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SIMPLE APPROXIMATION FOR SYNCHROTRON FREQUENCY

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> Accelerator Division Technical Note

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Abstract

The synchrotron frequency distribution for particles within the stationary bucket can be approximated with good accuracy by the formula

$$\Omega(\phi_{o}) = \omega_{o} \sqrt{1 - (\frac{\phi}{\pi})^{2}},$$

where $\underset{O}{\omega}$ is the usual synchrotron frequency for small amplitude, $_{O}^{\phi}$ is the particle amplitude angle.

1. Linear-Parametric Approximation

Within a stationary bucket in longitudinal phase space, each particle exercises a synchrotron oscillation with its own constant synchrotron frequency, which is higher toward the bucket center and lower for particles close to the separatrix. I will apply a term central (synchrotron) frequency for the particles of infinitesimal amplitude. That is what is usually called synchrotron frequency, because it comes from the equation

...

$$\phi + \omega_{o}^{2} \phi = 0 , \qquad (1)$$

approximating synchrotron oscillations of particles with such a small angular amplitude ϕ << 1, that satisfy

$$\phi = \sin \phi \quad (2)$$

For arbitrary $\phi,$ however, the approximate equation (1) is replaced by an exact one

$$\phi + \omega_o^2 \sin \phi = 0, \quad -\pi \leq \phi \leq \pi$$
 (3)

whose solution $\phi = \phi(t)$ should satisfy the initial conditions*

$$\phi(0) = \phi_0$$
, $\phi(0) = 0$. (4)

An approximation (2) is a part of a simple class of linear-parametric functions defined by

$$\sin \phi \simeq \phi p^2(\phi_0)$$
 (5)

and applicable to the problem (3), (4).

After the substitution of (5) to (3), the approximate synchrotron frequency

$$\Omega_{\mathbf{p}}(\phi_{\mathbf{o}}) = \omega_{\mathbf{o}} \mathbf{p}(\phi_{\mathbf{o}}) \tag{6}$$

should be compared with the exact one $\Omega = \Omega(\phi_0) = \omega_0 F(\phi_0)$. The latter can be found by the use of an elliptic integral¹ or by tracking the particle motion numerically. I did the tracking. In the next section, we will compare several approximating functions $p(\phi_0)$.

2. Comparison of Approximating Functions

Figures 1 to 4 show the computed results for four approximating functions defined as follows:

$$p_2^2(\phi_0) = 1 - \frac{\phi_0^2}{3!}$$
, (7)

^{*}Any other $(\phi(0) \neq 0)$ initial conditions can be reduced to (3) by the appropriate shifting of the time reference frame: $t \rightarrow t + t_0$.

 $p_3^2(\phi_0) = 1 - \frac{\phi_0^2}{3!} + \frac{\phi_0^4}{5!}$, (8)

$$p_{s}^{2}(\phi_{o}) = \frac{\sin \phi_{o}}{\phi_{o}}, \qquad (9)$$

$$p_{\pi}^2 = 1 - \left(\frac{\phi_0}{\pi}\right)^2$$
 (10)

All of the above originated from linear-parametric representation (5).

Let us now turn to the figures showing approximations and their errors. Each figure has three curves. Two of them are $F(\phi_0)$ (exact distribution) and $p_a(\phi_0)$ (approximate distribution (a = 2, 3, s, π) both starting at $\phi_0 = 0$, $F(0) = p_a(0) = 1$. The space between those two curves is shaded. The third curve $E(\phi_0)$, starting at $\phi_0 = 0$, E(0) = 0is relative error:

$$E(\phi_{o}) = \frac{F(\phi_{o}) - p_{a}(\phi_{o})}{F(\phi_{o})} \quad . \tag{11}$$

The worst approximation is the first which comes from obvious Taylor expansion up to the second term. Because this approximation is valid only for the short interval of argument ϕ_0 , Figure 1 is not completed for ϕ_0 close to π .

Figure 2 represents the approximation coming from the Taylor expansion up to the third term. The improvement in accuracy is not very dramatic for the price of increased complexity in p_3 .

Figure 3 shows that trying to avoid Taylor expansion does not pay in accuracy. Maybe the only profit from this approximation is the lower boundary estimation

$$\frac{\sin \phi_{o}}{\phi_{o}} \leq F^{2}(\phi_{o}) , \quad 0 \leq \phi_{o} \leq \pi .$$
 (12)

- 3 -

-4 --Pi



Figure 4 shows unexpectedly that a small correction in the second order Taylor expansion pays off very well: the relative error is less than 5% for at least 80% of the argument region.

It is interesting to compare the exact distribution $F(\phi_0)$ with its own expansion¹ up to the second term:

$$F(\phi_{o}) \simeq p_{E}(\phi_{o}) = \left(1 + \frac{\phi_{o}^{2}}{16}\right)^{-1}$$
 (13)

The comparison of F and $\rm p_E$ is shown in Figure 5. One can see $\rm P_E$ is good only for 60% of the region.

So, we choose as an approximate synchrotron frequency distribution

$$\Omega(\phi_{0}) = \omega_{0} \sqrt{1 - \left(\frac{\phi_{0}}{\pi}\right)^{2}} \quad . \tag{14}$$

Try it--you'll like it!

Reference

 L.D. Landau and E.M. Lifshitz, Mechanics, Third Ed., 1982, Pergamon Press, Para. 11, Problem 1.



