

SYNCHROTRON SIDEBAND AND SPIN RESONANCES IN A PROTON STORAGE RING

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August 1992

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USDOE Office of Science (SC)

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Technical Note

AGS/AD/Tech. Note No. 364

**SYNCHROTRON SIDEBAND AND SPIN RESONANCES
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August 6, 1992

Experiments at the IUCF Cooler Ring using a partial Siberian Snake and an rf solenoid have yielded graphs of deliberately induced depolarizing resonances including, in particular, higher order synchrotron sideband resonances. This is a short report to describe some theoretical formulas to fit those resonances, and to suggest tests to quantify the main predictions of the theory.

1 Introduction

This is a short note based on a very interesting recent preprint circulated by the experimental collaboration using a partial Siberian Snake at the IUCF Cooler Ring [1] which describes the use of an rf solenoid to induce depolarizing spin resonances in the circulating proton beam. In addition to the rf solenoid, a partial Siberian Snake was present in the ring, of strength s , where the spin rotation angle around the longitudinal axis is πs , so that $s = 1$ describes a full Snake. Values of s from 0 to 0.04 (a 4% Snake) were used.

Upon reading the preprint, I found that in addition to the main depolarizing resonance, where the rf solenoid frequency resonated with the spin tune, higher order resonances were also observed, viz. so-called synchrotron sideband resonances, where the rf solenoid frequency resonated with the synchrotron oscillations as well as the spin tune. Such resonances are analogous to synchrotron sidebands surrounding the betatron tune in the case of orbital motion, caused by tune modulation of the betatron oscillations by synchrotron oscillations and the chromaticity.

A fairly detailed description of the experiment and parameter values is provided in Ref. [1], but the fits to the data are restricted to the main resonances only. This is understandable, because there is no detailed theory to treat the higher order resonances. I realized, however, that formulas to describe such resonances could be adapted from the literature on electron storage rings, in particular a classic paper by Yokoya [2]. The IUCF Siberian Snake experiment affords a unique opportunity to bring these two subsets of physics together, to enhance our understanding of spin physics in storage rings, and this note is a short report in that direction.

2 Basic Formulas

The basic goal, in the theory, is to derive an expression for the \hat{n} axis for the various particles. The vector \hat{n} is the generalization of \hat{n}_0 , which is the spin closed orbit for a particle on the closed orbit. Following Yokoya [2], we can parameterize \hat{n} by writing

$$\hat{n} = \sqrt{1 - |\zeta|^2} \hat{n}_0 + \text{Re}(\zeta \vec{k}_0^*), \quad (2.1)$$

where ζ is a complex number and \vec{k}_0 is a solution of the Thomas-BMT equation on the closed orbit, and \vec{k}_0 is orthogonal to \hat{n}_0 . As is well known, \vec{k}_0 will thus precess around \hat{n}_0 at the spin tune ν_{sp} , and in particular it has the property

$$\vec{k}_0(\theta + 2\pi) = e^{i2\pi\nu_{\text{sp}}} \vec{k}_0(\theta), \quad (2.2)$$

so \vec{k}_0 is a complex vector. Of course $\hat{n}(\theta + 2\pi) = \hat{n}(\theta)$. Eq. (2.1) is just a way of representing a unit vector by a complex number ζ . The equation of motion for ζ is [2]

$$\frac{d\zeta}{d\theta} = -i\vec{\omega} \cdot \vec{k}_0 \sqrt{1 - |\zeta|^2} + i\vec{\omega} \cdot \hat{n}_0 \zeta, \quad (2.3)$$

where $\vec{\omega}$ is the spin precession vector which describes the perturbations, including the rf solenoid, and $\zeta = 0$ in the absence of perturbations ($\vec{\omega} = 0$). The above equation is solved using perturbation theory, assuming both $|\zeta|$ and $|\vec{\omega}|$ are small. To leading order, $\zeta \rightarrow \zeta_1$, where

$$\frac{d\zeta_1}{d\theta} = -i\vec{\omega} \cdot \vec{k}_0. \quad (2.4)$$

The perturbation we treat first is the rf solenoid, which oscillates at a frequency f , or a tune $\nu = f/f_c$, where f_c is the revolution frequency, and \vec{k}_0 oscillates at the spin tune ν_{sp} , so we can write

$$\frac{d\zeta_1}{d\theta} \simeq -i4\pi a \cos(\nu\theta) e^{i\nu_{\text{sp}}\theta} \delta(\theta - \theta_{\text{rf}}), \quad (2.5)$$

where a is a constant and the rf solenoid is localized at θ_{rf} . We can put $\theta_{\text{rf}} = 0$. We also do not need a detailed expression for a in terms of the accelerator lattice. Decomposing the delta function into a sum of Fourier harmonics, we find

$$\frac{d\zeta_1}{d\theta} \rightarrow -ia \sum_{p=-\infty}^{\infty} e^{i(\nu+\nu_{\text{sp}}+p)\theta} + e^{i(-\nu+\nu_{\text{sp}}+p)\theta}, \quad (2.6)$$

and a resonance will occur when one of the exponents goes to zero, i.e. the resonant frequencies are

$$f_r = f_c(p \pm \nu_{\text{sp}}), \quad (2.7)$$

in agreement with Eq. (2) of Ref. [1]. Picking out one such term, i.e. the term closest to resonance, and neglecting the rest, we obtain

$$\frac{d\zeta_1}{d\theta} \simeq -iae^{i(\nu-\nu_r)\theta}, \quad (2.8)$$

where $\nu_r = f_r/f_c$, and the solution which satisfies the appropriate boundary conditions on \hat{n} is

$$\zeta_1 = -a \frac{e^{i(\nu-\nu_r)\theta}}{\nu - \nu_r}. \quad (2.9)$$

Defining $\Delta = f - f_r$ and $A = -af_c$, we can write this as

$$\zeta_1 = \frac{A}{\Delta} e^{i(\nu-\nu_r)\theta}, \quad (2.10)$$

and so $|\zeta_1|^2 = |A|^2/\Delta^2$.

At this point we encounter a slight difficulty, which is that ζ_1 diverges at the resonance, whereas we know that \hat{n} is a unit vector and the full solution for ζ must satisfy $|\zeta| \leq 1$. We therefore adopt the following approximate strategy. For the polarization, the function of interest is the component along \hat{n}_0 , i.e. $\sqrt{1-|\zeta|^2}$, and we know that, for *small* $|\zeta_1|$,

$$1 - |\zeta_1|^2 \simeq \frac{1}{1 + |\zeta_1|^2}. \quad (2.11)$$

We use this approximation to express the polarization as

$$\frac{P}{P_0} \simeq \frac{1}{\sqrt{1 + |\zeta_1|^2}} \simeq \frac{|\Delta|}{\sqrt{\Delta^2 + |A|^2}}, \quad (2.12)$$

where P_0 is the initial polarization (a global constant). The r.h.s. will now be finite even when the rf solenoid frequency is close to the resonance, although of course the r.h.s. was derived by extrapolating an approximation. The above expression is similar to, but not exactly the same as, the form used in Eq. (5) of Ref. [1] to fit to the resonance in the polarization, which is instead

$$\frac{P}{P_0} = \frac{\Delta^2}{\Delta^2 + \Gamma^2}. \quad (2.13)$$

We see that the width Γ used in Ref. [1] corresponds to $|A|$ above.

It is not obvious which of Eqs. (2.12) or (2.13) is better to use. Let us follow Ref. [1] and use Eq. (2.13), so as to follow their work more closely. It is stated in Table 1 of Ref. [1] that $\Gamma = 0.62$ kHz for a 4% partial Snake. Using this value and $P_0 = 0.76$ (which is a value not published in Ref. [1] but was read off from their Fig. 4), yields the dashed curve shown in Fig. 1 below. The figure is a reproduction of Fig. 4 in Ref. [1]. The theory curves are calculated in this report, and are absent in the figure in Ref. [1]. We see that the resonance in the theory curve is too wide. In Eq. (2.13), Γ is the half width at half max of the resonance. However, if we interpret the value of

$\Gamma = 0.62$ kHz published in Table 1 of Ref. [1] as the full width at half max, then the equation for the polarization should read

$$\frac{P}{P_0} = \frac{\Delta^2}{\Delta^2 + \frac{1}{4}\Gamma^2}. \quad (2.14)$$

Using this equation, with $\Gamma = 0.62$ kHz, yields the solid curve shown in Fig. 1 below, which fits the resonance much better. Thus there may be some misunderstanding on my part as to the definition of Γ in Ref. [1]. Let us use Eq. (2.14) henceforth.

The next step is the really new substance of this report, viz. the fitting of the higher order synchrotron sideband resonances visible in Fig. 4 of Ref. [1], or Fig. 1 of this report. We follow Yokoya's treatment [2]. The synchrotron sidebands induce tune modulation of the spin precessions, via the term $\vec{\omega} \cdot \hat{n}_0$ in Eq. (2.3). Skipping most of the mathematics (a derivation is given in Ref. [2]), the result is that $\zeta \rightarrow \zeta_2$, where

$$\langle |\zeta_2|^2 \rangle \approx |a|^2 \sum_{m=-\infty}^{\infty} \frac{e^{-\lambda^2} I_m(\lambda^2)}{(\nu - \nu_r + m\nu_s)^2}, \quad (2.15)$$

where the angular brackets denote an average over the beam, and I_m is a modified Bessel function and the argument λ^2 is given by

$$\lambda^2 = \left(\frac{\sigma_E}{E_0} \frac{G\gamma}{\nu_s} \cos \alpha \right)^2, \quad (2.16)$$

where α will be explained below, and all the other symbols have their usual meanings, e.g. $G = (g - 2)/2 \simeq 1.792847$. The parameter λ^2 is sometimes called the "tune modulation index." The angle α is the tilt of the \hat{n}_0 axis away from the vertical, and one can show that

$$\tan \alpha = \frac{\tan(\pi s/2)}{\sin(\pi G\gamma)}, \quad (2.17)$$

where s is the partial Snake strength as explained above, so

$$\cos \alpha = \frac{\sin(\pi G\gamma)}{\sqrt{\tan^2(\pi s/2) + \sin^2(\pi G\gamma)}}. \quad (2.18)$$

The value of the polarization is plotted using the function

$$\frac{P}{P_0} = \frac{1}{1 + \langle |\zeta_2|^2 \rangle}, \quad (2.19)$$

with $\langle |\zeta_2|^2 \rangle$ given by Eq. (2.15), and recall $|a|^2 = \Gamma^2/(4f_c^2)$.

To fit the synchrotron sideband resonances, it is therefore necessary to know the value of λ^2 . Ref. [1] tells us that $G\gamma \simeq 2.0222$, using $G = 1.792847$ and a proton kinetic energy of $T = 120.02 \pm 0.03$ MeV and $\gamma = 1 + T/938.272$, and the partial Snake strength was $s = 0.04$, which tells us the value of $\cos \alpha$. The value of the synchrotron tune was [1] $f_s = 1.62 \pm 0.04$ kHz, and the revolution frequency was $f_c = 1,597,952$ Hz and $\nu_s = f_s/f_c$. However, the value of σ_E/E_0 is not given in Ref. [1]. Now

$$\frac{\sigma_E}{E_0} = \beta^2 \frac{\sigma_p}{p_0}, \quad (2.20)$$

where p is the momentum, and $\beta^2 = (v/c)^2 = 1 - \gamma^{-2}$. Further, a typical value of σ_p/p_0 is 10^{-3} , so let us *guess* that $\sigma_p/p_0 \simeq 0.001$, and σ_E/E_0 is given appropriately. All of this yields

$$\lambda^2 \simeq 0.1. \quad (2.21)$$

The fit to the data is shown in Fig. 2 below, where the data are again from Fig. 4 of Ref. [1], and the solid curve is the fit using Eq. (2.19) with $P_0 = 0.76$ and $\Gamma = 0.62$ kHz as before. In addition to the synchrotron sideband resonances at $f_r \pm f_s$, the resonance $f_r + 2f_s$ also lies in the range of the data, although it is very narrow and does not appear in the data. Perhaps a more detailed search might reveal it.

3 Discussion

The fit of the theory to the data in Fig. 2 is actually better than I had originally expected. This is naturally pleasing, but is tempered by various obvious approximations/assumptions that have clearly been made in deriving the theoretical fit. Let us now consider the implications of the above results. First, note that the experimental data show only the *transverse* polarization $\sqrt{P_r^2 + P_v^2}$, and not the full polarization, because the longitudinal component P_l at the polarimeter is unmeasurable. We adjust the global scale factor P_0 to deal with this, yielding in this case $P_0 = 0.76$. Second, Ref. [1] used a phenomenological function — basically a Lorentzian — to fit the shape of the main resonance. The same function was used in this report. A brief discussion of this topic was provided above, where a slightly different function was derived (but not used). However, the data show that the resonance is not symmetric around its center, which indicates that the theory is oversimplified in describing the resonance shape.

The values of most of the parameters have been clearly listed in Ref. [1], as well as a description of the experiment and the meaning of the graphs. Values of ancillary parameters such as the revolution frequency and the synchrotron frequency, were also provided. However, the value of an important parameter (from the point of view of this report), was not given, viz. the energy spread σ_E/E_0 . The value of this parameter had to be guessed above. This weakens the goodness of the fit.

There are several suggestions one can make for further study. It might be best to tabulate these in a list.

1. First, of course, it would be good to measure the energy spread of the beam, if possible. This would cross-check the value used above.
2. The theory claims that the widths of the synchrotron sideband resonances are all proportional to that of the main "parent" resonance. This can be checked by varying the peak value of the magnetic field integral of the rf solenoid. All the resonance widths should scale identically. Unfortunately, it appears that the field used was already at its maximum value in Ref. [1], and so one could only make the resonances narrower, not wider, which is a pity.
3. The theory also indicates that the *relative* widths of the synchrotron sideband resonances are all governed by only one parameter λ^2 , which is a combination of several quantities, viz. σ_E/E_0 and ν_s . One could test this claim, e.g. by varying the energy spread, and/or varying the synchrotron tune. One could also vary the value of $G\gamma$, but to achieve a significant change might require too large a change of beam energy, resulting in unacceptable loss of polarization at injection due to the mismatch between the injected polarization and the direction of the \hat{n}_0 axis in the ring. It is probably therefore better to vary σ_E/E_0 and ν_s . Note that changing ν_s would also change the *locations* of the resonances. Reducing ν_s (which might be easily achievable), would bring the resonances closer together, while also making them wider. Under these circumstances, it might be possible to observe the second synchrotron sidebands $f_r \pm 2f_s$.
4. The fits in this report have been restricted to only the data pertaining to the 4% partial Snake. There are graphs (Fig. 2 in Ref. [1]) which show that synchrotron sideband resonances are clearly visible at other values of the partial Snake strength, including $s = 0$ (no Snake).

However, the graphs are too small to resolve enough detail for a theoretical fit to the higher order resonances. It may be of interest to plot the data for $s = 0$, etc., on a larger scale, and to fit the higher order resonances. In particular, if we believe, as is plausible, that the energy spread and the synchrotron tune are the same in all cases, then it should be possible to fit all of the higher order resonances for all values of the partial Snake strength without introducing new parameters into the theory. This may be a test that the authors of Ref. [1] can perform without taking more data.

4 Conclusions

The experiments at IUCF with a full or partial Siberian Snake have yielded many results which have increased our understanding of polarized particle motion in storage rings, and in many cases have been ahead of theory. A fit to the data and a few theoretical predictions have been made above in one such area, viz. the behavior of the higher order synchrotron sideband resonances, and perhaps may help to set the theory on a firmer footing. In particular, it should be possible to check if the higher order synchrotron sideband resonances all depend on only one parameter, viz. the tune modulation index λ^2 .

References

- [1] V. A. Anferov et al., University of Michigan Preprint UM He 92-10 (July 1992), unpublished.
- [2] K. Yokoya, Part. Accel. **13**, 85 (1983).

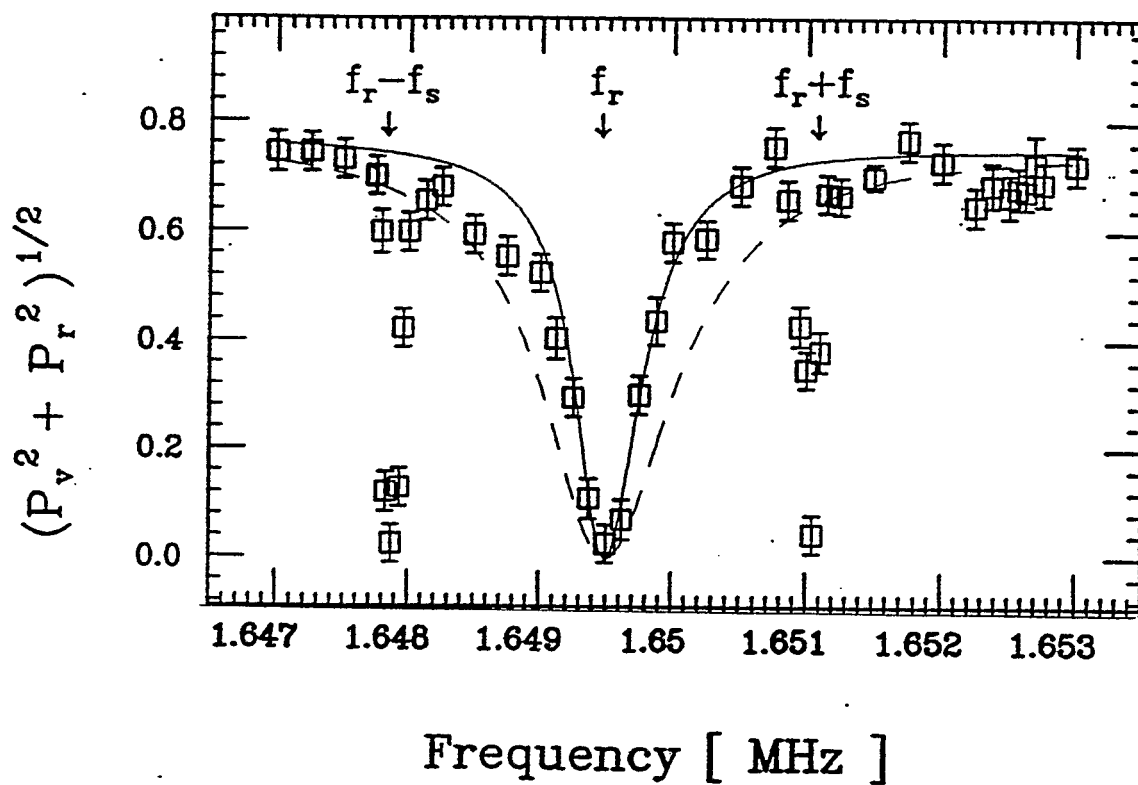


FIGURE 1

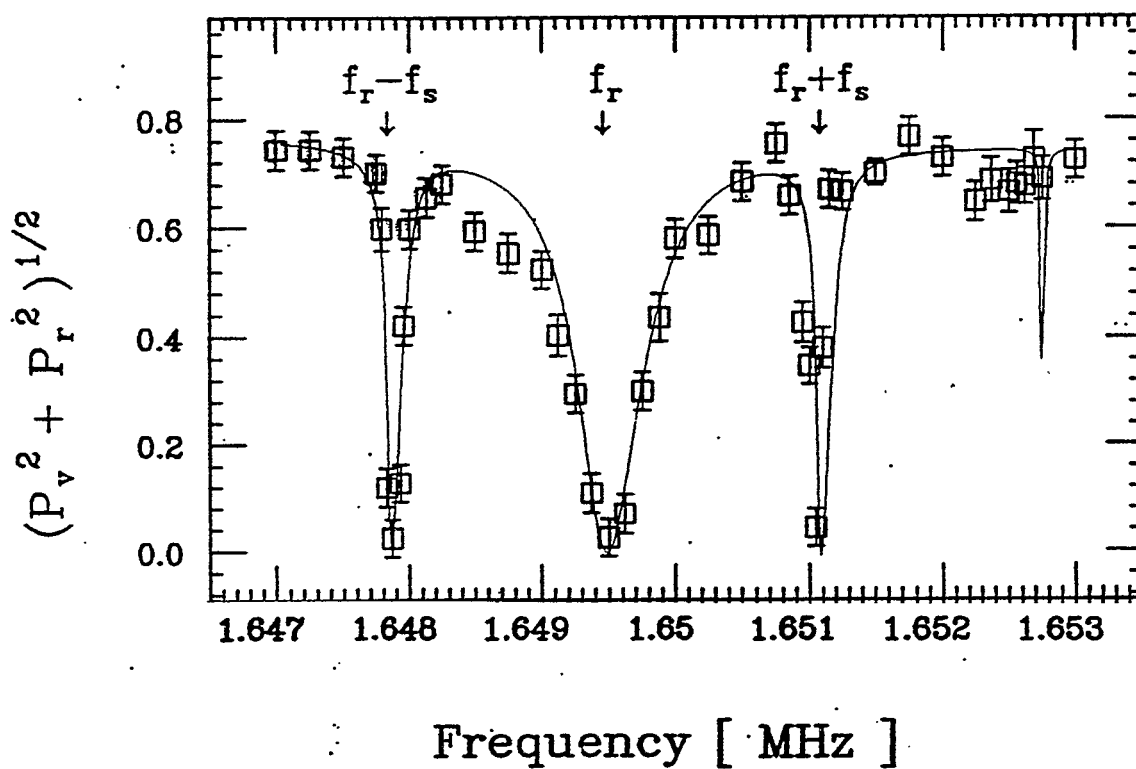


FIGURE 2