

INTRODUCTION TO FIELDS IN HOLLOW CYLINDRICAL PIPES AND CAVITY RESONATORS FOR RFQ (Part-1)

M. Puglisi

August 1985

Collider Accelerator Department
Brookhaven National Laboratory

U.S. Department of Energy

USDOE Office of Science (SC)

Notice: This technical note has been authored by employees of Brookhaven Science Associates, LLC under Contract No.DE-AC02-76CH00016 with the U.S. Department of Energy. The publisher by accepting the technical note for publication acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this technical note, or allow others to do so, for United States Government purposes.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or any third party's use or the results of such use of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof or its contractors or subcontractors. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Accelerator Division

BROOKHAVEN NATIONAL LABORATORY
 Associated Universities, Inc.
 Upton, New York 11973

INTRODUCTION TO FIELDS IN HOLLOW CYLINDRICAL PIPES AND CAVITY RESONATORS FOR
 RFQ (Part-1)

M. Puglisi

August 1985

Technical Note No. 219

Hypotheses

1. Linear, time invariant, lossless, isotropic medium (in our case the medium is the vacuum) bounded with uniform, perfectly conducting, cylindrical wall.

2. No charges and no currents in the medium.

Even upon the above hypotheses to find a general solution of the Maxwell Eq. inside this bounded medium would be a formidable task.

A very short way to arrive at significant results is to assume that the fields depend upon s and t as follows:

$$F = F(r, \theta) e^{(j\omega t - \gamma s)} \quad (1)$$

Upon substituting into the Maxwell equation we obtain the fundamental system:

$$\left\{ \begin{array}{l} 0 \quad +\gamma E_{\theta} \quad +j\omega\mu H_r \quad 0 \quad = -\frac{1}{r} \frac{\partial E_S}{\partial \theta} \\ -\gamma E_r \quad 0 \quad 0 \quad j\omega\mu H_{\theta} \quad = \frac{\partial E_S}{\partial r} \\ -j\omega\epsilon E_r \quad 0 \quad 0 \quad +\gamma H_{\theta} \quad = -\frac{1}{r} \frac{\partial H_S}{\partial \theta} \\ 0 \quad -j\omega\epsilon E_{\theta} \quad -\gamma H_r \quad 0 \quad = \frac{\partial H_z}{\partial z} \end{array} \right.$$

Solving with the Kramer rule and using the normal notation:

$$K_c^2 = \gamma^2 + \omega^2 \epsilon \mu$$

We obtain:

$$\left\{ \begin{array}{l} E_r = -\frac{1}{K_c^2} \left[\gamma \frac{\partial E_S}{\partial r} + j \frac{\omega \mu}{r} \frac{\partial H_S}{\partial \theta} \right] \\ E_\theta = -\frac{1}{K_c^2} \left[-\frac{\gamma}{r} \frac{\partial E_S}{\partial \theta} + j \omega \mu \frac{\partial H_S}{\partial r} \right] \\ H_r = -\frac{1}{K_c^2} \left[\frac{j\omega \epsilon}{r} \frac{\partial E_S}{\partial \theta} - \gamma \frac{\partial H_S}{\partial r} \right] \\ H_\theta = -\frac{1}{K_c^2} \left[j\omega \epsilon \frac{\partial E_S}{\partial r} + \frac{\gamma}{r} \frac{\partial H_S}{\partial \theta} \right] \end{array} \right. \quad (2)$$

This means that under the above hypotheses if the longitudinal field is known then the transverse fields can be obtained simply by derivation. Moreover because the system is linear then superposition applies and we can consider field configurations (modes) depending upon E_S or upon H_S .

The fields depending upon E_S pertains to the group of the accelerating field and are called TM modes. The fields depending upon H_S pertains to the group of the deflecting fields and are called TE modes.

Now our problem is to find a suitable expression for H_S because for the RFQ we are interested only in the TE modes.

Maxwell equation in vector notation are:

$$\begin{array}{ll} \nabla \cdot \mathbf{E} = 0 & \nabla \times \mathbf{H} = j\omega \epsilon \mathbf{E} \\ \nabla \cdot \mathbf{H} = 0 & \nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} \end{array}$$

It follows:

$$\left\{ \begin{array}{l} \nabla \times (\nabla \times \mathbf{H}) = j\omega \epsilon (\nabla \times \mathbf{E}) = \omega^2 \epsilon \mu \mathbf{H} \\ \nabla \times (\nabla \times \mathbf{H}) = \nabla (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} \end{array} \right.$$

Combining and noticing that the same result is valid for E we obtain the familiar wave equation.

$$\nabla^2 \begin{Bmatrix} H \\ E \end{Bmatrix} + \omega^2 \epsilon \mu \begin{Bmatrix} H \\ E \end{Bmatrix} = 0$$

Expanding the above equation and retaining the longitudinal S component we obtain:

$$\frac{\partial^2 H_S}{\partial r^2} + \frac{1}{r} \frac{\partial H_S}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_S}{\partial \theta^2} + K_c^2 H = 0 \quad (3)$$

If we substitute an assumed product solution and attempt to separate the variables in order to obtain two ordinary differential equations we should assume:

$$H_S = R\Theta$$

where R is a function of r alone and Θ is a function of θ alone.

Substituting and manipulating we obtain:

$$r^2 \frac{R''}{R} + r \frac{R'}{R} + K_c^2 = \frac{\Theta''}{\Theta}$$

The left side is function of r alone, the right of θ alone. Consequently, if both sides are to be equal for all values of r and θ then both sides must be equal to the same constant: for instance v^2 . Substituting we obtain:

$$\begin{cases} R'' + \frac{1}{r} R' (K_c^2 - \frac{v^2}{r^2}) R = 0 \\ \frac{\Theta''}{\Theta} = v^2 \end{cases} \quad (4)$$

The first equation is solved with the Bessel and Neuman functions of the order ν while the second is solved with sinusoids and we could write:

$$H_S = \left(A J_\nu(K_c r) + B N_\nu(K_c r) \right) \left(C \sin \nu \theta + D \cos \nu \theta \right)$$

For $r = 0$ H_S cannot be infinite, thus $B = 0$. On the other hand the fields should be the same every time we vary θ of a multiple of 2π . This means that ν must be an integer. Moreover, a proper selection of θ will allow us to use either the sine or the cosine.

So we obtain the general formula:

$$H_S = H_0 J_\nu(K_c r) \cos \nu \theta \quad (6)$$

On the inner surface of the cylindrical hollow pipe that contains the field the component E_θ must be zero.

From (2) we have:

$$E_\theta = \frac{j\omega\mu}{K_c^2} \frac{\partial H_S}{\partial r}$$

This means that if a is the radius of the pipe then:

$$J'_\nu(K_c a) = 0 \quad (7)$$

This condition determines K_c and we have an infinite number of solutions determining an infinite number of modes of the same family $TE_{\nu\ell}$.

If we are looking for a transverse field with four pole symmetry and no variation along S then we have to set:

$$\begin{cases} \gamma = 0 \\ \nu = 2 \end{cases}$$

and from (7), selecting the first zero we obtain:

$$\omega^2 \epsilon \mu a = 3.05424 \quad \text{or } f_c = \frac{145.8}{a} \text{ MHz} \quad (8)$$

where f_c is the so called cut-off frequency of the selected mode (Fig. 1).

This means that an infinitely long cylindrical lossless pipe with inner radius equal to a can support the wanted axially uniform four pole mode. The relationship between frequency and mode being defined by (8).

An infinitely long wave guide is not a practical device but it is perfectly possible to build a practical structure where for a long portion of the axis the field has four pole symmetry and is adequately uniform. (This structure is the RFQ-resonant cavity.)

In order to have some ideas about the cylindrical cavity resonators we suppose to short circuit, with a conducting wall normal to the axis, both the ends of our hollow pipe leaving a clearance L between the short circuits.

Now beside the above conditions on the cylindrical wall (Eq. (7)) the electric field should be normal or zero on the short circuiting surfaces and we have a third condition that enters in determining the cavity resonant frequency.

It is nearly obvious that both, for the TE and TM modes, this condition is fulfilled if and only if the distance L is an integer multiple of half wavelength of the field.

For the TE modes we have already seen that the condition (7) must be satisfied. For the TM modes is the field E_s that must be zero on the cylindrical wall. This in turn demands to fulfill the condition:

$$J_v(K_c a) = 0 \quad (9)$$

Let us call $R_{\nu\ell}$ the value for the argument that satisfies Eq. (7) or (9).

Consequently we have:

$$\gamma^2 + \omega^2 \epsilon \mu = \left(\frac{R_{\nu\ell}}{a} \right)^2$$

Because we need propagation then γ should be imaginary and we put $\gamma = j\beta$. If we call λ_g the wavelength inside the pipe then it is rather obvious that $\beta = 2\pi/\lambda_g$. In fact when we pass through a length equal to λ_g the field has to repeat because the argument of $e^{j\beta\lambda_g}$ changes of 2π . Substituting in the above equation we obtain

$$\left(\frac{2\pi}{\lambda} \right)^2 = \left(\frac{R_{\nu\ell}}{a} \right)^2 + \left(\frac{2\pi}{\lambda_g} \right)^2$$

Rearranging and introducing the third condition, that the resonator length L can be only equal to an integer number, say p , of half guide wavelength we obtain:

$$\lambda = \frac{2L}{\sqrt{p^2 + \frac{R_{\nu\ell}}{2\pi a} \cdot (2L)^2}}$$

where λ is the free space wavelength of a cylindrical resonator of radius a and length L operating in the $TE_{\nu\ell p}$ or $TM_{\nu\ell p}$ mode. (Three examples are given in Fig. 2.)

Up to this point, we considered only the elementary cylindrical resonator where the fields E_S or H_S are completely described with only one eigenfunction and the resonant frequency is the corresponding eigenvalue.

However the cavity resonators, used as accelerators, even maintaining the cylindrical symmetry, are very often much more complicated and in order to sat-

isfy the boundary conditions dictated by a technical resonator the complete set of the cylindrical eigenfunctions is normally required.

Even a simple outline of the general theory would be beyond the purpose of this rather intuitive treatment, vice versa it is important to know that many powerful computer programs are now available for analyzing, with good accuracy, practically any useful cylindrical resonator.

The cavity for an RFQ originates from a TE_{211} cylindrical resonator which is loaded with four V-shaped vanes symmetrically connected to the cylindrical wall as shown in Figs. 3&4.

The vanes terminate at some distance from the short circuiting wall and consequently the central vane section is symmetrically coupled to the two end sections (we note, in passing, that this resonator is no longer uniform along the abscissas).

The boundary conditions provided by the end sections allows the whole cavity to resonate in a very complicated manner where the fields are nearly uniform along the axis of the vane section. This condition is obtained if the TE_{21} cut-off frequency of the guide represented by the vane section* is slightly below the operating frequency of the whole cavity.

*For this reason some authors write that the cavity is resonating in the TE_{210} mode. Only an infinitely long resonator could support this mode. In a physical cylindrical resonator the TE electric field must be zero at both the ends so the last index of the mode cannot be zero and consequently the guide wavelength should be finite.

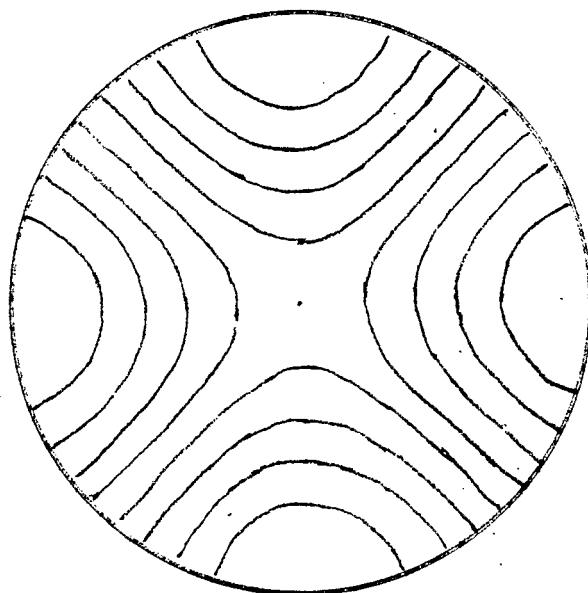
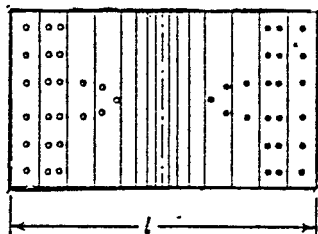
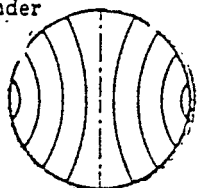


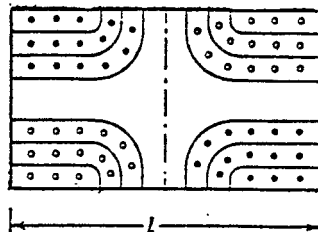
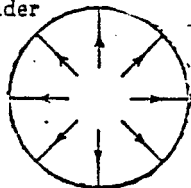
Fig. 1. Lines of force of the electric field in a cylindrical wave guide at the cut-off frequency for the TE_{21} mode.

TE_{111} , Cylinder



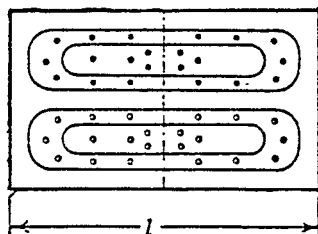
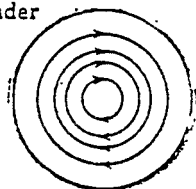
$$\lambda_1 = \frac{2l}{\sqrt{1 + \left(\frac{2l}{3.41a}\right)^2}}$$

TM_{011} , Cylinder



$$\lambda_1 = \frac{2l}{\sqrt{1 + \left(\frac{2l}{2.61a}\right)^2}}$$

TE_{011} , Cylinder



$$\lambda_1 = \frac{2l}{\sqrt{1 + \left(\frac{2l}{1.64a}\right)^2}}$$

Fig. 2. Cross and longitudinal sections of the same cylindrical resonator excited in three different modes.

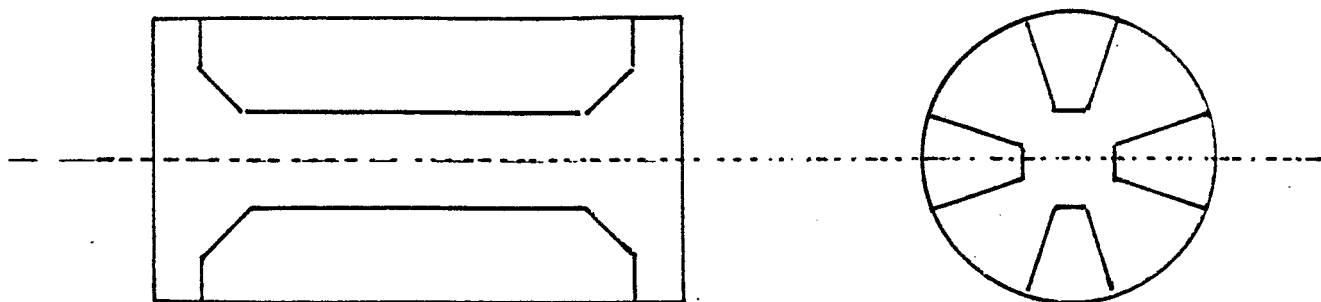


Fig. 3. Simplified scheme of an RFQ resonant cavity.

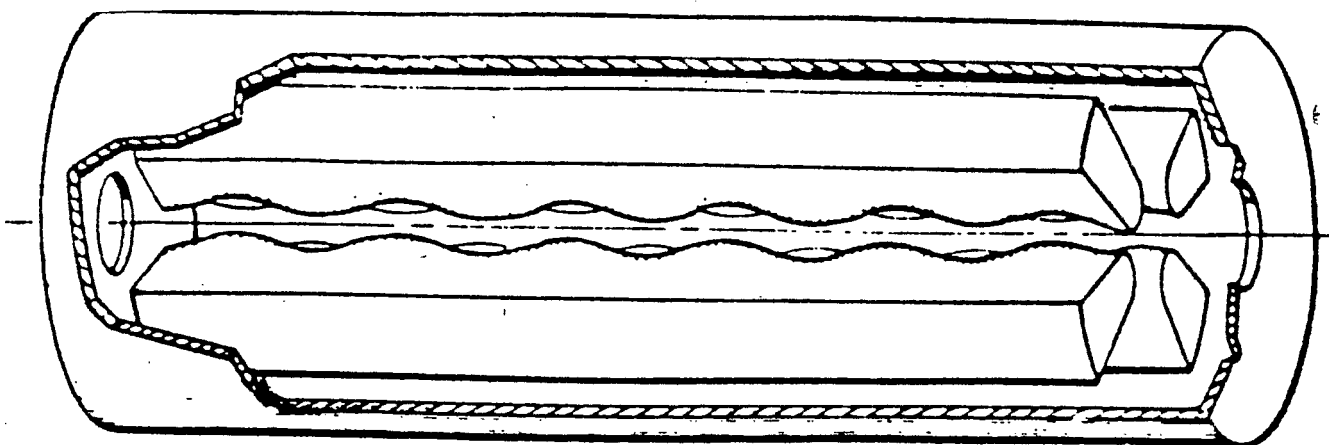


Fig. 4. Perspective view of an RFQ resonator. The serration on the vane tips is also shown.