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## SIMPLE APPROXIMATION FOR SYNCHROTRON FREQUENCY

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Abstract

The synchrotron frequency distribution for particles within the stationary bucket can be approximated with good accuracy by the formula

$$\Omega(\phi_0) = \omega_0 \sqrt{1 - \left(\frac{\phi_0}{\pi}\right)^2},$$

where  $\omega_0$  is the usual synchrotron frequency for small amplitude,  $\phi_0$  is the particle amplitude angle.

1. Linear-Parametric Approximation

Within a stationary bucket in longitudinal phase space, each particle exercises a synchrotron oscillation with its own constant synchrotron frequency, which is higher toward the bucket center and lower for particles close to the separatrix. I will apply a term central (synchrotron) frequency for the particles of infinitesimal amplitude. That is what is usually called synchrotron frequency, because it comes from the equation

$$\ddot{\phi} + \omega_0^2 \phi = 0, \quad (1)$$

approximating synchrotron oscillations of particles with such a small angular amplitude  $\phi \ll 1$ , that satisfy

$$\phi = \text{Sin } \phi. \quad (2)$$

For arbitrary  $\phi$ , however, the approximate equation (1) is replaced by an exact one

$$\ddot{\phi} + \omega_0^2 \sin \phi = 0, \quad -\pi \leq \phi \leq \pi \quad (3)$$

whose solution  $\phi = \phi(t)$  should satisfy the initial conditions\*

$$\phi(0) = \phi_0, \quad \dot{\phi}(0) = 0. \quad (4)$$

An approximation (2) is a part of a simple class of linear-parametric functions defined by

$$\sin \phi \approx \phi p^2(\phi_0) \quad (5)$$

and applicable to the problem (3), (4).

After the substitution of (5) to (3), the approximate synchrotron frequency

$$\Omega_p(\phi_0) = \omega_0 p(\phi_0) \quad (6)$$

should be compared with the exact one  $\Omega = \Omega(\phi_0) = \omega_0 F(\phi_0)$ . The latter can be found by the use of an elliptic integral<sup>1</sup> or by tracking the particle motion numerically. I did the tracking. In the next section, we will compare several approximating functions  $p(\phi_0)$ .

## 2. Comparison of Approximating Functions

Figures 1 to 4 show the computed results for four approximating functions defined as follows:

$$p_2^2(\phi_0) = 1 - \frac{\phi_0^2}{3!}, \quad (7)$$

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\*Any other ( $\dot{\phi}(0) \neq 0$ ) initial conditions can be reduced to (3) by the appropriate shifting of the time reference frame:  $t \rightarrow t + t_0$ .

$$p_3^2(\phi_o) = 1 - \frac{\phi_o^2}{3!} + \frac{\phi_o^4}{5!} , \quad (8)$$

$$p_s^2(\phi_o) = \frac{\text{Sin } \phi_o}{\phi_o} , \quad (9)$$

$$p_\pi^2 = 1 - \left(\frac{\phi_o}{\pi}\right)^2 . \quad (10)$$

All of the above originated from linear-parametric representation (5).

Let us now turn to the figures showing approximations and their errors. Each figure has three curves. Two of them are  $F(\phi_o)$  (exact distribution) and  $p_a(\phi_o)$  (approximate distribution ( $a = 2, 3, s, \pi$ )) both starting at  $\phi_o = 0$ ,  $F(0) = p_a(0) = 1$ . The space between those two curves is shaded. The third curve  $E(\phi_o)$ , starting at  $\phi_o = 0$ ,  $E(0) = 0$  is relative error:

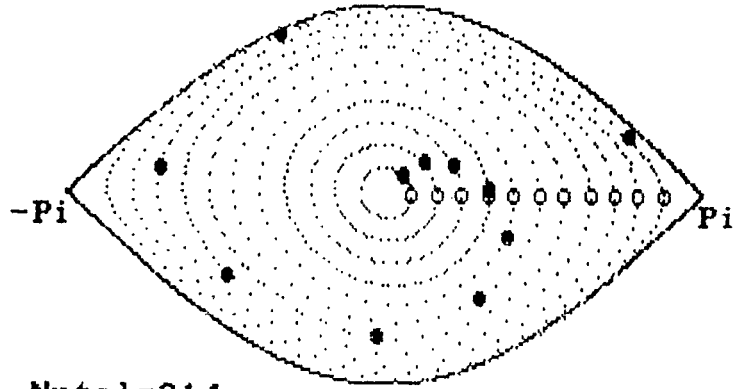
$$E(\phi_o) = \frac{F(\phi_o) - p_a(\phi_o)}{F(\phi_o)} . \quad (11)$$

The worst approximation is the first which comes from obvious Taylor expansion up to the second term. Because this approximation is valid only for the short interval of argument  $\phi_o$ , Figure 1 is not completed for  $\phi_o$  close to  $\pi$ .

Figure 2 represents the approximation coming from the Taylor expansion up to the third term. The improvement in accuracy is not very dramatic for the price of increased complexity in  $p_3$ .

Figure 3 shows that trying to avoid Taylor expansion does not pay in accuracy. Maybe the only profit from this approximation is the lower boundary estimation

$$\frac{\text{Sin } \phi_o}{\phi_o} \leq F^2(\phi_o) , \quad 0 \leq \phi_o \leq \pi . \quad (12)$$



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Figure 4 shows unexpectedly that a small correction in the second order Taylor expansion pays off very well: the relative error is less than 5% for at least 80% of the argument region.

It is interesting to compare the exact distribution  $F(\phi_0)$  with its own expansion<sup>1</sup> up to the second term:

$$F(\phi_0) \approx p_E(\phi_0) = \left(1 + \frac{\phi_0^2}{16}\right)^{-1}. \quad (13)$$

The comparison of  $F$  and  $p_E$  is shown in Figure 5. One can see  $p_E$  is good only for 60% of the region.

So, we choose as an approximate synchrotron frequency distribution

$$\Omega(\phi_0) = \omega_0 \sqrt{1 - \left(\frac{\phi_0}{\pi}\right)^2}. \quad (14)$$

Try it--you'll like it!

Reference

1. L.D. Landau and E.M. Lifshitz, Mechanics, Third Ed., 1982, Pergamon Press, Para. 11, Problem 1.

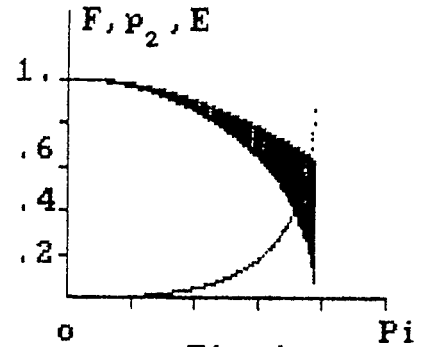


Fig.1

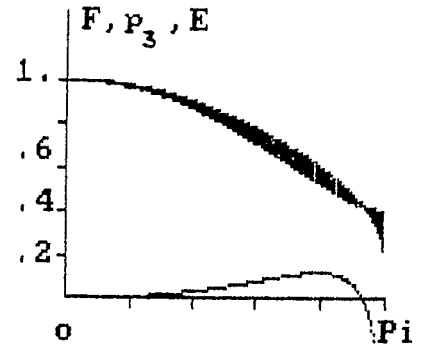


Fig.2

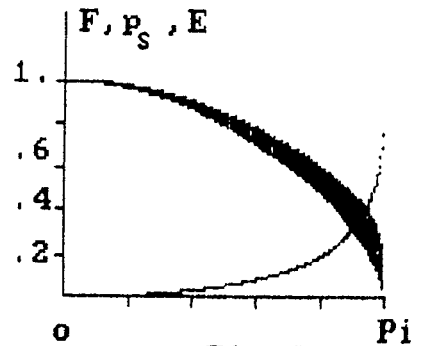


Fig.3

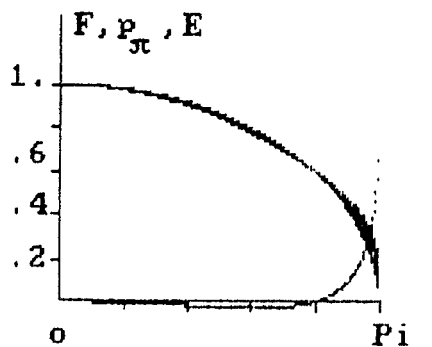


Fig.4

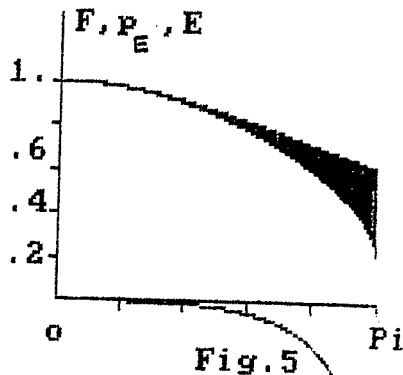


Fig.5