



BNL-104630-2014-TECH

AGS/AD/Tech Note No. 203;BNL-104630-2014-IR

# FORCES AND DEFELECTIONS ON NEUTRINO HORN END PLATES

L. E. Repeta

June 1984

Collider Accelerator Department  
**Brookhaven National Laboratory**

**U.S. Department of Energy**

USDOE Office of Science (SC)

Notice: This technical note has been authored by employees of Brookhaven Science Associates, LLC under Contract No.DE-AC02-76CH00016 with the U.S. Department of Energy. The publisher by accepting the technical note for publication acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this technical note, or allow others to do so, for United States Government purposes.

## **DISCLAIMER**

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or any third party's use or the results of such use of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof or its contractors or subcontractors. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Accelerator Department  
BROOKHAVEN NATIONAL LABORATORY  
Associated Universities, Inc.  
Upton, New York 11973

AGS Division Technical Note  
No. 203

FORCES AND DEFELECTIONS ON  
NEUTRINO HORN END PLATES

L.E. Repeta

June 25, 1984

INTRODUCTION

When neutrino horns are designed it is essential that the high energy particles (secondaries) pass through a minimum thickness of horn material to minimize their absorption, and yet the apparatus must maintain its structural integrity. Since the equations which describe the effects of a variable symmetrical electrical force applied to a flat circular plate are not presented in the literature it was necessary to develop the required equations. The purpose of this note is to document this work.

## NOMENCLATURE

L	self inductance	henry = $\frac{\text{volt sec}}{\text{ampere}}$
W	energy stored in the magnetic field	joules
$\mu_0$	magnetic permeability of free space ( $4\pi \times 10^{-7}$ )	henry/meter = $\frac{\text{weber}}{\text{amp meter}}$
I	current	amperes
k	constant force	newtons; pounds
F	force	newtons; pounds
B	magnetic flux density	weber/meter <sup>2</sup>
J	current density	ampere/meter <sup>2</sup>
r	radius	inches
a	radius of plate I.D.	meters; inches
b	radius of plate O.D.	meters; inches
ℓ	length	meters; inches
V	shear force per unit circumferential length	pounds/inch
φ	bending angle	radians
U	poisson's ratio	
ω	Deflection	inches
D	flexural rigidity of the plate	pound inches
t	plate thickness	inches
M	bending moment per unit length	pound inches/inches
E	modulus of elasticity	pound/inch <sup>2</sup>

## ELECTRICAL FORCE

The first task is to determine the shape of the force distribution and second to evaluate its magnitude.

where, force  $\int J \times B$

$$B \propto \left(\frac{1}{r}\right) \dots$$

$$J \propto \left(\frac{1}{r}\right) \dots$$

therefore,

$$F \propto \frac{1}{r^2} \dots$$

so,

$$F = \frac{k}{r^2}$$

The electrical force is variable and symmetrical about the cylindrical axis as shown in Figure I.

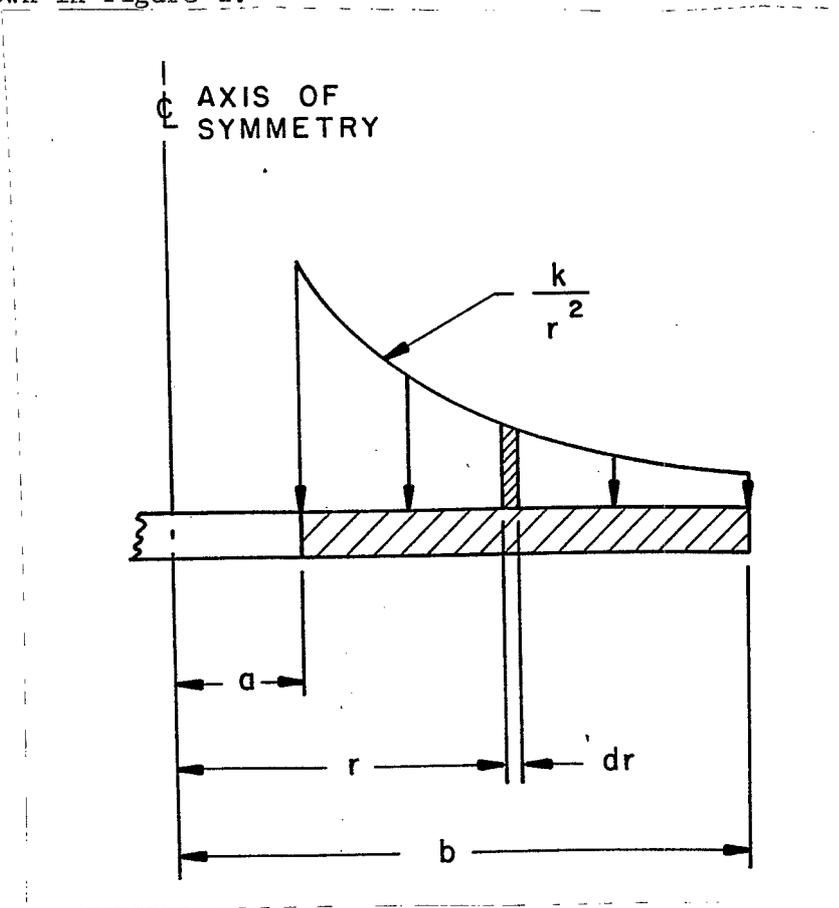


FIGURE I

$$F_{\text{total}} = \int_a^b 2\pi r dr \frac{k}{r^2}$$

$$F_{\text{total}} = 2\pi k \int_a^b \frac{dr}{r} = 2\pi k \ln \frac{b}{a} \quad (\text{a})$$

The stored energy  $W$  in the magnetic field of a self-inductance  $L$  due to a current  $I$  flowing is

$$W = \frac{1}{2} LI^2$$

where,

$$L = \frac{1}{2\pi} \mu_0 \ell \ln \frac{b}{a}$$

therefore,

$$W/\ell = \frac{\mu_0 I^2}{4\pi} \ln \frac{b}{a} \quad (\text{see also Appendix I}) \quad (\text{b})$$

$$F = W/\ell$$

Equate (a) and (b) and solve for  $k$

$$2\pi k \ln \frac{b}{a} = \frac{\mu_0 I^2}{4\pi} \ln \frac{b}{a}$$

$$k = \frac{\mu_0 I^2}{4\pi} \ln \frac{b}{a} / 2\pi \ln \frac{b}{a}$$

$$k = \frac{\mu_0 I^2}{8\pi^2} \quad (\text{newtons}) \quad (\text{c})$$

The force per unit area is

$$F = \frac{k}{r^2} = \frac{\mu_0 I^2}{8\pi^2 r^2} \quad (\text{newtons/meter}^2) \quad (\text{d})$$

## ASSUMPTIONS

1. The circular plate is perfectly elastic, homogeneous and isotropic.
2. The plate is of uniform thickness.
3. The plate is initially flat.
4. The deflection of the plate is small compared to its thickness ( $\omega \leq 1/2t$ ).
5. Deformation is symmetrical about the cylindrical axis.
6. All forces, loads and reactions are parallel to the cylindrical axis.
7. The shear effect on bending is negligible.

## MATHEMATICAL CONCEPTS

From statics the shearing force per unit tangential length  $V$  at any radius within the load distribution region,  $a \leq r \leq b$  (see Figure I), is given as

$$V = \frac{1}{2\pi r} \int_a^r \frac{k}{r^2} 2\pi r dr = \frac{k}{r} \ln \frac{r}{a} \quad (1)$$

and the shear force for the unloaded region,  $0 \leq r \leq a$ , is

$$V = 0 \quad (2)$$

The specific slope, deflection and bending moment equations obtained here are based on the derivation developed by Timoshenko in Chapter 3 of his volume Theory of Plates and Shells.<sup>1</sup> Equation 54 from Timoshenko (rewritten below as equation 3)<sup>2</sup> can be used to determine the slope of the deflection surface,  $\phi$ , and the deflection  $\omega$  of the plate.

$$\frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \frac{\phi}{r^2} = -\frac{V}{D} \quad (3)$$

where

$$D = \frac{Et^3}{12(1 - \nu^2)}$$

The equations for the loaded region are obtained by substituting equation (1) in (3)

$$\frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \frac{\phi}{r^2} = -\frac{1}{D} \left( \frac{k}{r} \ln \frac{r}{a} \right)$$

$$\frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r\phi) \right) = -\frac{1}{D} \left( \frac{k}{r} \ln \frac{r}{a} \right)$$

integrate

$$\frac{d}{dr} (r\phi) = -\frac{kr}{D} \left( \frac{1}{2} (\ln r)^2 - \ln r (\ln a) \right) + c_1 r$$

$$\phi = -\frac{d\omega}{dr} = -\frac{kr}{4d} \left( (\ln \frac{r}{a})^2 - \ln \frac{r}{a} - (\ln a)^2 + \frac{1}{2} \right) +$$

$$\frac{c_1 r}{2} + \frac{c_2}{r} \quad (4)$$

$$\omega = \frac{kr^2}{8D} \left( (\ln \frac{r}{a} - 1)^2 - (\ln a)^2 + \frac{1}{2} \right) -$$

$$\frac{c_1 r^2}{4} - c_2 \ln r + c_3 \quad (5)$$

The equations for the unloaded region are obtained by substituting equation (2) in (3)

$$\frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (\phi r) \right) = -\frac{1}{D} (0)$$

integrate

$$\frac{d}{dr} (r\phi) = c_4 r$$

$$\phi = -\frac{dw}{dr} = -\frac{C_4 r}{2} + \frac{C_5}{r} \quad (6)$$

$$\omega = -\frac{C_4 r^2}{4} - C_5 \ln r + C_6. \quad (7)$$

The bending moments for the loaded region of the plate are obtained by substituting the expressions for  $\phi$  (equation 4) and  $d\phi/dr$  (equation 8) in Timoshenko's equations (52) and (53) rewritten below as equations (9) and (10).<sup>3</sup>

$$\begin{aligned} \frac{d\phi}{dr} = & -\frac{kr}{4D} \left[ 2\left(\ln \frac{r}{a}\right) \frac{1}{r} \left(\frac{1}{a} - \frac{1}{r}\right) - \right. \\ & \left. \frac{k}{4D} \left[ \left(\ln \frac{r}{a}\right)^2 - \ln \frac{r}{a} - \left(\ln a\right)^2 + \frac{1}{2} \right] + \right. \\ & \left. \frac{C_1}{2} - \frac{C_2}{r^2} \right] \\ \frac{d\phi}{dr} = & -\frac{kr}{4D} \left[ 2\left(\ln \frac{r}{a}\right) \left(\frac{1}{r} - \frac{1}{a}\right) - \right. \\ & \left. \frac{k}{4D} \left[ \left(\ln \frac{r}{a}\right)^2 - \ln \frac{r}{a} - \left(\ln a\right)^2 + \frac{1}{2} \right] + \right. \\ & \left. \frac{C_1}{2} - \frac{C_2}{r^2} \right] \end{aligned} \quad (8)$$

$$M_r = D \left( \frac{d\phi}{dr} + \frac{\nu}{r} \phi \right) \quad (9)$$

$$M_t = D \left( \frac{\phi}{r} + \nu \frac{d\phi}{dr} \right) \quad (10)$$

The constants of integration  $C_1$ ,  $C_2$  and  $C_3$  are determined from the boundary conditions at the edge and center of the circular plate. The neutrino horn end plates have their outer edge fixed and supported, and the inner edge fixed as depicted in Figure II.

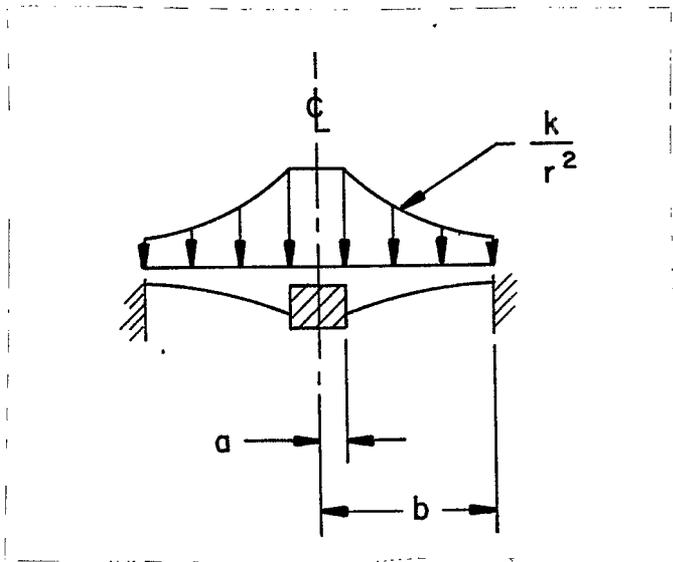


FIGURE II

The boundary conditions for this case are

$$\frac{d\omega}{dr} = 0 \text{ when } r = a$$

$$\frac{d\omega}{dr} = 0 \text{ when } r = b$$

$$\omega = 0 \text{ when } r = b$$

The three equations to be solved to obtain the constants of integration for this case are

$$0 = -\frac{ka}{4d} \left( \frac{1}{2} - (\ln a)^2 \right) + \frac{C_1 a}{2} + \frac{C_2}{a}$$

$$0 = -\frac{kb}{4D} \left\{ \left( \ln \frac{b}{a} \right)^2 - \ln \frac{b}{a} - (\ln a)^2 + \frac{1}{2} \right\} + \frac{C_1 b}{2} + \frac{C_2}{b}$$

$$0 = \frac{kb^2}{8D} \left\{ \left( \ln \frac{b}{a} - 1 \right)^2 - (\ln a)^2 + \frac{1}{2} \right\} - \frac{C_1 b^2}{4} - C_2 \ln b + C_3$$

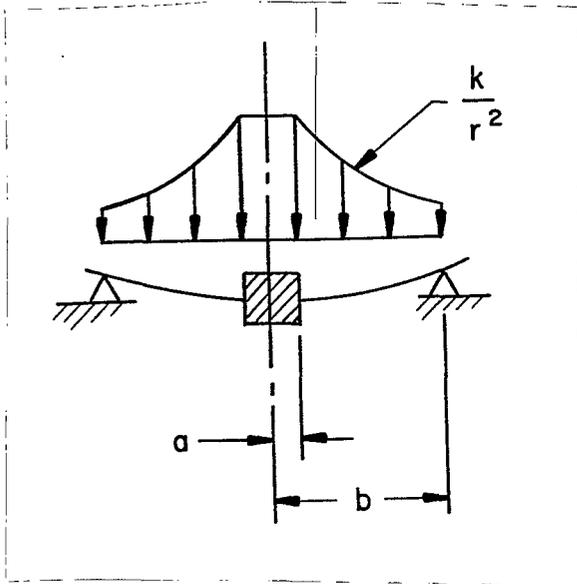
## CONCLUSIONS

Since the equations for the slope, deflection and bending moments are best solved by computer no attempt has been made to substitute equations (4) and (8) in (9) and (10), or substitute the constants of integration in equations (4) and (5) and simplify the results. A large amount of time would be required to do these manipulations; the chances for error in this process are large, and the time saved doing the calculation by computer is negligible.

After the maximum bending moment has been determined the maximum stress is evaluated using<sup>4</sup>

$$S_{\max} = \frac{6 M_{\max}}{t^2}$$

The boundary conditions for three other edge support conditions are listed below.



### EDGE CONDITIONS

Outer edge simply supported.

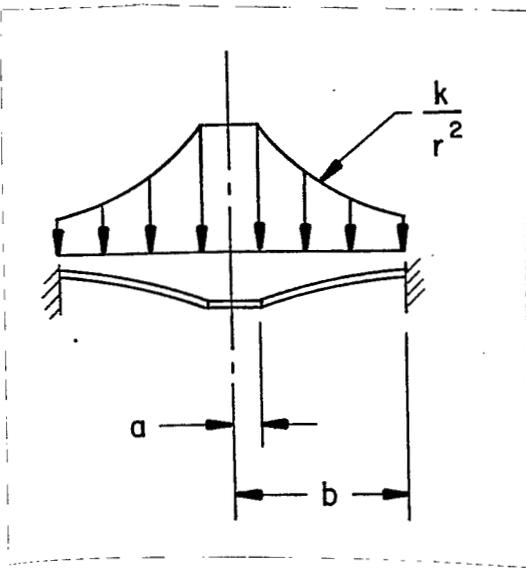
Inner edge fixed.

### BOUNDARY CONDITIONS

$$\omega = 0 \text{ when } r = b$$

$$M_r = 0 \text{ when } r = b$$

$$\frac{d\omega}{dr} = 0 \text{ when } r = a$$



BOUNDARY CONDITIONS

$\omega = 0$  when  $r = b$

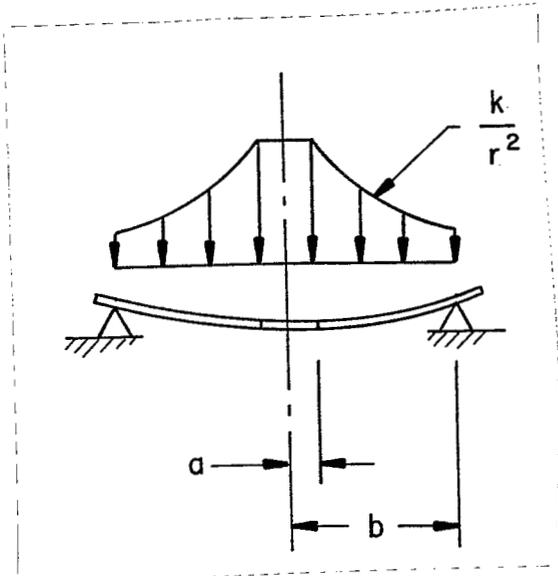
$\frac{d\omega}{dr} = 0$  when  $r = b$

$M_r = 0$  when  $r = a$

EDGE CONDITIONS

Outer edge supported and fixed.

Inner edge free.



EDGE CONDITIONS

Outer edge simply supported.

Inner edge free.

BOUNDARY CONDITIONS

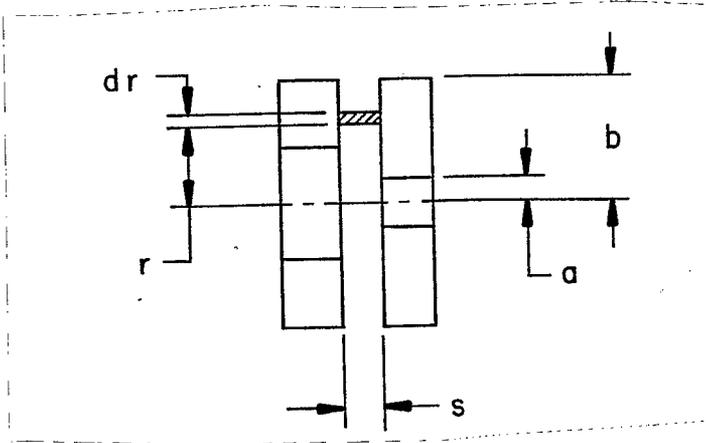
$\omega = 0$  when  $r = b$

$M_r = 0$  when  $r = b$

$M_r = 0$  when  $r = a$

APPENDIX I

FORCE ON END PLATE<sup>5</sup>



$w_m$  = Energy density of magnet field (joules/m<sup>3</sup>)

$H$  = Magnetic field intensity (amp/m)

$v$  = volume (m<sup>3</sup>)

$$W = \int_v w_m dv = \frac{1}{2} \int_v \mu_o H^2 dv$$

$$W = \frac{1}{2} \int_a^b \int_0^{2\pi} \mu_o H^2 S r dr d\theta$$

$$W = \frac{1}{2} S \int_a^b \mu_o H^2 r dr \Big|_0^{2\pi}$$

$$W = \frac{1}{2} (2) \pi S \mu_o \int_a^b H^2 r dr = \pi S \mu_o \int_a^b H^2 r dr$$

$$B(\text{at } r) = \frac{\mu_o I}{2\pi r}$$

$$H = \frac{B}{\mu_o} = \frac{I}{2\pi r}; H^2 = \frac{I^2}{4\pi^2 r^2}$$

$$W = \pi S \mu_o \frac{I^2}{4\pi^2} \int_a^b \frac{r}{r^2} dr = \frac{\mu_o S I^2}{r\pi} \int_a^b \frac{dr}{r}$$

$$W = \frac{\mu_o S I^2}{4\pi} \ln \frac{b}{a}$$

then,

$$F = - \frac{dW}{dS} = - \frac{\mu_o I^2}{4\pi} \ln \frac{b}{a} \text{ (newtons)}$$

FOOTNOTES

1. Timoshenko, S. and Woinowsky-Krieger, S., The Theory of Plates and Shells, McGraw-Hill Book Co., Inc., 1959, pp. 51-78.
2. IBID., p. 53.
3. IBID., p. 52.
4. IBID., p. 42
5. Kraus, John D., Electromagnetics, McGraw Hill Book Co., Inc. 1953, pp. 264-265.

WPC

Distribution: Dept. S&P