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LONGITUDINAL MOTION - DATA PRESENTATION

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1 Introduction

This note attempts to show a straightforward way to numerically calculate quantities related to buckets and bunches with simple codes, without either linearizing the equation of motion or looking up solutions in tables. The bucket and bunch are characterized by ϕ_s, the synchronous phase, and ϕ_2, one of the extreme phases of the bunch ($\phi_2 > \phi_s$.). Functions $\alpha_b(\phi_s, \phi_2)$, $Y_b(\phi_s, \phi_2)$, $\Gamma_b(\phi_s, \phi_2)$ are introduced. Numerical algorithms used in this note are Newton's method, Gauss quadrature and Chebyshev quadrature. Newton's method is generally utilized to evaluate their inverse functions. Gauss quadrature is used to calculate well behaved integrands, and Chebyshev quadrature is used to evaluate integrands with two integrable singularities.

Bunch matching is also discussed, some conditions are given.

2 Synchrotron equations and Hamiltonian

Let a synchronous particle with charge e have synchronous energy E_s (β_s and γ_s), synchronous phase ϕ_s, angular revolution frequency ω_0 and an energy gain eV_0 per turn. While eV_0 is greater than zero, acceleration is taking place; deceleration otherwise. Consider an arbitrary RF accelerating voltage $V(\phi)$ (see Figure 1) with angular frequency ω_{rf} in a synchrotron with transition at γ_{tr}, where $\phi = h\theta \pmod{2\pi}$, θ is the azimuthal coordinate of a particle around the ring and $h = \dfrac{\omega_{rf}}{\omega_0}$ is the harmonic number. A non-synchronous particle is in general denoted as having energy E and phase ϕ whose conjugate variable is $W = \dfrac{E - E_s}{\omega_{rf}} = \dfrac{\Delta E}{\omega_{rf}}$. The equation of motion [1] of a non-synchronous particle is

$$\frac{dW}{dt} = \frac{e}{2\pi h}[V(\phi) - V_0], \tag{1}$$

$$\frac{d\phi}{dt} = \frac{\eta \omega_{rf}^2}{E_s \beta_s^2} W, \tag{2}$$

where $\eta = \dfrac{1}{\gamma_{tr}^2} - \dfrac{1}{\gamma_s^2}$ is the freqency slip factor.

The Hamiltonian [1] describing the motion of a non-synchronous particle is

$$\mathcal{H}(\phi, W) = \frac{\eta \omega_{rf}^2}{2E_s \beta_s^2} W^2 + \frac{eV_{rf}}{2\pi h} U(\phi, \phi_s), \tag{3}$$

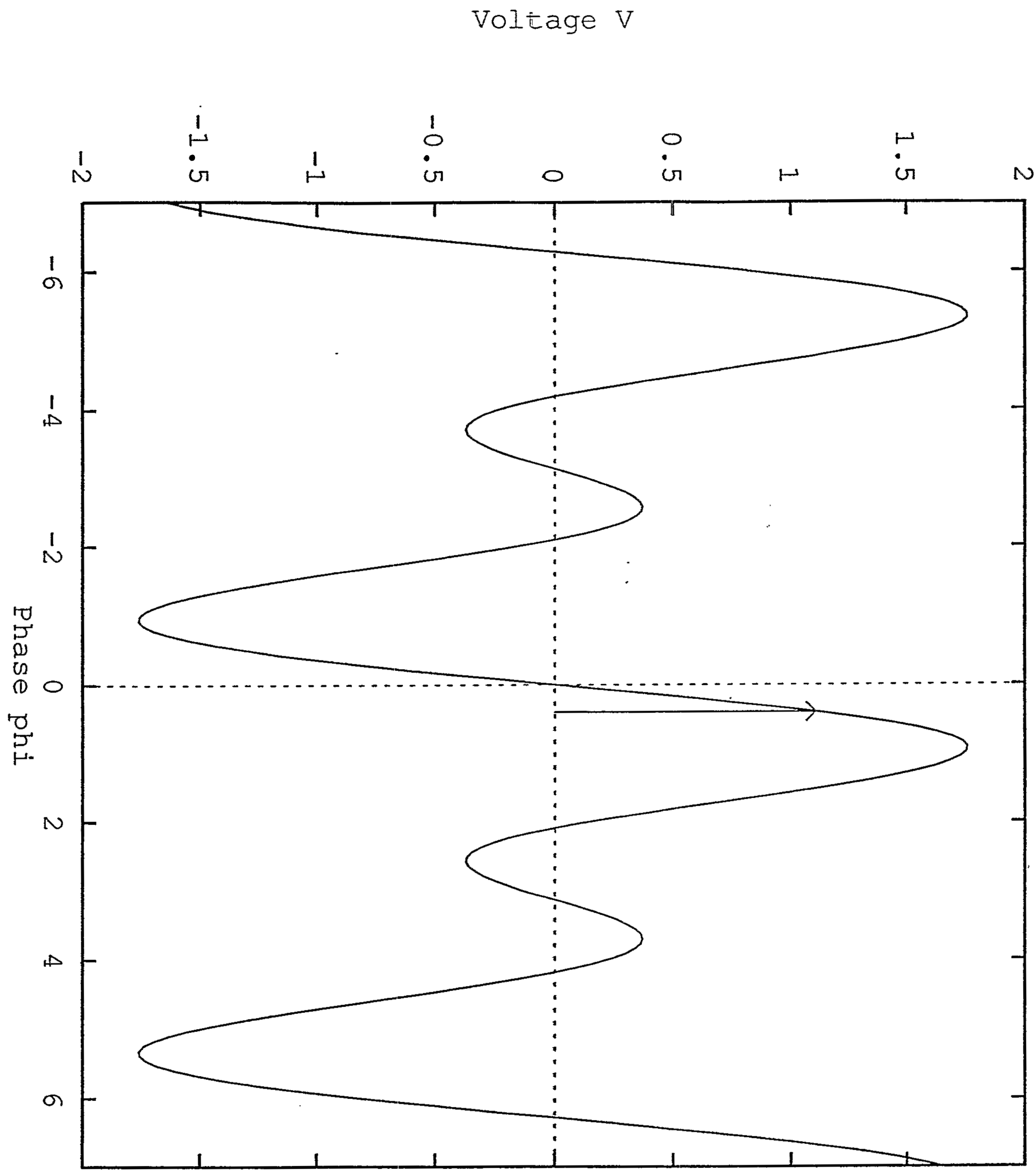

Figure 1: Arbitrary voltage and synchronous phase.

where V_{rf} is the peak voltage in one turn and $U(\phi, \phi_s)$ is the RF potential, which is expressed as

$$U(\phi, \phi_s) = -\frac{1}{V_{rf}} \int_{\phi_s}^{\phi} [V(\phi') - V_0] d\phi'. \tag{4}$$

The equation of motion and the Hamiltonian are inadequate when η approaches zero. Other treatments are needed in that case, which are not considered in this note.

If the Hamiltonian does not explicitly depends on time, it is a conserved system. The trajectory of a non-synchronous particle lies on a constant $\mathcal{H}$ curve.

3 Phase stability

The stable fixed points of the Hamiltonian, in phase space spanned by (ϕ, W), underlies the principle of phase stability, which maintains that a non-synchronous particle can make small oscillations around a stable phase. The fixed points of the Hamiltonian are found by solving the following equations

$$\frac{\partial \mathcal{H}}{\partial W} = 0, \frac{\partial \mathcal{H}}{\partial \phi} = 0. \tag{5}$$

It is apparent that the fixed points have the form $(\phi, W) = (\phi_f, 0)$. The fixed points lie on the ϕ axis, it is sufficient to denote a fixed point by ϕ_f. The motion in the neighborhood of the fixed points dictates the nature of a fixed point, specifically the eigenvalues of the following characteristic matrix evaluated at the fixed points determine the nature of the fixed points [2]

$$\begin{pmatrix} \frac{\partial^2 \mathcal{H}}{\partial W \partial \phi} & \frac{\partial^2 \mathcal{H}}{\partial W^2} \\ -\frac{\partial^2 \mathcal{H}}{\partial \phi^2} & -\frac{\partial^2 \mathcal{H}}{\partial \phi \partial W} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\eta \omega_{rf}^2}{E_s \beta_s^2} \\ \frac{e}{2\pi h} V'(\phi_f) & 0 \end{pmatrix}, \tag{6}$$

where $V'(\phi_f)$ stands for the derivative of V with respect to ϕ evaluated at the fixed point ϕ_f, it follows that the two eigenvalues $\lambda_{1,2}$ can be written formally as

$$\lambda_{1,2} = \pm \sqrt{\frac{e \eta \omega_{rf}^2 V'(\phi_f)}{2\pi h E_s \beta_s^2}}. \tag{7}$$

The eigenvalues are either pure imaginary (complex conjugate to each other) or real, corresponding to elliptic fixed point or hyperbolic fixed point respectively. Elliptic fixed points are stable fixed points, close to which a non-synchronous particle would make small oscillations; and hyperbolic fixed points are unstable, from which a non-synchronous particle would move away.

A stable region and a unstable region is divided by a separatrix, which is determined by the minimum Hamiltonian evaluated at two neighboring unstable fixed points ϕ_l and ϕ_r $(\phi_l < \phi_f < \phi_r)$

$$\mathcal{H}(\phi, W) = \min(\mathcal{H}(\phi_l, 0), \mathcal{H}(\phi_r, 0)). \tag{8}$$

Particles residing in stable regions around the stable fixed points can be accelerated to raise their energies, which is the principle of phase stability. However, the hyperbolic fixed points have been employed for beam manipulation purposes. If $V'(\phi_f)$ is zero, the phase stability is lost. Also notice that as η changes sign (transition crossing), the nature of a fixed point changes accordingly to equation (7), a fixed point ϕ_f changes from a stable fixed point to an unstable fixed point, and vice versa. An accelerating voltage with only stable fixed points is not suitable to accelerate particles through transition crossing.

In the next section, a sinusoidal RF accelerating voltage is closely examined. The formalism developed there applies to a general accelerating voltage $V(\phi)$ as well.

4 Sinusoidal RF voltage

In practice, the accelerating voltage is generally sinusoidal volt in synchrotrons

$$\begin{aligned} V(\phi) &= V_{rf}\sin\phi, \\ V_0 &= V_{rf}\sin\phi_s, \end{aligned} \tag{9}$$

where ϕ_s is the synchronous phase and is contained in a domain $[0,\pi]$.In the rest of this note, sinusoidal RF voltage is assumed.

Let's briefly mention how to find the synchronous phase ϕ_s. ϕ_s is determined by the energy gain per turn $eV_{rf}\sin\phi_s$ and the rate of change of magnetic dipole field $\dot{B}$

$$V_{rf}\sin\phi_s = C\rho\dot{B} \tag{10}$$

where C is the circumference of the synchrotron, ρ is the bending radius of the magnetic dipole field.

The Hamiltonian for a sinusoidal RF voltage can be written as

$$\mathcal{H}(\phi, W) = \frac{1}{2}AW^2 + B[\cos\phi - \cos\phi_s + (\phi-\phi_s)\sin\phi_s], \tag{11}$$

where $A = \frac{\eta\omega_{rf}^2}{E_s\beta_s^2}$ and $B = \frac{eV_{rf}}{2\pi h}$. The fixed points are located at ϕ_s and $(\pi-\phi_s)$. The characteristic matrix evaluated at a fixed point ϕ_f is

$$\begin{pmatrix} 0 & A \\ B\cos\phi_f & 0 \end{pmatrix}. \tag{12}$$

The eigenvalues can be formally written as $\lambda_{1,2} = \pm\sqrt{AB\cos\phi_f}$. Without losing generality, assume $eV_{rf} > 0$. To find the stable fixed point at which the eigenvalues are imaginary, it requires $AB\cos\phi_s < 0$, or $\eta\cos\phi_s < 0$. It is immediately seen that while ϕ_s is a stable fixed point, $(\pi-\phi_s)$ is an unstable fixed point, and vice versa. The stable fixed phase points are $0 \le \phi_s < \frac{\pi}{2}$ for $\eta < 0$, and $\frac{\pi}{2} < \phi_s \le \pi$ for $\eta > 0$.

The Hamiltonian is invariant under the transformation

$$\begin{aligned} \phi &\rightarrow \pi - \phi, \\ \phi_s &\rightarrow \pi - \phi_s, \\ \eta &\rightarrow -\eta, \end{aligned} \tag{13}$$

which establishes that the motion around the stable fixed phase point $\frac{\pi}{2} < \phi_s \le \pi$ and $\eta > 0$ can be found from the motion around the the stable fixed point $(\pi - \phi_s)$ and $\eta < 0$ by making such a transformation. As a result, we will concentrate on the motion around fixed phase point $0 \le \phi_s < \frac{\pi}{2}$ and $\eta < 0$,or $A < 0$. The unstable fixed phase point is thus $(\pi-\phi_s)$, and the equation of the separatrix is

$$\begin{aligned} \mathcal{H}(\phi, W) &= \mathcal{H}(\pi-\phi_s, 0) \\ &= B[-2\cos\phi_s + (\pi - 2\phi_s)\sin\phi_s]. \end{aligned} \tag{14}$$

Note the categorization of fixed points excludes the case when $\eta = 0$ or $\phi_s = \frac{\pi}{2}$. When $\phi_s = \frac{\pi}{2}$, it is rather unusual. It is a situation when a stable fixed point collapses with an unstable fixed point, the stable region possesses a zero area which has no practical applications.

Suppose the Hamiltonian does not explicitly depends on time, so $\mathcal{H}$ is conserved. The trajectory of a non-synchronous particle with initial condition $(\phi_2, 0)$ $(\phi_s < \phi_2 \leq \pi - \phi_s)$ lies on a constant $\mathcal{H}(\phi_2, 0)$

$$\mathcal{H}(\phi, W) = \mathcal{H}(\phi_2, 0), \tag{15}$$

or

$$\frac{1}{2}AW^2 + Bf(\phi, \phi_s, \phi_2) = 0, \tag{16}$$

where the function $f(\phi, \phi_s, \phi_2)$ is

$$f(\phi, \phi_s, \phi_2) = \cos\phi - \cos\phi_2 + (\phi - \phi_2)\sin\phi_s. \tag{17}$$

This is simply a function of ϕ with two parameters ϕ_s and ϕ_2, nevertheless it plays an important role in the analysis.

One notices that for given values of $\sin\phi_s$ and ϕ_2 (or a constant bunch length), the trajectory depends only on the ratio of $\frac{B}{|A|}$. This is to be consider in Section 7 concerning with matching.

The other extreme phase point $(\phi_1, 0)$ $(-\pi \leq \phi_1 < \phi_s)$ can be found from

$$f(\phi_1, \phi_s, \phi_2) = f(\phi_2, \phi_s, \phi_2) = 0, \tag{18}$$

which leads to the equation to find ϕ_1

$$\cos(\phi_1) + \phi_1 \sin\phi_s = \cos\phi_2 + \phi_2 \sin\phi_s. \tag{19}$$

There are no analytic solutions for $\phi_1(\phi_s, \phi_2)$, which has a range of $[-\pi, \frac{\pi}{2})$. For any given values of ϕ_s and ϕ_2, ϕ_1 can be easily calculated by Newton's method.

Some other useful quantities associated with the trajectory are the area enclosed $A_b(\phi_s, \phi_2)$, the half height $H_b(\phi_s, \phi_2)$, the width $\Delta_b(\phi_s, \phi_2)$ and the period of the trajectory $T_b(\phi_s, \phi_2)$. Before we calculate these quantities, two auxiliary functions $\alpha_b(\phi_s, \phi_2)$ and $Y_b(\phi_s, \phi_2)$ are needed.

The following integral properly defines $\alpha_b(\phi_s, \phi_2)$

$$\begin{aligned} \alpha_b(\phi_s, \phi_2) &= \frac{\sqrt{2}}{8}\int_{\phi_1}^{\phi_2} \sqrt{f(\phi, \phi_s, \phi_2)}\, d\phi \\ &= \frac{\sqrt{2}}{8}\int_{\phi_1}^{\phi_2} \sqrt{\cos\phi - \cos\phi_2 + (\phi - \phi_2)\sin\phi_s\, d\phi}, \end{aligned} \tag{20}$$

$\alpha_b(\phi_s, \phi_2)$ spans a possible domain $[1, 0)$, the integration is easily done with Gauss quadrature.

The function $Y_b(\phi_s, \phi_2)$ is rather simple

$$\begin{aligned} Y_b(\phi_s, \phi_2) &= \frac{1}{\sqrt{2}}\sqrt{f(\phi_s, \phi_s, \phi_2)} \\ &= \frac{1}{\sqrt{2}}\sqrt{\cos\phi_s - \cos\phi_2 + (\phi_s - \phi_2)\sin\phi_s} \end{aligned} \tag{21}$$

$Y_b(\phi_s, \phi_2)$ ranges in a possible domain $[1, 0)$.

The area $A_b(\phi_s, \phi_2)$, half height $H_b(\phi_s, \phi_2)$ and width $\Delta_b(\phi_s, \phi_2)$ then can be written as

$$\begin{aligned} A_b(\phi_s, \phi_2) &= A_0 \alpha_b(\phi_s, \phi_2), \\ H_b(\phi_s, \phi_2) &= Y_0 Y_b(\phi_s, \phi_2), \\ \Delta_b(\phi_s, \phi_2) &= \phi_2 - \phi_1(\phi_s, \phi_2), \end{aligned} \tag{22}$$

where $A_0 = 16\sqrt{\frac{B}{|A|}}$ and $Y_0 = 2\sqrt{\frac{B}{|A|}}$, both are in unit eVs.

The function $\Gamma_b(\phi_s, \phi_2)$ is defined by

$$\Gamma_b(\phi_s, \phi_2) = \int_{\phi_1}^{\phi_2} \frac{1}{\sqrt{f(\phi, \phi_s, \phi_2)}} d\phi. \tag{23}$$

Note the integrand in the definition of $\Gamma_b(\phi_s, \phi_2)$ has integrable singularities at ϕ_1 and ϕ_2, Chebyshev quadrature is suitable in this case to carry out the numerical integration. Γ_b tends to infinity on the separatrix.

The period $T_b(\phi_s, \phi_2)$ is

$$T_b(\phi_s, \phi_2) = \sqrt{\frac{2}{|A|B}} \Gamma_b(\phi_s, \phi_2) \tag{24}$$

The motion is extremely slow as ϕ_2 approaches to the unstable fixed phase $(\pi - \phi_s)$.

For an arbitrary voltage $V(\phi)$, the function $f(\phi, \phi_s, \phi_2)$ would be

$$f(\phi, \phi_s, \phi_2) = U(\phi, \phi_s) - U(\phi_2, \phi_s), \tag{25}$$

Similar functions for $\alpha_b(\phi_s, \phi_2)$ and $\Gamma_b(\phi_s, \phi_2)$ can be defined. But $Y_b(\phi_s, \phi_2)$ may lose its applicability.

5 Buckets

The region enclosed by a separatrix around a stable fixed point is called a bucket in the jargon of accelerator physics. For $\phi_s = 0$ or π, such a bucket is termed as a stationary bucket, in which a synchronous particle is not accelerated. For other values of ϕ_s, the bucket is called a moving bucket. The state of the accelerator determines the A, B and ϕ_s which fully describe a bucket, we can calculate the bucket area, half height and width.

It is known immediately that for the bucket $\phi_2 = \pi - \phi_s$. The auxiliary functions defined in the above sections are generally denoted by $\alpha(\phi_s)$ and $Y(\phi_s)$ [3]. The bucket area $A(\phi_s)$, half height $H(\phi_s)$ and width $\Delta(\phi_s)$ are

$$\begin{aligned} A(\phi_s) &= A_0 \alpha(\phi_s), \\ H(\phi_s) &= Y_0 Y(\phi_s), \\ \Delta(\phi_s) &= \phi_2 - \phi_1, \\ &= \pi - \phi_s - \phi_1(\phi_s). \end{aligned} \tag{26}$$

For a stationary bucket, $\alpha(0) = Y(0) = 1$. A_0 and Y_0 are the area and half height of a stationary bucket. The bucket width is 2π.

The inverse functions $\alpha^{-1}(\phi_s)$ and $Y^{-1}(\phi_s)$ can also be utilized to find the synchronous phase ϕ_s. $\alpha^{-1}(\phi_s)$ and $Y^{-1}(\phi_s)$ are calculated by Newton's method.

5.1 Known bucket area $A(\phi_s)$

Suppose it is desirable to have the bucket area $A(\phi_s)$ known during an acceleration cycle, for instance a constant bucket area. Then, the question is: what are the programs for $V_r f$ and magnetic dipole field ? Before we answer this question, let's take a look at function $\alpha_{sin}(\phi_s)$ defined by

$$\alpha_{sin}(\phi_s) = \frac{\alpha(\phi_s)}{\sqrt{\sin \phi_s}}. \tag{27}$$

There is no general analytic solution for the inverse function $\alpha_{sin}^{-1}(\phi_s)$, but it can be found numerically by Newton's method.

If the magnetic field program is known, so is $|A|$. From equations (26) and (10), we find

$$\alpha_{sin}(\phi_s) = \frac{A(\phi_s)}{16}\sqrt{\frac{|A|}{eC\rho\dot{B}}} \tag{28}$$

The quantities on the left hand side are known, so is $\alpha_{sin}^{-1}(\phi_s)$. Then from equation (10), V_{rf} can be determined.

On the other hand, if the voltage $V_r f$ (or $\mathcal{B}$) program is known, and $\mathcal{B}$ (or $|A|$) is known at a particular moment t, what is $\dot{\mathcal{B}}$ then? From equation (26)

$$\alpha(\phi_s) = \frac{A(\phi_s)}{16}\sqrt{\frac{|A|}{\mathcal{B}}} \tag{29}$$

So $\alpha^{-1}(\phi_s)$ is known. Then from equation (10), $\dot{\mathcal{B}}$ is known. Hence, at the next moment $t + dt$ $\mathcal{B}$ is known, repeat the above process the program of magnetic dipole field program can be determined.

6 Bunches

In a synchrotron, particles are accelerated in bunches. To characterize a bunch in the phase space spanned by (ϕ, W), we choose a particle on the perimeter of a bunch in the phase space, whose trajectory will encompass all other particles' trajectories in the phase space. The formalism developed in Section 4 is used to calculate the bunch area $A_b(\phi_s, \phi_2)$, half height $H_b(\phi_s, \phi_2)$ and width $\Delta_b(\phi_s, \phi_2)$. The phase space area occupied by a bunch is also termed as longitudinal emittance, it is an adiabatic invariant quantity.

From equation (22), it is clear that if programs V_{rf} and $\mathcal{B}$ are predetermined, then, ϕ_2 is needed to calculate the longitudinal emittance. If somehow the bunch length $\Delta_b(\phi_s, \phi_2)$ can be measured, then, combining it with equation (19), ϕ_2 can be calculated. In other words, the inverse function $\Delta_b^{-1}(\phi_2)$ is numerically calculatable for given values of Δ_b and ϕ_s.

Suppose $A_b(\phi_s, \phi_2)$, an adiabatic invariant, is known along with ϕ_s, the inverse function $A_b^{-1}(\phi_2)$, *i. e.* , ϕ_2 can be calculated by Newton's method. So is the bunch length Δ_b.

Let's point out some interesting results for a special case when $\dot{\mathcal{B}}$ is held at a constant. The value of A varies as a function of γ_s as

$$A \propto \frac{1}{\gamma_{tr}^2 \gamma_s} - \frac{1}{\gamma_s^3}. \tag{30}$$

It possesses an extreme point at $\gamma_s = \sqrt{3}\gamma_{tr}$, at which after transition the bucket has a minimum area, the bunch has a maximum bunch length.

7 Matching

Since the bucket is the stable region, it has to be large enough to contain a bunch. Suppose a bunch having half height H_b and length Δ_b matches in a bucket, the matching voltage V_{rf} then satisfies

$$\frac{\mathcal{B}}{|A|} = \frac{eV_{rf}}{2\pi h|A|} = \left(\frac{H_b}{2Y_b(\phi_s, \phi_2)}\right)^2, \tag{31}$$

Other values of V_{rf} will distort the trajectory.

7.1 $\omega_{\text{rf1}} = \omega_{\text{rf2}}$

As mentioned in Section 4, the bunch shape trajectory for fixed values of $\sin\phi_s$ and ϕ_2 (or constant bunch length) depends only on the ratio of $\frac{B}{|A|}$. One implication of this result is to transfer bunches between two synchrotron. If the two synchrotrons operate at the same ϕ_s and RF frequency $\omega_{rf1} = \omega_{rf2}$, then, to maintain an unperturbed bunch would require to keep $\frac{B}{|A|}$ the same in two synchrotrons. Translate it explicitly for the voltages, it is

$$\left(\frac{eV_{rf}}{h|\eta|}\right)_1 = \left(\frac{eV_{rf}}{h|\eta|}\right)_2, \tag{32}$$

where the subscripts 1 and 2 denote the sending and receiving synchrotrons respectively. The condition for having the same RF frequency can be expressed in terms of the circumferences C or radii R and the harmonic numbers in the sending and receiving synchrotrons as

$$\frac{C_1}{C_2} = \frac{R_1}{R_2} = \frac{h_1}{h_2} \tag{33}$$

For example, it is possible to transfer unperturbed bunches between the *Booster* operating at $h = 3$ and *AGS* operating at $h = 12$, since the circumference of the *Booster* is a quarter that of *AGS*.

7.2 $\omega_{\text{rf1}} \neq \omega_{\text{rf2}}$

Consider a bunch and two buckets, the RF frequencies of these two buckets are ω_{rf1} to ω_{rf2}. Is it possible to transfer the bunch between these two buckets without disturbing the trajectory of the bunch? This is a situation that can happen between two synchrotrons, for instance, transferring bunches between the *AGS* and the *RHIC*; or in the same synchrotron, for instance, the voltage is operated alternatively at ω_{rf2} and ω_{rf1} . Put the question in other words, is it possible to conserve the equation of motion of the bunch ?

Let's consider transferring bunches in stationary buckets [4]. Put $\phi_s = 0$, and rewrite equation (3) ,

$$\mathcal{H}(\phi, W) = \frac{1}{2}AW^2 + BU(\phi, 0), \tag{34}$$

then, for a bunch with a phase point $(\phi_2, 0)$, the equation of motion follows

$$\frac{AW^2}{2BU(\phi_2,0)} + \frac{U(\phi,0)}{U(\phi_2,0)} = 1 \tag{35}$$

To switch the RF frequency from ω_{rf1} to ω_{rf2} is to make a transformation

$$\begin{aligned} \phi &\rightarrow a\phi, \\ W &\rightarrow \frac{W}{a}, \end{aligned} \tag{36}$$

where $a = \dfrac{\omega_{rf2}}{\omega_{rf1}}$. To conserve the equation of motion under the transformation would demand the equation (35)

$$\frac{AW^2}{2a^2bBU(\phi_2,0)} + \frac{U(\phi,0)}{U(\phi_2,0)} = 1, \tag{37}$$

where b is a constant. A necessary condition would require that the potential $U(\phi, 0)$ be a homogeneous function of degree r

$$U(a\phi, 0) = a^r U(\phi, 0), \tag{38}$$

so the constant b is simply a^r. Notice the first term in the Hamiltonian is a homogeneous function of degree 2 in W. The matching condition can be written down

$$\left(\frac{B\omega_{rf}^{2+r}}{A}\right)_1 = \left(\frac{B\omega_{rf}^{2+r}}{A}\right)_2. \tag{39}$$

By examining the equation (11), it is readily concluded that the potential term for a sinusoidal accelerating voltage is not a homogeneous function of any sort. In general, it is impossible to transfer bunches between two synchrotrons operating at different RF frequencies without disturbing the trajectory of bunches. It is inevitable that the process itself will blow up the longitudinal emittance. The best strategy rests on finding a voltage to minimize the longitudinal emittance blow-up [5], which is consider later in this section.

Let's consider the same situation in a slightly different context, in which the bunch length is small in both buckets. In this case, the Hamiltonian, namely the equation (11) can be linearized around ϕ_s

$$\mathcal{H}(\phi, W) = \frac{1}{2}AW^2 + \frac{1}{2}B\phi^2. \tag{40}$$

The potential term is clearly a homogeneous function of degree 2 ($r = 2$). A perfect matching can indeed succeed.

The condition for a perfect matching of a short bunch comes from equation (39) expressed in a form with explicit dependency on V_{rf} is

$$\left(\frac{eV_{rf}}{h\eta}\omega_{rf}^2\right)_1 = \left(\frac{eV_{rf}}{h\eta}\omega_{rf}^2\right)_2. \tag{41}$$

One possible solution to match a bunch perfectly in two buckets with different RF frequencies is to make the bunch short, so that it falls in the linear regions in both buckets. However, a short bunch probably may have some other unpleasant features. If these two frequencies are identical, then, the condition above reduces to equation (32), the most general case.

Although these matching conditions are derived under the assumption that $\phi_s = 0$, they are also valid for transferring a bunch from a $\phi_s = 0$ bucket to a $\phi_s = \pi$ bucket, and vice versa.

Let's consider the more general question to minimize the emittance blow-up (or filamentation). For the sake of simplicity, we consider transferring a bunch in two stationary buckets below transitions. The bunch equation in the first bucket satisfies

$$\frac{1}{2}\frac{A_1}{\omega_{rf1}}\Delta E^2 + B_1\cos(\omega_{rf1}t) = B_1\cos(\omega_{rf1}t_2), \tag{42}$$

where $\omega t_2 = \phi_2$, the right extreme phase point. An arbitrary point $(t_i, \Delta E_i)$ on the trajectory, then, satisfies the following equation

$$\frac{1}{2}\frac{A_1}{\omega_{rf1}}\Delta E_i^2 + B_1\cos(\omega_{rf1}t_i) = B_1\cos(\omega_{rf1}t_{i2}), \tag{43}$$

where $t_{i2} = t_2$. The arbitrary point $(t_i, \Delta E_i)$ also satisfy the above equation at the instant the bunch is transferred into the second bucket.

The Hamiltonian for the bunch in the second stationary bucket is

$$\mathcal{H}(a\phi, W_2) = \frac{1}{2}A_2W_2^2 + B_2(\cos(\omega_{rf2}t) - 1) \tag{44}$$

where $W_2 = \frac{\Delta E}{\omega_{rf2}}$ and $a = \frac{\omega_{rf2}}{\omega_{rf1}}$. After a new equilibrium distribution settles down, the arbitrary point $(t_i, \Delta E_i)$ will satisfy

$$\frac{1}{2}\frac{A_2}{\omega_{rf2}}\Delta E^2 + B_2\cos(\omega_{rf2}t) = \frac{A_2}{A_1}\frac{\omega_{rf1}^2}{\omega_{rf2}^2}B_1(\cos(\omega_{rf1}t_{i2}) - \cos(\omega_{rf1}t_i)) + B_2\cos(\omega_{rf2}t_i) \tag{45}$$

Note $\omega_{rf2}t_{i2}$ must be less than π for the above equations to be true.

In the first bucket, the extreme point $(t_{i2}, 0)$ on the bunch moves to the half height point $(0, \Delta E_{imax})$ in a quarter of a synchrotron period (see Figure 2). Let's find out what the voltage in the second bucket should be in order to accomplish the same action. This voltage is the lowest voltage, denoted as B_{2L}, that will make the bulk of the bunch not engage in the process of filamentation. We will call it a nominal matching voltage. The condition is readily found out as follows

$$\frac{B_2\omega_{rf2}^2}{|A_2|} = \frac{B_1\omega_{rf1}^2}{|A_1|}\frac{1-\cos(\omega_{rf1}t_{i2})}{1-\cos(\omega_{rf2}t_{i2})}. \tag{46}$$

However, any other points, say, $(t_i, \Delta E_i)$ will spread out to make the emittance larger. For short bunches (small t_{i2}), we can expand the right hand side of the above equation, it recovers to equation (41).

Let's find the point $(\phi_i, \Delta E_i)$ whose final trajectory covers the largest area at a given voltage $(B_2 > B_{2L})$, the trajectory contains all the other trajectories. In other words, we need to have

$$\frac{d\Delta E}{dt} = \frac{d\Delta E_i}{dt_i}, \tag{47}$$

which states that the new trajectory in the second bucket is tangential to the trajectory in the first bucket at that point $(t_i, \Delta E_i)$, and from which we have

$$\frac{B_2\omega_{rf2}^3}{|A_2|} = \frac{B_1\omega_{rf1}^3}{|A_1|}\frac{\sin(\omega_{rf1}t_i)}{\sin(\omega_{rf2}t_i)}, \tag{48}$$

which recovers to equation (41) for short bunches. As expected that for short bunches all points on the perimeter of the bunch in the first bucket are equivalent, they all lie on the same trajectory in the second bucket. This equation can also serve to find a voltage, denoted as B_{2M}, above which the bunch length measured in time is conserved, if we choose $t_i = t_{i2}$.

To find an optimal voltage $(B_{2L} < B_2 < B_{2M})$, we first choose the point $(t_i, \Delta E_i)$ determined by equation (48). Then, we find that the emittance S will be

$$\begin{aligned} S &= 2\int_{-t_2}^{t_2} \Delta E_2\, dt \\ &= 4\int_0^{t_2} \sqrt{\frac{2B_2}{|A_2|}(\cos(\omega_{rf2}t) - \cos(\omega_{rf1}t_i)) + \frac{2B_1}{|A_1|}\frac{1}{a^2}(\cos(\omega_{rf1}t_i) - \cos(\omega_{rf1}t_{i2}))}\, dt, \end{aligned} \tag{49}$$

where $\omega_{rf2}t_2 = \phi_2$, the bunch's extreme phase point in the second bucket, note that t_{i2} is that in the first bucket. The minimum emittance, then, is determined by the extreme condition

$$\frac{dS}{dB_2} = 0. \tag{50}$$

This equation can not be written down as an analytic function. Numerical method has to be employed to find the optimal voltage.

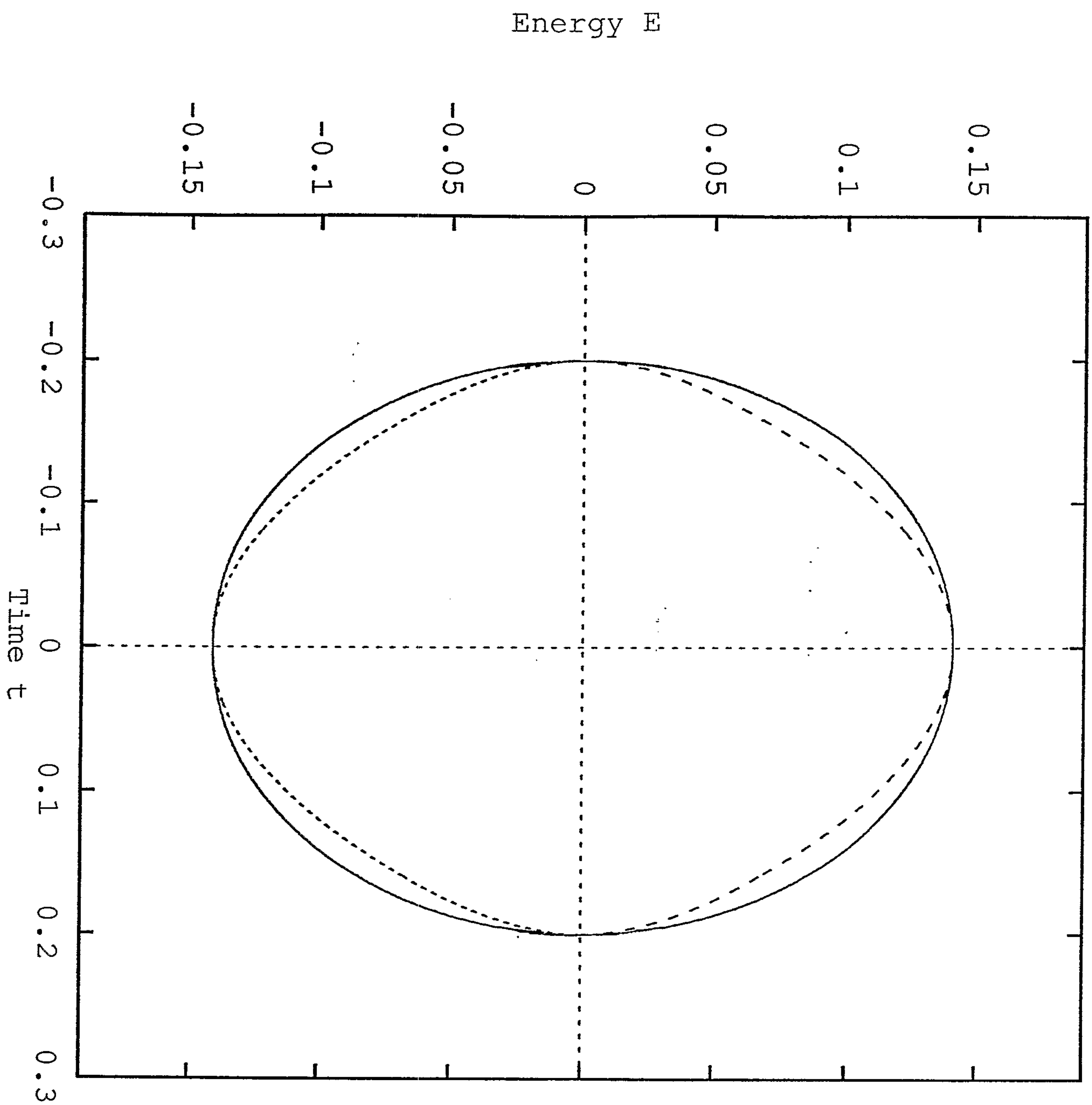

Figure 2: Bunch shape in the first synchrotron (in solid line). Phase trajectory of point $(t_{i2}, 0)$ in the second synchrotron to find the nominal matching voltage (in dash line).

8 Conclusions

The author finds it is convenient to characterize a bunch and a bucket by ϕ_s and ϕ_2 ($\phi_2 = \pi - \phi_s$ in a bucket). Then, the task of evaluating a bunch or a bucket area, half height, length and period is reduced to evaluate some well defined functions. The inverse functions are computed by Newton's method. Perfect matching is examined, and is always possible for two synchrotrons operating at the same RF frequencies. Perfect matching of short bunches can be realized for different RF freqencies, but is impossible for long bunches. Conditions for minimizing the blowup of emittance, and conserve the bunch length are given.

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[4] The discussion is also valid for transferring a bunch from ϕ_s bucket to $a\phi_s$ bucket.

[5] J. M. Brennan proposed this scheme.