B_1(x^2 - y^2)/2 + C_n F_n(x,y) \qquad (18)$$

where A, B, and B_1 are the values of B_x, B_y, and $\partial B_y/\partial x$ along the reference trajectory,

$$h = \frac{eB_0}{cp_0}, \tag{19}$$

and B_0 is the value of the vertical guide field for which a particle of momentum p_0 follows the reference trajectory. The departure of B from B_0 and of A from 0 are the errors in the vertical and horizontal fields along the reference trajectory. $C_nF_n(x,y)$ is the vector potential for the multipole under consideration.

Thus we have

$$W = \frac{p_0}{p}\left\{gy - \frac{B}{B_0}(hx - \frac{1}{2}h^2x^2) + \frac{1}{2}nh^2(x^2 - y^2) + K_nF_n(x,y)\right\}, \tag{20}$$

where

$$g = \frac{eA}{cp_0}, \quad nh^2 = -\frac{eB_1}{cp_0}, \quad K_n = \frac{eC_n}{cp_0}, \tag{21}$$

and the hamiltonian is then

$$H = H_0 + H_n, \tag{22}$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}\frac{p_0}{p}\left(\frac{B}{B_0} - n\right)h^2x^2 + \frac{1}{2}\frac{p_0}{p}nh^2y^2 \\ &+ \frac{Bp_0 - B_0p}{B_0p}hx - \frac{p_0}{p}gy - \frac{p_0}{p}hgxy, \end{aligned} \tag{23}$$

and

$$H_n = -\frac{p_0}{p}K_nF_n(x,y). \tag{24}$$

Here we have omitted the term proportional to $hxF_n(x,y)$ in H_n, and only terms to second order in coordinates and momenta have been retained in the expression for H_0. The nh^2x^2 and nh^2y^2 terms in (23) are responsible for horizontal and vertical focusing, and the factor p_0/p multiplying these terms gives rise to the variation of the tunes with momentum. The hx term is responsible for horizontal dispersion and for the distortion of the horizontal closed orbit due to errors in the guide field. Likewise the gy term is responsible for the distortion of the vertical closed orbit. The $hgxy$ term couples the oscillations in the horizontal and vertical planes whenever the curvature, h, and the horizontal field are nonzero ($g \neq 0$) along the

reference trajectory. (This could occur if, for example, the horizontal bending magnets were rolled by some amount about the longitudinal axis thereby producing a horizontal field in a region of nonzero curvature.) Note that by setting $p = p_0$, $B = B_0$, and $g = 0$ in (23) one obtains

$$H_0 = \frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}(1-n)h^2x^2 + \frac{1}{2}nh^2y^2 \tag{25}$$

which is the usual hamiltonian for linear oscillations about the reference trajectory.

The equations of motion, obtained from the canonical equations

$$x' = \frac{\partial H}{\partial P_x}, \quad P_x' = -\frac{\partial H}{\partial x}, \quad y' = \frac{\partial H}{\partial P_y}, \quad P_y' = -\frac{\partial H}{\partial y} \tag{26}$$

are

$$x'' + \frac{p_0}{p}\left(\frac{B}{B_0} - n\right)h^2x + \frac{Bp_0 - B_0p}{B_0p}h - \frac{p_0}{p}hgy - \frac{p_0}{p}K_n\frac{\partial F_n}{\partial x} = 0, \tag{27}$$

and

$$y'' + \frac{p_0}{p}nh^2y - \frac{p_0}{p}g(1+hx) - \frac{p_0}{p}K_n\frac{\partial F_n}{\partial y} = 0. \tag{28}$$

Defining

$$\delta p = p - p_0, \quad \delta B = B - B_0, \tag{29}$$

we have

$$\frac{p_0}{p} = 1 - \frac{\delta p}{p}, \quad \frac{B}{B_0} = 1 + \frac{\delta B}{B_0}, \tag{30}$$

and, to first order in $\delta p/p$ and $\delta B/B_0$,

$$\frac{p_0}{p}\left(\frac{B}{B_0} - n\right) = \left(1 - \frac{\delta p}{p}\right)(1-n) + \frac{\delta B}{B_0}, \quad \frac{Bp_0 - B_0p}{B_0p} = \frac{\delta B}{B_0} - \frac{\delta p}{p}. \tag{31}$$

Equations (27–28) then become

$$\begin{aligned} x'' &+ \left(1 - \frac{\delta p}{p}\right)(1-n)h^2x + \frac{\delta B}{B_0}h^2x + \left(\frac{\delta B}{B_0} - \frac{\delta p}{p}\right)h \\ &- \left(1 - \frac{\delta p}{p}\right)hgy - \left(1 - \frac{\delta p}{p}\right)K_n\frac{\partial F_n}{\partial x} = 0, \end{aligned} \tag{32}$$

and

$$y'' + \left(1 - \frac{\delta p}{p}\right)nh^2y - \left(1 - \frac{\delta p}{p}\right)g(1+hx) - \left(1 - \frac{\delta p}{p}\right)K_n\frac{\partial F_n}{\partial y} = 0. \tag{33}$$

4 The Distorted Closed Orbit

The deviation of the distorted closed orbit from the reference trajectory is given by the periodic solutions, $x_c(s)$ and $y_c(s)$, of equations (32) and (33). The lowest-order solutions are

$$x_c(s) = \frac{\delta p}{p} D(s) + d_x(s), \quad y_c(s) = d_y(s) \tag{34}$$

where

$$D'' + (1-n)h^2 D - h = 0, \quad d_x'' + (1-n)h^2 d_x + \frac{\delta B}{B_0} h = 0, \tag{35}$$

$$d_y'' + nh^2 d_y - g = 0. \tag{36}$$

Here $D(s)$ is the horizontal dispersion and $d_x(s)$ and $d_y(s)$ are the distortions of the horizontal and vertical closed orbits due to dipole errors. Thus both the horizontal dispersion and horizontal dipole errors contribute to x_c while only vertical dipole errors contribute to y_c.

5 Hamiltonian for Oscillations about the Distorted Closed Orbit

Following Ref.[4] we introduce new coordinates and momenta

$$q_1 = x - x_c, \quad p_1 = P_x - x_c', \quad q_2 = y - y_c, \quad p_2 = P_y - y_c', \tag{37}$$

which represent the deviations of the particle trajectory from the distorted closed orbit. This transformation is canonical and is generated by

$$F_2(x, p_1, y, p_2) = (x - x_c)(p_1 + x_c') + (y - y_c)(p_2 + y_c'). \tag{38}$$

Thus we have

$$P_x = \frac{\partial F_2}{\partial x} = p_1 + x_c', \quad q_1 = \frac{\partial F_2}{\partial p_1} = x - x_c \tag{39}$$

$$P_y = \frac{\partial F_2}{\partial y} = p_2 + y_c', \quad q_2 = \frac{\partial F_2}{\partial p_2} = y - y_c \tag{40}$$

and the new hamiltonian is

$$G = H + \frac{\partial F_2}{\partial s} = H - x_c'(p_1 + x_c') + q_1 x_c'' - y_c'(p_2 + y_c') + q_2 y_c''. \tag{41}$$

Using (23–24) and (27–28) in (41) we find

$$G = G_0 + G_n, \tag{42}$$

where

$$G_0 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\frac{p_0}{p}\left(\frac{B}{B_0} - n\right) h^2 q_1^2 + \frac{1}{2}\frac{p_0}{p} n h^2 q_2^2 - \frac{p_0}{p} h g q_1 q_2, \tag{43}$$

$$G_n = -\frac{p_0}{p} K_n \left[F_n(q_1 + x_c, q_2 + y_c) - q_1 \frac{\partial F_n}{\partial x}(x_c, y_c) - q_2 \frac{\partial F_n}{\partial y}(x_c, y_c)\right]. \tag{44}$$

Using (11) and (14–15) in (44) we find, for the case in which $F_n = U_n$,

$$\begin{aligned} G_n &= -\frac{p_0}{p} K_n [U_n(q_1, q_2) + x_c U_{n-1}(q_1, q_2) - y_c V_{n-1}(q_1, q_2) \\ &+ \frac{1}{2}(x_c^2 - y_c^2) U_{n-2}(q_1, q_2) - x_c y_c V_{n-2}(q_1, q_2) + \cdots], \end{aligned} \tag{45}$$

and for the case in which $F_n = V_n$,

$$\begin{aligned} G_n &= -\frac{p_0}{p} K_n [V_n(q_1, q_2) + x_c V_{n-1}(q_1, q_2) + y_c U_{n-1}(q_1, q_2) \\ &+ \frac{1}{2}(x_c^2 - y_c^2) V_{n-2}(q_1, q_2) + x_c y_c U_{n-2}(q_1, q_2) + \cdots]. \end{aligned} \tag{46}$$

Here we see that the displacements, x_c and y_c, of the closed orbit in the multipoles U_n and V_n give rise to lower order normal and skew multipoles which are proportional to x_c, y_c, x_c^2, y_c^2, and $x_c y_c$. Using the approximate expressions (34) for x_c and y_c we obtain the following hamiltonians for various multipoles.

Sextupole:

$$G_3 = -K_3 \left[U_3 + \frac{\delta p}{p} D U_2 + d_x U_2 - d_y V_2\right] + \cdots \tag{47}$$

Skew Sextupole:

$$G_3 = -K_3 \left[V_3 + \frac{\delta p}{p} D V_2 + d_x V_2 + d_y U_2\right] + \cdots \tag{48}$$

Octupole:

$$G_4 = -K_4 \left[U_4 + \frac{\delta p}{p} D U_3 + d_x U_3 - d_y V_3\right] + \cdots \tag{49}$$

Skew Octupole:

$$G_4 = -K_4 \left[V_4 + \frac{\delta p}{p} D V_3 + d_x V_3 + d_y U_3 \right] + \cdots \tag{50}$$

Thus the strength of the normal (skew) $2(n-1)$-pole arising from the horizontal displacement of the closed orbit in a normal (skew) $2n$-pole magnet is

$$K(s) = -K_n(s) \left[\frac{\delta p}{p} D(s) + d_x(s) \right], \tag{51}$$

and the strength of the skew (normal) $2(n-1)$-pole arising from the vertical displacement of the closed orbit in a normal (skew) $2n$-pole magnet is

$$K(s) = \pm K_n(s) d_y(s). \tag{52}$$

6 Resonance Excitation

Transverse resonances [7] are defined by the equation

$$mQ_x + nQ_y = N \tag{53}$$

where Q_x and Q_y are the horizontal and vertical tunes, and m, n, and N are integers. If m and n have opposite signs, the resonance is called a difference resonance; otherwise it is called a sum resonance. The order, l, of the resonance is

$$l = |m| + |n|. \tag{54}$$

The resonance condition (53) arises from the first-order perturbation treatment of the vector potential terms $x^{|m|} y^{|n|}$ associated with the $2l$-pole field. Resonance excitation can occur only if the tunes are sufficiently close to the resonance and the excitation coefficient,

$$\kappa = \int_0^{2\pi r} \beta_x^{|m/2|} \beta_y^{|n/2|} K(s) e^{i\psi} ds, \tag{55}$$

is nonzero. Here

$$\psi(s) = m(\mu_x - Q_x\theta) + n(\mu_y - Q_y\theta) + N\theta, \quad \theta = s/r, \tag{56}$$

$$\mu_x(s) = \int_0^s \frac{ds'}{\beta_x(s')}, \quad \mu_y(s) = \int_0^s \frac{ds'}{\beta_y(s')}, \quad K(s) = \frac{e}{cp} \left(\frac{\partial^{(l-1)} B}{\partial x^{(l-1)}} \right), \tag{57}$$

and $B = B_y$ (B_x) if n is even (odd). Since

$$\psi(s) = m\mu_x + n\mu_y \approx N\theta, \tag{58}$$

when the tunes are near the resonance, we see that the excitation coefficient is essentially the Nth harmonic, in azimuth θ, of the multipole strength $K(s)$. Thus if the tunes are sufficiently close to the resonance, i.e. if they are within the resonance stopband, and if n is even (odd), the resonance will be excited by the Nth harmonic of the normal (skew) $2l$-pole fields present in the machine. The width of the stopband is proportional to the strength of the $2l$-pole field, and for resonances of order 3 and higher also depends on the amplitudes of the betatron oscillations.

In the AGS Booster we therefore have the following possibilities:

The resonances $2Q_x = 9$ and $2Q_y = 9$ can be excited by the ninth harmonic present in

$$K(s) = -K_3(s)\left[\frac{\delta p}{p}D(s) + d_x(s)\right], \tag{59}$$

where $K_3(s)$ is the sextupole strength along the reference trajectory, or by the ninth harmonic present in

$$K(s) = -K_3(s)d_y(s), \tag{60}$$

where $K_3(s)$ is the skew sextupole strength along the reference trajectory.

The resonance $Q_x + Q_y = 9$ can be excited by the ninth harmonic present in

$$K(s) = -K_3(s)\left[\frac{\delta p}{p}D(s) + d_x(s)\right], \tag{61}$$

where $K_3(s)$ is the skew sextupole strength along the reference trajectory, or by the ninth harmonic present in

$$K(s) = K_3(s)d_y(s), \tag{62}$$

where $K_3(s)$ is the sextupole strength along the reference trajectory.

The resonances $3Q_x = 14$ and $Q_x + 2Q_y = 14$ can be excited by the 14th harmonic present in

$$K(s) = -K_4(s)\left[\frac{\delta p}{p}D(s) + d_x(s)\right], \tag{63}$$

where $K_4(s)$ is the octupole strength along the reference trajectory, or by the 14th harmonic present in

$$K(s) = -K_4(s)d_y(s), \tag{64}$$

where $K_4(s)$ is the skew octupole strength along the reference trajectory.

The resonances $3Q_y = 14$ and $Q_y + 2Q_x = 14$ can be excited by the 14th harmonic present in

$$K(s) = -K_4(s)\left[\frac{\delta p}{p}D(s) + d_x(s)\right], \tag{65}$$

where $K_4(s)$ is the skew octupole strength along the reference trajectory, or by the 14th harmonic present in

$$K(s) = K_4(s)d_y(s), \tag{66}$$

where $K_4(s)$ is the octupole strength along the reference trajectory.

The contributions of D and d_x to K(s) in the AGS Booster are discussed in Ref.[3].

7 References

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