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Using the Minimal Normal Form Method With Discrete Maps

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Using the Minimal Normal Form Method With Discrete Maps

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We modify the method of minimal normal forms to treat discrete maps rather than continuous differential equations. We start with a general description of the nature and aims of the method, then give a simple non-trivial example, followed by a general scheme. Various models are used to discuss some of the constraints imposed by the need for symplecticity.

1 Introduction

In an accelerator with many components, it is very cumbersome and time consuming to calculate the transfer map of each component in a long-term tracking study. Instead, the concept of the one-turn map [1] was proposed to both improve the precision of the calculation and to save computing time. In calculating the one-turn map, the condition of symplecticity, and higher order effects, and convergence are of paramount importance. Here, a new method is proposed to improve the convergence of the calculation. Its comparison with other methods will be given in a later report.

We present a progress report on work done on applying the method of "minimal normal forms" to discrete maps. The method is adapted from that developed for use with differential equations by Kahn and Zarmi [2]. However, for accelerators, or, specifically, synchrotrons and storage rings, we are more interested in applications to *maps*, especially the one-turn map. In this report, we start with a general description of the nature and aims of the method. To make the statements more precise and visualizable, we then give a simple but nontrivial example. We then present the general scheme, and we use it to treat more complicated models. At the end, we discuss some of the consequences of the symplectic condition, i.e. the implications for the minimal normal form method. Throughout this report, we shall only treat one coordinate and one momentum variable, i.e. a two dimensional phase space. The method will be extended to cover a four or six dimensional phase space in a later report.

2 General Remarks

The method of minimal normal forms is a method for solving nonlinear equations of motion, originally developed for use with differential equations, but obviously it has applications to particle tracking. It is hoped, but not proved, that the method has advantages over other techniques, e.g. a larger radius of convergence in terms of the strength of the nonlinearities. In this section we give a brief description of the procedure. The general scheme will be given later in this report. Suppose we have a coordinate-momentum pair (x, p), and, in the absence of nonlinearities, they satisfy the one-turn relation

$$\begin{pmatrix} x \\ p \end{pmatrix}_{n+1} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}_n \tag{1}$$

with an obvious notation. We define $z = x + ip$, so that $z_{n+1} = \lambda\, z_n$, with $\lambda = e^{-i\mu}$. We now write the more general equation of motion as

$$\begin{aligned} z_{n+1} &= \lambda\, z_n + \epsilon Z_1(z_n, z_n^*) + \epsilon^2 Z_2(z_n, z_n^*) + \cdots \\ &= \lambda\, z_n + \sum_{k=1}^{\infty} \epsilon^k Z_k(z_n, z_n^*)\,, \end{aligned} \tag{2}$$

where ϵ is the "small parameter" which characterizes the strength of the nonlinearities. (If there are several small parameters, we scale them all by ϵ for now, and worry about a multi-parameter expansion later.) The functions Z_k are homogenous polynomials of degree $k+1$ in z and z^*, and we define

$$Z_k(z, z^*) = \sum_{p+q=k+1} Z_{pq}\, z^p z^{*\,q}\,, \tag{3}$$

so that, for example,

$$\begin{aligned} Z_2(z, z^*) &= Z_{20} z^2 + Z_{11} z z^* + Z_{02} z^{*\,2}\,, \\ Z_3(z, z^*) &= Z_{30} z^3 + Z_{21} z^2 z^* Z_{12} z z^{*\,2} + Z_{03} z^{*\,3}\,, \end{aligned} \tag{4}$$

etc. We now introduce the normal form variable u, for which we hope to get a "simple" equation of motion, via

$$z = u + \epsilon T_1(u, u^*) + \cdots = u + \sum_{k=1}^{\infty} \epsilon^k T_k(u, u^*)\,. \tag{5}$$

As with the Z_k, the T_k are homogenous polynomials of degree $k+1$, but in the variables u and u^*, and we define

$$T_k(u, u^*) = \sum_{p+q=k+1} T_{pq}\, u^p u^{*\,q}\,. \tag{6}$$

To borrow some terminology from matrices, u is the variable in terms of which the one-turn map is "diagonalized". The next step is to specify a procedure to determine the T_{pq}. We substitute Eq. (5) into Eq. (2), using also Eqs. (3) and (6). This gives an equation for u_{n+1} in terms of u_n and u_n^*:

$$u_{n+1} = \lambda\, u_n + O(\epsilon) + O(\epsilon^2) + \cdots \tag{7}$$

We then choose the T_{pq} to *cancel as many terms on the r.h.s. of Eq. (7) as possible.* This will be made more precise below. It turns out that this fixes most, but not all, of the T_{pq}. Extra criteria have to be set to determine the remaining so-called "free terms". The terms which remain on the r.h.s. above yield the equation for u_{n+1}. We shall see below that they have the form $u^{k+1}u^{*\,k}$, yielding an equation of the form

$$u_{n+1} \;=\; \lambda\, u_n + \epsilon^2 \widetilde{U}_2 u^2 u^* + \epsilon^4 \widetilde{U}_4 u^3 u^{*\,2} + \cdots \;=\; \lambda\, u_n \left[1 + \sum_{k=1}^{\infty} \epsilon^{2k} \frac{\widetilde{U}_{2k}}{\lambda} (uu^*)^k \right]. \tag{8}$$

The "free terms", i.e. the undetermined T_{pq}, are of the form $T_{k+1\,k}$, i.e. the coefficients of $u^{k+1}u^{*\,k}$ in Eq. (6). It would be nice if we could choose the free terms to make the $\widetilde{U}_{2k}$ all vanish, but unfortunately this turns out to be not possible, so instead we choose the free terms to make the series on the r.h.s. above *add up to an oscillating exponential series*, viz.

$$u_{n+1} \;=\; \lambda\, u_n \exp\left(i\epsilon^2\, k\, uu^*\right), \tag{9}$$

where k is a *real* constant, which in fact equals $-i\widetilde{U}_2/\lambda$. *If* this can be achieved, then the solution for u_n can be obtained in closed form:

$$u_n = \rho\, e^{-i\psi_n}, \tag{10}$$

with ρ constant and

$$\psi_{n+1} \;=\; \psi_n + \mu - \epsilon^2\, k\, \rho^2, \tag{11}$$

i.e.

$$u_n = \rho\, \exp\Big[-in\,(\mu - \epsilon^2 k \rho^2) + i\psi_0\Big], \tag{12}$$

which implies an amplitude dependent tuneshift. Substitution into the transformation from z to u then gives the solution for the original variables x_n and p_n.

3 Sextupole Map

The above discussion was purely abstract. We make it more concrete by solving a special example, one well known to all accelerator physicists, viz. a one-turn linear map plus one thin-lens sextupole. Without loss of generality, the equation of motion is

$$\begin{pmatrix} x \\ p \end{pmatrix}_{n+1} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}_n + \epsilon \begin{pmatrix} 0 \\ x_{n+1}^2 \end{pmatrix} . \tag{13}$$

In terms of z, this reads

$$z_{n+1} = \lambda\, z_n + \frac{i\,\epsilon}{4}\,(\,\lambda z_n + \lambda^* z_n^*\,)^2 . \tag{14}$$

To $O(\epsilon)$, we have $z = u + \epsilon T_1$, and substituting into Eq. (14), we obtain

$$u_{n+1} + \epsilon\, T_1(u_{n+1}, u_{n+1}^*) = \lambda\,(\,u + \epsilon\, T_1(u_n, u_n^*)\,) + \frac{i\,\epsilon}{4}\,(\,\lambda u_n + \lambda^* u_n^*\,)^2 . \tag{15}$$

On the l.h.s. we can put $u_{n+1} = \lambda u_n$ in T_1, which yields

$$\begin{aligned} u_{n+1} &= \lambda\, u_n + \epsilon \left[T_{20}(\lambda - \lambda^2)\, u_n^2 + T_{11}(\lambda - 1)\, u_n u_n^* + T_{02}(\lambda - \lambda^{*2})\, u_n^{*2} \right] \\ &\quad + \frac{i\,\epsilon}{4}\,(\,\lambda u_n + \lambda^* u_n^*\,)^2 . \end{aligned} \tag{16}$$

We want all the $O(\epsilon)$ terms to vanish, which yields the solutions

$$T_{20} = \frac{i}{4}\frac{\lambda}{\lambda - 1}, \qquad T_{11} = -\frac{i}{2}\frac{1}{\lambda - 1}, \qquad T_{02} = -\frac{i}{4}\frac{1}{\lambda^3 - 1}, \tag{17}$$

after a little algebra. The equation of motion for u_n is thus $u_{n+1} = \lambda\, u_n + O(\epsilon^2)$, with no term of $O(\epsilon)$. At the next order, we put $z = u + \epsilon T_1 + \epsilon^2 T_2$, and calculate T_{30}, T_{21}, T_{12} and T_{03}. We find that

$$u_{n+1} + \epsilon^2 T_2(\lambda u_n, \lambda^* u_n^*) = \lambda \left(u + \epsilon^2 T_2(u_n, u_n^*) \right) + \frac{i\,\epsilon^2}{2}\,(\lambda u_n + \lambda^* u_n^*)(\lambda T_1 + \lambda^* T_1^*), \tag{18}$$

which leads to

$$\begin{aligned} &\epsilon^2 \Big[(\lambda^3 - \lambda) T_{30} u^3 + (\lambda - \lambda) T_{21} u^2 u^* \\ &\quad + (\lambda^* - \lambda) T_{12} u u^{*2} + (\lambda^{*3} - \lambda) T_{03} u^{*3} \Big] = \frac{i\,\epsilon^2}{2} \Big[u^3\,(\,\lambda^2 T_{20} + T_{02}^*) \\ &\qquad + u^2 u^*\,(\,\lambda^2 T_{11} + T_{11}^* + T_{20} + \lambda^{*2} T_{02}^*\,) \\ &\qquad + u u^{*2}\,(\,\lambda^2 T_{02} + T_{20}^* + T11 + \lambda^{*2} T_{11}^*\,) \\ &\qquad + u^{*3}\,(\,T_{02} + \lambda^{*2} T_{20}^*\,) \Big] . \end{aligned} \tag{19}$$

We now notice that we cannot determine T_{21} because its coefficient vanishes, and consequently we cannot eliminate the u^2u^* term on the r.h.s. Hence T_{21} is a "free function". The solutions for the other functions are

$$
\begin{aligned}
T_{30} &= \frac{i}{2}\,\frac{\lambda^2 T_{20} + T_{02}^*}{\lambda^3 - \lambda}, \\
T_{12} &= \frac{i}{2}\,\frac{\lambda^2 T_{02} + T_{20}^* + T_{11} + \lambda^{*2} T_{11}^*}{\lambda^* - \lambda}, \\
T_{03} &= \frac{i}{2}\,\frac{T_{02} + \lambda^{*2} T_{20}^*}{\lambda^{*3} - \lambda}.
\end{aligned}
\tag{20}
$$

The term which appears in the equation for u_n is $\epsilon^2 \widetilde{U}_2 u^2 u^*$, and

$$
\widetilde{U}_2 = \frac{i}{2}\left(\lambda^2 T_{11} + T_{11}^* + T_{20} + \lambda^{*2} T_{02}^* \right), \tag{21}
$$

so

$$
u_{n+1} = \lambda\, u_n + \epsilon^2\, \widetilde{U}_2\, u^2 u^* + \cdots \tag{22}
$$

This is the beginning of the exponential series. We see that the constant k is

$$
\begin{aligned}
k = -i\,\frac{\widetilde{U}_2}{\lambda} &= \frac{1}{2\lambda}\left[-\frac{1}{2}\frac{\lambda^2}{\lambda - 1} + \frac{1}{2}\frac{1}{\lambda^* - 1} + \frac{1}{4}\frac{\lambda}{\lambda - 1} + \frac{1}{4}\frac{\lambda^{*2}}{\lambda^{*3} - 1} \right] \\
&= -\frac{i}{8}\,\frac{2\lambda^3 + 3\lambda^2 + 3\lambda + 2}{\lambda^3 - 1}.
\end{aligned}
\tag{23}
$$

It is straightforward to verify that k is indeed real using the property $\lambda^* = 1/\lambda$.

The fact that k is real is not trivial. Notice that we had no way to adjust the value of $\widetilde{U}_2$, so if k had turned out not to be real, then the minimal normal form method would fail. If k were complex, the equation for u_{n+1} would contain a real exponential, leading to damping or growth of the phase space.

Although not proved above, it can easily be verified that, in general, there is no control over the value of $\widetilde{U}_2$. There is therefore something outside the minimal normal form method which constrains $\widetilde{U}_2$. The matter will be discussed below when studying the constraints imposed by the symplectic condition.

The next step is to calculate T_3 and T_4. As with T_{21}, we find that all the T_{pq} can be determined except T_{32}. The algebra is tedious but the results for T_3 are:

$$
\begin{aligned}
T_{40} &= -\frac{i}{4}\frac{1}{\lambda-\lambda^4}\left[(\lambda T_{20}+\lambda^* T_{02}^*)^2 + 2\lambda(\lambda T_{30}+\lambda^* T_{03}^*)\right], \\
T_{31} &= -\frac{i}{4}\frac{1}{\lambda-\lambda^2}\Big[2(\lambda T_{20}+\lambda^* T_{02}^*)(\lambda T_{11}+\lambda^* T_{11}^*), \\
&\qquad +2\lambda(\lambda T_{21}+\lambda^* T_{12}^*) + 2\lambda^*(\lambda T_{30}+\lambda^* T_{03}^*) - 8\lambda^2 k T_{20}\Big], \\
T_{22} &= -\frac{i}{4}\frac{1}{\lambda-1}\Big[(\lambda T_{11}+\lambda^* T_{11}^*)^2 + 2(\lambda T_{20}+\lambda^* T_{02}^*)(\lambda T_{02}+\lambda^* T_{20}^*) \\
&\qquad +2\lambda(\lambda T_{12}+\lambda^* T_{21}^*) + 2\lambda^*(\lambda T_{21}+\lambda^* T_{12}^*)\Big], \\
T_{13} &= -\frac{i}{4}\frac{1}{\lambda-\lambda^{*2}}\Big[2(\lambda T_{02}+\lambda^* T_{20}^*)(\lambda T_{11}+\lambda^* T_{11}^*) \\
&\qquad +2\lambda^*(\lambda T_{12}+\lambda^* T_{21}^*) + 2\lambda(\lambda T_{03}+\lambda^* T_{30}^*) + 8\lambda^{*2} k T_{02}\Big], \\
T_{04} &= -\frac{i}{4}\frac{1}{\lambda-\lambda^{*4}}\left[(\lambda T_{02}+\lambda^* T_{20}^*)^2 + 2\lambda^*(\lambda T_{03}+\lambda^* T_{30}^*)\right].
\end{aligned} \tag{24}
$$

There is no need to write out T_4 explicitly. The object of principal interest is $\widetilde{U}_4$, the second term in the exponential series for the amplitude dependent tuneshift, the coefficient of u^3u^{*2}. We find

$$
\begin{aligned}
\widetilde{U}_4 &= \frac{i}{2}\Big[\lambda(\lambda T_{22}+\lambda^* T_{22}^*) + \lambda^*(\lambda T_{31}+\lambda^* T_{13}^*) \\
&\qquad +(\lambda T_{20}+\lambda^* T_{02}^*)(\lambda T_{12}+\lambda^* T_{21}^*) + (\lambda T_{11}+\lambda^* T_{11}^*)(\lambda T_{21}+\lambda^* T_{12}^*) \\
&\qquad +(\lambda T_{02}+\lambda^* T_{20}^*)(\lambda T_{30}+\lambda^* T_{03}^*) - 2\lambda\, k\, T_{21}\Big].
\end{aligned} \tag{25}
$$

The problem facing us is: can we somehow adjust this so that (1) $\widetilde{U}_4/\lambda$ is purely real, and (2) it then fits to the exponential series, i.e. $\mathrm{Re}(\widetilde{U}_4/\lambda) = -k^2/2$?

We have, at our disposal, the free function T_{21}. It could not be used to control $\widetilde{U}_2$, but we see now that it can be used to control $\widetilde{U}_4$. This feature will persist, i.e. the free functions cannot be constrained at the lowest order of perturbation theory at which they appear, but they can be determined by using them to control *higher* order $\widetilde{U}_{2k}$ terms. The way we exercise this control is the key feature of the minimal normal form method.

Returning to $\widetilde{U}_4$, we discover three remarkable and significant facts. The first is that only the *real* part of T_{21} appears in $\widetilde{U}_4/\lambda$, the second is that it only appears in the *imaginary* part of $\widetilde{U}_4/\lambda$:

$$\frac{\widetilde{U}_4}{\lambda} = \frac{ik}{8}\,(\,T_{21} + T_{21}^*\,) + \cdots \tag{26}$$

where k is the same constant that appeared in $\widetilde{U}_2$. Hence only the real part of T_{21} is constrained by $\widetilde{U}_4$ — to constrain $\mathrm{Im}(T_{21})$ we shall have to go to $\widetilde{U}_6$ — and, more importantly, we have *no control* over the real part of $\widetilde{U}_4/\lambda$: we are *unable* to adjust it to equal $-k^2/2$ as desired. This is the third remarkable fact: $\mathrm{Re}(\widetilde{U}_4/\lambda)$ equals $-k^2/2$ *without any adjustment*: the above expression for $\widetilde{U}_4$, when simplified, yields

$$\frac{\widetilde{U}_4}{\lambda} = -\frac{k^2}{2} + \frac{ik}{8}\,(T_{21} + T_{21}^*) + (\text{imaginary})\,. \tag{27}$$

Hence everything is all right, we use $\mathrm{Re}(T_{21})$ to make $\mathrm{Im}(\widetilde{U}_4/\lambda)$ vanish, and proceed to $\widetilde{U}_6$. At that point, we find the following: (1) only $\mathrm{Re}(T_{32})$ appears in $\widetilde{U}_6$, and only in the imaginary part of $\widetilde{U}_6/\lambda$, but $\mathrm{Im}(T_{21})$ appears in the real part of $\widetilde{U}_6/\lambda$, so we have enough degrees of freedom to control $\widetilde{U}_6$ completely. Similarly, we determine $\mathrm{Im}(T_{32})$ and $\mathrm{Re}(T_{43})$ by adjusting the value of $\widetilde{U}_8$, and so on for all the other $\widetilde{U}_{2k}$.

We have thus established that it is indeed possible to choose the T_{pq}, in particular the free functions, to yield a simple equation for u_n, which can be solved exactly (in closed form), reducing the effects of the nonlinearity to an amplitude dependent tuneshift. In so doing, we discovered two significant difficulties, which fortunately were non-problems in the above example: we had no control over $\widetilde{U}_2$ and $\mathrm{Re}(\widetilde{U}_4/\lambda)$, and had to hope that $\widetilde{U}_2/\lambda$ was pure imaginary, and $\mathrm{Re}(\widetilde{U}_4/\lambda)$ was equal to $(1/2)(\widetilde{U}_2/\lambda)^2$. Both hopes were fulfilled. If the minimal normal form method is to work in general, though, it must be verified that $\widetilde{U}_2$ and $\mathrm{Re}(\widetilde{U}_4/\lambda)$ behave as desired. The matter will be discussed again below, in the section on symplecticity.

4 General Formalism

Having given a concrete example and illustrated the application of the technique and some of its unresolved problems, we now give the general implementation of the minimal normal form method for discrete maps. We start with an equation of the form

$$z_{n+1} \;=\; \lambda\, z_n + \sum_{k=1}^{\infty} \epsilon^k\, Z_k(z_n, z_n^*)\,, \tag{28}$$

with

$$Z_k \;=\; \sum_{p+q=k+1} Z_{pq}\, z^p\, z^{*\,q}\,, \tag{29}$$

and we put

$$z \;=\; u + \sum_k T_k(u, u^*)\,, \tag{30}$$

with

$$T_k \;=\; \sum_{p+q=k+1} T_{pq}\, u^p\, u^{*\,q}\,. \tag{31}$$

We substitute into the equation of motion for z, and collect terms in powers of ϵ, to obtain

$$u_{n+1} + \sum_k \epsilon^k T_k(u_{n+1}, u_{n+1}^*) \;=\; \lambda\left[u_n + \sum_k \epsilon^k T_k(u_n, u_n^*)\right] + \sum_k \epsilon^k N_k(u, u^*)\,, \tag{32}$$

with new monomials

$$N_k \;=\; \sum_{p+q=k+1} N_{pq}\, u^p\, u^{*\,q}\,. \tag{33}$$

This can be performed quite easily using a symbolic manipulator program, or using differential algebra. We then *postulate* that the equation for u_{n+1} is

$$u_{n+1} \;=\; \lambda\, u_n\, \exp(\, i\epsilon^2\, k\, u_n u_n^*\,) \tag{34}$$

where $k = -i\widetilde{U}_2/\lambda$ is assumed to be real (recall that we have no control over $\widetilde{U}_2$). Using this ansatz, we find that

$$\begin{aligned}\sum_{p+q=k+1} \epsilon^k T_{pq}\,(\,\lambda^{p-q} - \lambda\,)\, u^p u^{*\,q} \;&=\; \sum_{p+q=k+1} \epsilon^k N_{pq} u^p u^{*\,q} \\ &\quad - \sum_{p+q=k+1} \epsilon^k\, \lambda^{p-q} u^p u^{*\,q} \sum_{r\geq 1} T_{p-r\,q-r}\, \frac{[i(p-q)k]^r}{r!}\,.\end{aligned} \tag{35}$$

The solution, except for the free terms (those for which $p - q = 1$), is

$$T_{pq} \;=\; \frac{1}{\lambda^{p-q} - \lambda}\left[N_{pq} - \lambda^{p-q} \sum_{r\geq 1} T_{p-r\,q-r}\, \frac{[i(p-q)k]^r}{r!}\right]. \tag{36}$$

It is understood that the sum over r terminates when the indices reach zero or negative values. Formally, we simply set all such $T_{p-r\,q-r}$ to zero. The free functions are determined by requiring that the terms in the series expansion for the equation for u_{n+1} add up to the complex exponential $\exp(\, i\epsilon^2\, k\, u_n u_n^*)$.

When expressed in this concise form, all that remains is to supply the values of the Z_{pq}. Everything else follows automatically. As explained in the previous section, though, this disguises some unresolved problems about our ability to obtain the desired exponential series. Anyway, armed with the above general formalism, we can now treat more complicated (more "realistic") models.

5 FODO Cell

5.1 General

A good model to study is a FODO cell, made up of thin lenses, with some nonlinear elements. A sketch of the cell is shown below in Fig. 1.

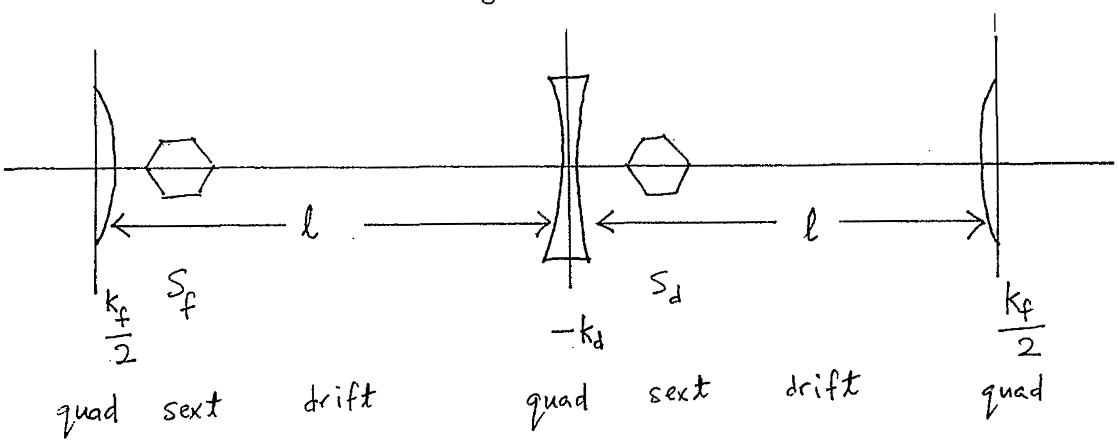

Fig. 1 Thin lens FODO cell with two nonlinear elements.

The cell consists of half a focusing quadrupole, with focusing strength $k_f/2$, a drift of length l, a defocusing quad of strength $-k_d$, another drift of length l and half a focusing quadrupole of strength $k_f/2$. This has the advantage that the ends of the cell are symmetry points of the lattice.

Thin lens nonlinear elements, e.g. sextupoles, are placed immediately after the quadrupoles, as indicated. The map is given by

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = M \begin{pmatrix} x \\ x' \end{pmatrix}_n \tag{37}$$

where

$$M = \begin{pmatrix} 1 + l(k_d - k_f) - k_f k_d l^2/2 & l\,(2 + k_d l\,) \\ (1 - k_f l/2\,)\,(\,k_d - k_f - k_f k_d l/2\,) & 1 + l(k_d - k_f) - k_f k_d l^2/2 \end{pmatrix}, \tag{38}$$

which yields, with the usual notation,

$$\begin{aligned} \cos\mu &= 1 + l(k_d - k_f) - \frac{k_f k_d l^2}{2}, \\ \alpha &= 0, \\ \beta \sin\mu &= l\,(2 + k_d l\,), \\ \gamma \sin\mu = \frac{\sin\mu}{\beta} &= (1 - \frac{k_f l}{2})\frac{1 - \cos\mu}{l}. \end{aligned} \tag{39}$$

The conjugate momentum is $p = \alpha x + \beta x' = \beta x'$. In terms of x and p, we recover the usual rotation matrix

$$\begin{pmatrix} x \\ p \end{pmatrix}_{n+1} = R \begin{pmatrix} x \\ p \end{pmatrix}_n \tag{40}$$

with

$$R = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}. \tag{41}$$

5.2 Sextupoles

If the nonlinear elements are two sextupoles, with strengths $\epsilon\, s_f$ and $-\epsilon\, s_d$ respectively, so that $\Delta x = 0$ and $\Delta x' = \pm s_{f,d}\, x^2$ respectively, then the map is

$$\begin{aligned} x_{n+1} &= \cdots - \epsilon\, s_f\, \beta \sin\mu\, x^2 + \epsilon\, s_d l \left[\left(1 - \frac{k_f l}{2}\right) x + \frac{lp}{\beta} - \epsilon\, s_f\, l\, x^2 \right]^2, \\ p_{n+1} &= \cdots - \epsilon\, s_f\, \beta \cos\mu\, x^2 + \epsilon\, s_d \beta \left(1 - \frac{k_f l}{2}\right) \left[\left(1 - \frac{k_f l}{2}\right) x + \frac{lp}{\beta} - \epsilon\, s_f\, l\, x^2 \right]^2, \end{aligned} \tag{42}$$

where the dots denote the linear terms. This yields the following equation for $z = x + ip$:

$$z_{n+1} = \lambda\, z_n - \frac{i\,\epsilon\, s_f \beta \lambda}{4}\,(z + z^*)^2 + \frac{i\,\epsilon\, s_d \beta C}{4} \left[C z_n + C^* z_n^* - \frac{\epsilon\, s_f l}{2}\,(z_n + z_n^*)^2 \right]^2. \tag{43}$$

Here C is a constant given by

$$C = 1 - \frac{k_f l}{2} - i\frac{l}{\beta} \tag{44}$$

and has the property $C = C^* \lambda$.

If we put $s_d = 0$, then the map reduces to

$$z_{n+1} = \lambda z_n - \frac{i \epsilon s_f \beta \lambda}{4} (z + z^*)^2 . \tag{45}$$

This has almost the form treated earlier. If we put $z = \lambda w$, then we find

$$w_{n+1} = \lambda w_n - \frac{i \epsilon s_f \beta}{4} (\lambda w_n + \lambda^* w_n^*)^2 \tag{46}$$

which is exactly the form of the equation already solved above, with $\epsilon \to -\epsilon s_f \beta$. Hence we can transcribe the solution of that problem to this model.

If we put $s_f = 0$ instead, we obtain the equation

$$z_{n+1} = \lambda z_n + \frac{i \epsilon s_d \beta C}{4} (C z_n + C^* z_n^*)^2 . \tag{47}$$

This time we use the transformation $\lambda w = C z$, which yields

$$w_{n+1} = \lambda w_n + \frac{i \epsilon s_d \beta C^2}{4\lambda} (\lambda w_n + \lambda^* w_n^*)^2 . \tag{48}$$

Using the relation $C = C^* \lambda$, we deduce that $C^2 = CC^* \lambda$, so $C^2/\lambda = CC^*$, hence

$$w_{n+1} = \lambda w_n + \frac{i \epsilon s_d \beta C C^*}{4} (\lambda w_n + \lambda^* w_n^*)^2 , \tag{49}$$

and so the equation is again of the type treated earlier, with $\epsilon \to \epsilon s_d \beta C C^*$.

Hence, if there is only one sextupole, it is unnecessary to solve the problem afresh; we can read off the solution for u_n and the T_{pq} from above. If *both* sextupoles are present, then there is no simple transformation to bring the equation into the idealized form treated earlier; in particular, there are cubic and quartic terms in the nonlinearity, i.e. $O(\epsilon^3)$ and $O(\epsilon^4)$ terms.

Use of the general formalism presented above enables us to attack the problem in a systematic fashion. Nevertheless, it is too tedious to write out the solution analytically by hand; instead a computer program was written to evaluate the N_{pq} and T_{pq}, and the tuneshift parameter k, given the Z_{pq} as input. The program was run with several different values of μ, s_f, and s_d, and it was verified that, in all cases, $k = -i\widetilde{U}_2/\lambda$ was real, and $\mathrm{Re}(\widetilde{U}_4/\lambda) = -k^2/2$. It was also verified that only $\mathrm{Re}(T_{21})$ appeared in $\widetilde{U}_4$, and it only affected $\mathrm{Im}(\widetilde{U}_4/\lambda)$. This is an "empirical" argument; it would be good to have an analytical proof. A symbolic manipulator program would presumably be able to handle the task easily. Such work is under way; it will be reported elsewhere.

5.3 Sextupoles and Octupoles

Now that we have both a general formalism *and* a numerical program to implement it, nothing stops us from treating other nonlinear multipoles. If we use a sextupole s_f and an octupole o_d, we get a map of the form

$$z_{n+1} = \lambda z_n - \frac{i\,\epsilon\, s_f \beta\lambda}{4}(z+z^*)^2 + \frac{i\,\epsilon^2\, o_d\beta C}{8}\left[C z_n + C^* z_n^* - \frac{\epsilon\, s_f l}{2}(z_n+z_n^*)^2\right]^3 . \quad (50)$$

The program again indicates that the $\widetilde{U}_{2k}$ behave as desired. There is no point in studying more multipoles; the minimal normal form method seems to work, and it is now more important to investigate more fundamental issues, specifically, *why* does it work? *Why* do the $\widetilde{U}_{2k}$ behave in a "nice" way?

6 Symplecticity

6.1 General

The distinguishing feature of the maps of interest to accelerator physicists is that they are symplectic. Otherwise the maps could be completely arbitrary. A symplectic map preserves the value of the Poisson Bracket: $[x_n, p_n] = 1$, or $[z_n, z_n^*] = -2i$, independent of n. Let us therefore investigate the constraints imposed by the symplectic condition. This turns out to be a big undertaking; we shall therefore only scratch the surface of this subject below.

Let us first verify that the maps studied previously really are symplectic. Let us consider a map of the form

$$z_{n+1} \;=\; \lambda\, z_n + i\epsilon\,(\,\lambda z_n + \lambda z_n^*\,)^N\,, \tag{51}$$

where N is any integer. Any global constants in the perturbation can be absorbed into ϵ, as long as they are real, because we assume that ϵ is real. Then

$$z_{n+1}^* \;=\; \lambda^*\, z_n^* - i\epsilon\,(\,\lambda z_n + \lambda z_n^*\,)^N\,, \tag{52}$$

and so

$$\begin{aligned}[\,z_{n+1},\, z_{n+1}^*\,] \;&=\; \lambda\lambda^*\,[\,z_n,\, z_n^*\,] - i\epsilon\,[\,\lambda\, z_n,\, (\,\lambda z_n + \lambda z_n^*\,)^N\,] + i\epsilon[\,(\,\lambda z_n + \lambda z_n^*\,)^N,\, \lambda\, z_n^*\,] \\ &=\; [\,z_n,\, z_n^*\,] - i\epsilon\,[\,\lambda\, z_n + \lambda^* z_n^*,\, (\,\lambda z_n + \lambda z_n^*\,)^N\,]\,.\end{aligned} \tag{53}$$

The last term vanishes for arbitrary N, which shows that the Poisson Bracket is indeed preserved:

$$[\,z_{n+1},\, z_{n+1}^*\,] \;=\; [\,z_n,\, z_n^*\,]\,. \tag{54}$$

Clearly, the map will be symplectic if the nonlinearity is any polynomial or power series in $\lambda z_n + \lambda^* z_n^*$, with real coefficients. In particular, this establishes that the cubic (Duffing) map studied by Kahn and Zarmi [3] is symplectic.

6.2 Quadratic Perturbation

Let us now proceed in the *opposite* direction. Let us assume that the map is of the form

$$z_{n+1} \;=\; \lambda\left[\,z_n + \epsilon Z_1\,\right] \tag{55}$$

where we have scaled out λ from the whole r.h.s. for convenience. We *demand* that the map be symplectic and see what this implies for the Z_{pq}. Recall that

$$Z_1 \;=\; Z_{20} z^2 + Z_{11} z z^* + Z_{02} z^{*\,2}\,, \tag{56}$$

and we shall drop the subscript n for clarity. Then

$$[\,z_{n+1},\, z_{n+1}^*\,] \;=\; [\,z,\, z^*\,] + \epsilon\,[\,z,\, Z_1^*\,] - \epsilon\,[\,z^*,\, Z_1\,] + \epsilon^2\,[\,Z_1,\, Z_1^*\,]\,. \tag{57}$$

All terms beyond the first must vanish exactly, not merely to some high power of ϵ. The $O(\epsilon)$ term implies that

$$
\begin{aligned}
0 &= [\, z,\, Z_{20} z^{*\,2} + Z_{11} z z^* + Z_{02} z^2 \,] - [\, z^*,\, Z_{20} z^2 + Z_{11} z z^* + Z_{02} z^{*\,2} \,] \\
&= [\, z,\, z^* \,] \Big\{ (\, Z_{11}^* + 2 Z_{20} \,)\, z + (\, Z_{11} + 2 Z_{20}^* \,)\, z^* \Big\} . \qquad (58)
\end{aligned}
$$

which requires that

$$
Z_{11} = -2\, Z_{20}^* . \qquad (59)
$$

The $O(\epsilon^2)$ term implies that

$$
\begin{aligned}
0 &= [\, Z_{20} z^2 + Z_{11} z z^* + Z_{02} z^{*\,2},\, Z_{20} z^{*,2} + Z_{11} z z^* + Z_{02} z^2 \,] \\
&= [\, z,\, z^* \,] \Big\{ z^2 (\, 2\, Z_{20} Z_{11}^* - 2\, Z_1 1 Z_{02}^* \,) + z^{*\,2} (\, 2\, Z_{20}^* Z_{11} - 2\, Z_1 1^* Z_{02} \,) \\
&\qquad + 4 z z^* (\, Z_{20} Z_{20}^* - Z_{02} Z_{02}^* \,) \Big\} . \qquad (60)
\end{aligned}
$$

This leads to two independent but compatible constraints

$$
\begin{aligned}
Z_{02} &= Z_{20}^* Z_{11} / Z_{11}^* , \\
|\, Z_{20} \,|^2 &= |\, Z_{02} \,|^2 . \qquad (61)
\end{aligned}
$$

The overall conclusion is that

$$
Z_{20} = -Z_{11}^*/2 , \qquad Z_{02} = -Z_{11}^2/(2 Z_{11}^*) . \qquad (62)
$$

and so

$$
z_{n+1} = \lambda\, z_n - \frac{\epsilon \lambda}{2 Z_{11}^*} (\, Z_{11}^*\, z_n - Z_{11}\, z_n^* \,)^2 . \qquad (63)
$$

Transforming via $\lambda w_n = i Z_{11}^* z_n$, we obtain

$$
w_{n+1} = \lambda w_n + \frac{i\,\epsilon}{2} (\, \lambda w_n + \lambda^* w_n^* \,)^2 . \qquad (64)
$$

This is of the form studied above, where it is known that both $\widetilde{U}_2$ and $\widetilde{U}_4$ behave properly. We have therefore shown that the requirement of symplecticity forces these functions to have the properties we desire.

6.3 Cubic Perturbation

To study a general nonlinearity is difficult, at the present stage of development of this work. Let us consider another homogenous polynomial perturbation, viz:

$$z_{n+1} = \lambda \left[z_n + \epsilon^2 Z_2 \right], \tag{65}$$

where

$$Z_2 = Z_{30} z^3 + Z_{21} z^2 z^* + Z_{12} z z^{*2} + Z_{03} z^{*3}. \tag{66}$$

We obtain, similarly to before,

$$[\, z_{n+1}, z^*_{n+1} \,] = [\, z, z^* \,] + \epsilon^2 [\, z, Z_2^* \,] - \epsilon^2 [\, z^*, Z_2 \,] + \epsilon^4 [\, Z_2, Z_2^* \,], \tag{67}$$

and demand that all the higher order terms vanish. At $O(\epsilon^2)$, this implies

$$z^2 (\, Z_{12}^* + 3 Z_{30} \,) + 2 z z^* (\, Z_{21} + Z_{21}^* \,) + z^{*2} (\, Z_{12} + 3 Z_{30}^* \,) = 0. \tag{68}$$

This yields the two independent constraints

$$\begin{aligned} Z_{21} + Z_{21}^* &= 0, \\ Z_{12}^* + 3 Z_{30} &= 0. \end{aligned} \tag{69}$$

The $O(\epsilon^4)$ term implies

$$\begin{aligned} Z_{30}^* Z_{12} - Z_{03} Z_{21}^* &= 0, \\ Z_{30}^* Z_{21} - Z_{03} Z_{12}^* &= 0, \\ 3 |\, Z_{30} \,|^2 + |\, Z_{21} \,|^2 - |\, Z_{12} \,|^2 - 3 |\, Z_{03} \,|^2 &= 0. \end{aligned} \tag{70}$$

All of these conditions leave us with only two independent parameters. We can put $Z_{21} = ik$, where k is arbitrary but real, and then we can put $Z_{12} = ik\, e^{i\theta}$, where θ is also arbitrary but real. It then follows that $Z_{30} = ik\, e^{i\theta}/3$ and $Z_{03} = ik\, e^{i2\theta}/3$. We then deduce that

$$z_{n+1} = \lambda\, z_n + \frac{i\, \epsilon^2 k \lambda}{3}\, e^{i\theta/2} (\, e^{-i\theta/2} z_n + e^{i\theta/2} z_n^* \,)^3. \tag{71}$$

Putting $\lambda w = e^{-i\theta/2} z$ transforms this to

$$w_{n+1} = \lambda w_n + \frac{i\epsilon^2 k}{3} (\lambda w_n + \lambda^* w_n^*)^3, \tag{72}$$

which is the form of the equation treated in Ref. [3], where it was shown that $\widetilde{U}_2$ and $\widetilde{U}_4$ behave as desired. The constant k above is the tuneshift parameter:

$$u_{n+1} = \lambda u_n \exp(i\epsilon^2 k\, u_n u_n^*). \tag{73}$$

We have, as stated above, only scratched the surface of the question of symplecticity, but we have managed to show that, for nonlinearities consisting of homogenous polynomials of degrees two and three, the symplectic condition forces $\widetilde{U}_2$ and $\widetilde{U}_4$ to have the behavior required for the minimal normal form method to work. It is, however, reasonable to expect that the symplectic condition will force $\widetilde{U}_2$ and $\widetilde{U}_4$ to behave properly in more general cases. Thus the success (and validity) of the general formalism presented above, for applying the minimal normal form method to maps, is not fortuitous.

7 Conclusion

We have described the application of the minimal normal form method to discrete maps, studied a specific model analytically, developed a general formalism to apply the method, written a computer program to implement it, and indicated some of the outstanding unresolved problems (unproved foundations) of the method. Evidence has been presented that the symplectic condition is the key which secures those foundations, and guarantees the success of the method. Admittedly, no graphs or plots have been presented in this report to show how well the analytical results, to a given order of ϵ, actually approximate the exact solution (particle tracking output). Such work will be reported at a later date.

There are other outstanding unresolved problems, basically concerned with more practical matters: since it is of course a perturbation theoretic technique, is the method any better than others already available in the literature? We can at least say that in this method, we are able to obtain a closed form solution for the normal form variable u. Hence, to obtain a solution for x and p to high orders in ϵ, we only need to calculate the transformation function from $z = x + ip$ to u (the T_{pq} coefficients) to higher powers of ϵ. In other methods, there is no closed form solution, in general, for the diagonalized map, and so we need to calculate *both* the transformation function *and* the solution of the diagonalized map, to higher and higher orders in ϵ.

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