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ALGORITHMS FOR RECOVERING PHASE DENSITY

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Abstract

Two algorithms are presented for recovering the longitudinal density distribution of particles in a stationary bunch from the experimentally obtained line density. The algorithms can be used as alternatives to the analytical theory.

1. INTRODUCTION

The knowledge of particles' density distribution in longitudinal phase space is important for the study of various instabilities, and for computer simulations which always start from an assumed initial distribution. This initial density profile should be as close to experiment as possible.

In longitudinal phase space, any particle is characterized by its energy, E, and phase angle, φ, while the density distribution is some function $\rho=\rho(E,\varphi)$ which can also depend on time t.

Experimentally, we cannot directly observe the phase density distribution. What we can see (Fig.1), is the so-called *line density* λ or "mountain range", which is an integral of phase density, ρ, over all particles with the given phase angle φ :

$$\lambda(\varphi) = \int_{E_{min}(\varphi)}^{E_{max}(\varphi)} \rho(E,\varphi)dE \ . \qquad (1)$$

Thus, the problem arises:

how to recover an unknown phase density distribution, $\rho(E,\varphi)$, from a given line density $\lambda(\varphi)$?

Generally speaking, this problem has no unique solution. However, there are practical cases where a unique solution, ρ, can be found.

A review of some cases along with the analytical theory is found in [1].

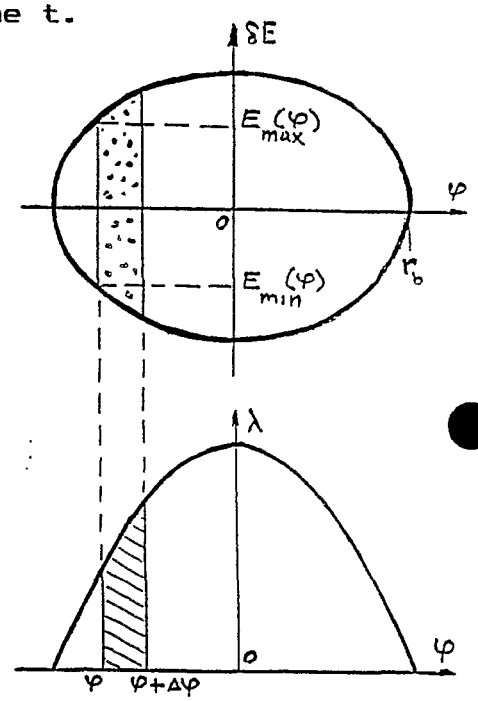

Fig.1. Line density as a projection of a phase density. When $\Delta\varphi\to 0$ *then* $\frac{1}{\Delta\varphi}\int_{\varphi}^{\varphi+\Delta\varphi}\lambda d\varphi\to\lambda(\varphi)$.

In this report we deal with a stationary bunch whose local density, ρ, can be reduced to the function of the Hamiltonian **H** (Fig.2):

$$\rho=\rho(E,\varphi)=\rho(\mathbf{H}) \qquad (2)$$

for which the line density is an even function:

$$\lambda(-\varphi)=\lambda(\varphi). \qquad (3)$$

We will present algorithms, which can recover a local density of type (2) from line density (1). The latter can be represented either by an analytic function or just by a table.

In section 2 we recapitulate the dynamic theory of the stationary bunch.
In section 3 we suggest the recovery algorithms.

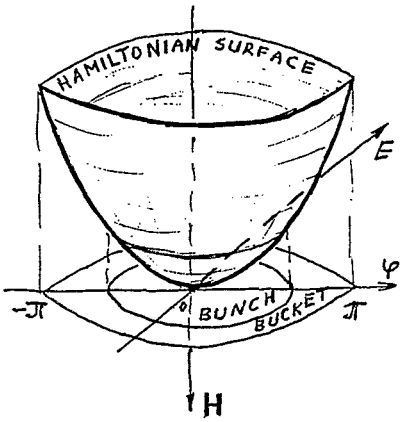

Fig.2. Stationary bucket, bunch, and Hamiltonian.

2. A STATIONARY BUNCH

In this section we review the basic equations and parameters of the stationary bunch, which we will need later to find the recovery algorithms. More detailed treatment can be found in Ref.2.

Synchrotron motion of non-interacting particles within a stationary bunch below transition energy is governed by the equations:

$$-2\frac{\partial H}{\partial \varphi} = 2\delta\dot{E} = \mathrm{Sin}\varphi\ , \qquad |\delta E| \leqslant 1, \tag{1}$$

$$\frac{\partial H}{\partial \delta E} = \dot{\varphi} = -2\delta E\ , \qquad |\varphi| \leqslant \pi, \tag{2}$$

with the corresponding Hamiltonian

$$H(\delta E,\varphi) = -\delta E^2 - \mathrm{Sin}^2\frac{\varphi}{2}\ . \tag{3}$$

The system (1)-(3) above is written in dimensionless form, where dimensionless time τ is measured in units of the synchrotron period T_o

$$\tau = \Omega_o t, \qquad T_o = 2\pi/\Omega_o, \qquad \cdot \equiv \frac{d}{d\tau} = \frac{1}{\Omega_o}\frac{d}{dt}\ , \tag{4}$$

$$\delta E = (E-E_s)/\Delta E_o\ , \qquad \Delta E_o = \sqrt{2\frac{qV}{\pi}|e_s|} = 2\frac{\Omega_o}{\omega}|e_s|\ , \tag{5}$$

$$e_s = \beta_s^2 E_s/h\eta, \quad \eta = 1/\gamma_t^2 - 1/\gamma_s^2\ , \quad \gamma_s = E_s/E_r, \quad \beta_s^2 = 1-1/\gamma_s^2\ . \tag{6}$$

Subscripts r,s,t refer to the rest, synchronous and transition energy; knot $_o$ refers to the bucket (with the exception of T_o,Ω_o), so, ΔE_o is bucket half-height; subscript $_b$ refers to the bunch; q is the particle charge; V is the (total) peak voltage of the RF system, driven with the frequency f which is synchronized with the particle's revolution frequency, ω, by

$$h\omega = 2\pi f \ , \tag{7}$$

where h is machine harmonic number (h=12 for the AGS).

The separatrix (bucket) equation $\delta E=\delta E(\varphi)$ in phase space is just

$$\delta E(\varphi) = \pm \mathrm{Cos}\frac{\varphi}{2} \ . \tag{8}$$

The bucket emittance is

$$\varepsilon_o = 4\int_o^{\pi} \mathrm{Cos}\frac{\varphi}{2} d\varphi = 8. \tag{9}$$

The bucket half-height (dimensionless) is

$$\delta E_o = \mathrm{Cos}0 = 1 \qquad (\text{or } \delta E_o = \Delta E_o / \Delta E_o = 1). \tag{10}$$

The trajectory equation of a particle with the initial conditions $\delta E_i, \varphi_i$ and with the Hamiltonian

$$\mathbf{H}_i = -\delta E_i^2 - \mathrm{Sin}^2\frac{\varphi_i}{2} = -\mathrm{Sin}^2\frac{\tau}{2} \tag{11}$$

can be written as

$$\delta E = \pm \sqrt{\mathrm{Sin}^2\frac{\tau}{2} - \mathrm{Sin}^2\frac{\varphi}{2}} \ . \tag{12}$$

If (12) is the equation of the bunch boundary, then the bunch half-height is

$$\delta E_b = \sqrt{m} = \mathrm{Sin}\frac{\tau_b}{2}, \tag{13}$$

the bunch half-length is τ_b, and the bunch emittance is

$$\varepsilon_b = \varepsilon(\tau_b) = 4\int_o^{\tau_b} \sqrt{\mathrm{Sin}^2\frac{\tau_b}{2} - \mathrm{Sin}^2\frac{\varphi}{2}}\, d\varphi = 8\left[E(m) - (1-m)K(m)\right] \ , \tag{14}$$

where $K(m)$ and $E(m)$ are complete elliptic integrals of the 1st and 2nd kind (see Appendix C in Ref. 2 for more details).

For a short bunch ($\mathrm{Sin}\frac{\tau_b}{2} \cong \frac{\tau_b}{2}$), a simplified formula is valid:

$$\varepsilon_b = \frac{\pi\tau_b^2}{2} \ . \tag{15}$$

The dimensional bunch half-height, half-length, and emittance are

$$\Delta E_b=\Delta E_o \delta E_b, \quad r_b, \quad A=\Delta E_o \varepsilon_b, \tag{16}$$

where ΔE_o is a bucket half-height according to (5).

3. THE RECOVERY ALGORITHM

The recovery algorithm is a procedure in which a number of *steps* are followed in a certain order. Each step consists of several *blocks*, and each block prescribes certain mathematical computations. Below, we will describe each individual block and then the whole algorithm. Actually we will suggest two algorithms--A and B. As A is a simple modification of B, we will first describe B and later A.

We need a few notations from set theory: $\mathcal{P}\in\mathcal{E}$ means that element $\mathcal{P}$ belongs to set $\mathcal{E}$; $\emptyset$ means an empty set, $\mathcal{E}\neq\emptyset$ means that set $\mathcal{E}$ is not empty; and $\mathcal{R}\cap\mathcal{E}\neq\emptyset$ means that the intersection of the set $\mathcal{R}$ and set $\mathcal{E}$ is not empty.

3.1 THE BLOCKS

Block 1. *The grid.*

In longitudinal phase space $(\varphi,\delta E)$, a stationary bunch, $\mathbb{B}$, can be inscribed into a rectangle, $\mathbb{R}$, of size $2r_b\times 2\delta E_b$, as shown in Fig.3. If M,N are two arbitrary integers, $L=M\cdot N$ and

$$h^r=\frac{r_b}{M}, \quad h^x=\frac{h^r}{N}, \quad h^y=\frac{\delta E_b}{2L}, \tag{1}$$

then introducing a grid $\mathbb{G}_{MN}$

$$\varphi_i=h^x\cdot i\ , \quad i=0,\pm1,\pm2,\ldots,\pm L\ , \tag{2}$$

$$E_j=h^y\cdot j\ , \quad j=0,\pm1,\pm2,\ldots,\pm 2L \tag{3}$$

we will cover the bunch, $\mathbb{B}$, by small rectangles $\mathcal{R}_{ij}$, with vertices(φ_i,E_j).

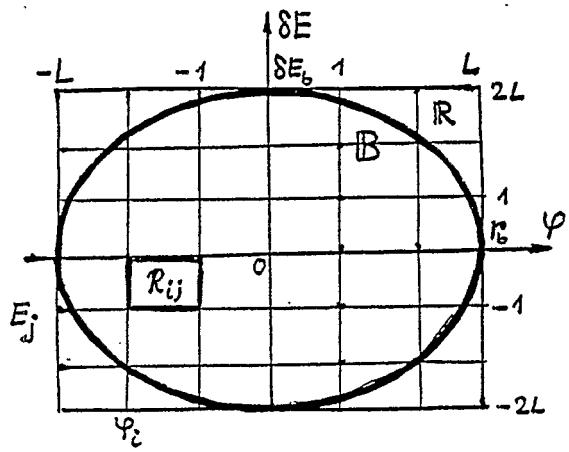

Fig.3. Grid $\mathbb{G}_{MN}$ superposed onto the bunch $\mathbb{B}$.

Block 2. *The rings.*

We now partition the bunch, $\mathbb{B}$, into M elliptic-like rings, $\mathcal{E}_k$, bounded by two closed trajectories, $\delta E_k(\varphi)$ and $\delta E_{k-1}(\varphi)$, $k=1,2,\ldots,M$.

$$\delta E_k = \pm \sqrt{\mathrm{Sin}^2\frac{\varphi_k}{2} - \mathrm{Sin}^2\frac{\varphi}{2}}\ , \qquad -\varphi_k \leqslant \varphi \leqslant \varphi_k\ . \tag{4}$$

$$\varphi_k = -r_b + h^x \cdot k\ , \qquad k=0,1,\ldots,M. \tag{5}$$

$$\mathbb{B} = \sum_{k=1}^{M} \mathcal{E}_k\ . \tag{6}$$

Block 3. *Random particle.*

Within any rectangle, $\mathcal{R}_{ij}$, we choose a random (particle) point $\mathcal{P} = \mathcal{P}(\bar{\varphi}_i, \bar{E}_j)$ with coordinates

$$\bar{\varphi}_i = \varphi_i + h^x \cdot \mathbf{RND}(i), \qquad \delta\bar{E}_j = \delta E_j + h^y \cdot \mathbf{RND}(j), \tag{7}$$

where **RND** is a generator of random numbers homogeneously distributed between 0 and 1:

$$0 \leqslant \mathbf{RND} \leqslant 1\ . \tag{8}$$

Block 4. *The ring cover.*

For any ring $\mathcal{E}_k$ we can find all rectangles, $\mathcal{R}_{ij}$, intersecting that ring. We denote those rectangles with a bar:

$$\bar{\mathcal{R}}_{ij} = \mathcal{R}_{ij} \cap \mathcal{E}_k \neq \emptyset\ . \tag{9}$$

Applying Block 3 to all such rectangles we'll find a set $\mathbb{S}_k$ of random particles $\mathcal{P} \in \mathbb{S}_k$. Some of these particles lie within the ring $\mathcal{E}_k$, and some of them are outside of the ring (Fig.4).
We check whether or not all the particles $\mathcal{P}(\bar{\varphi}_i, \bar{E}_j)$ lie within a ring $\mathcal{E}_k$.
Those which do lie in $\mathcal{E}_k$ should satisfy satisfy the inequality

$$\mathrm{Sin}^2\frac{\varphi_{k-1}}{2} \leqslant \delta\bar{E}_j^2 + \mathrm{Sin}^2\frac{\bar{\varphi}_i}{2} \leqslant \mathrm{Sin}^2\frac{\varphi_k}{2}\ , \tag{10}$$

where φ_k determined by (5).

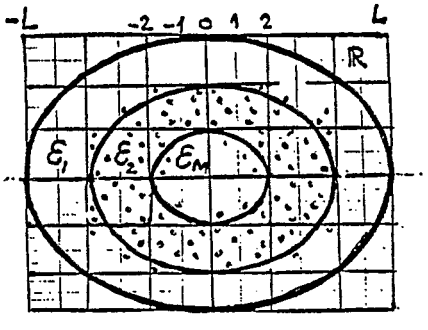

Fig.4. The rings and the ring ($\mathcal{E}_2$) cover ($M=3, N=2$).

Those particles which do not satisfy (10) should be discarded from further treatment. As a result, the ring $\mathcal{E}_k$ will be fairly uniformly filled by the set of the particles

$$\mathbb{P}_k = \left\{ \mathcal{P} \in \mathcal{E}_k \right\}, \tag{11}$$

whose density distribution is (approximately) constant.

We call this procedure *ring cover*, which leads to establishing a set $\mathbb{P}_k$ of uniformly distributed particles for the given ring $\mathcal{E}_k$.

An arbitrary ring, $\mathcal{E}_k$, is composed of small rectangles, $\mathcal{R}_{ij}$, many of which are cut by the ring boundaries. Such rectangles are *truncated*, contrary to *full* rectangles which are not truncated.

The result of the ring covering procedure is that every full rectangle contains exactly *one* particle, while some of the truncated rectangles may have *one* particle or *no* particles at all due to criteria (10).

Subject to the covering procedure we will denote a *one-fold* covered ring with the notation $\mathcal{E}_k^1$. Repeating this filling procedure to the same ring q times we will get a q-fold covered ring, again having a homogeneous particle distribution of q times higher density. The homogeneity of the distribution is approximate due to the presence of truncated rectangles lying along two borders of the ring. During the q-fold repetition each full rectangle will receive exactly q particles, while each truncated rectangle will receive an arbitrary number of particles between 0 and q, depending how big is the cut part of truncated rectangle. Thus, the smaller the size of $\mathcal{R}_{ij}$, the better the approximation. If we can neglect the influence of all truncated rectangles on the density distribution, then it is obvious that for a q-fold covered ring the local density is proportional to q :

$$\rho(\mathcal{E}_k^q) = q / A_R \ , \tag{12}$$

where the area of the full rectangle is

$$A_R = h^x \cdot h^y \tag{13}$$

and similarly for all the rings in the bunch.

It is obvious that the local densities, ρ, of two different rings satisfy the inequality

$$\rho(\mathcal{E}_k^p) \leqslant \rho(\mathcal{E}_l^q), \qquad \text{if } p \leqslant q \ . \tag{14}$$

Block 5. *The projection.*

Along with the set of rings $\mathbb{E}=\{\mathcal{E}_k\}$, determined by Block 2, we need some construction connecting those rings with the line density $\lambda(\varphi)$. The latter is usually obtained experimentally as a table. We will assume that we possess an interpolating algorithm able to interpolate a value of $\lambda(\varphi)$ for any φ within $-r_b \leqslant \varphi \leqslant r_b$, where r_b is the bunch half-length.

As we have seen, altogether we have M rings covering the bunch. Let us consider k consecutive rings ($k \leqslant M$). Some of them are covered by particles as was described in Block 4. So we have ring sequence

$$\mathcal{E}_1^{q_1},\ \mathcal{E}_2^{q_2},\ \ldots,\ \mathcal{E}_k^{q_k}, \qquad (15)$$

where $q_i \geqslant 0$. Any ring, i, has two closed boundary curves intersecting the axis φ at two pairs of symmetric points.

On the left-hand side of $\varphi=0$ consider two points with $\varphi=\varphi_{k-1}$ and $\varphi=\varphi_k$ which are both taken from (5). Drawing through those two points two vertical lines, we will intersect the bunch as well as the line density graph. Now let's find out how many particles lie between the two verticals. These particles come from all k rings (15).
Suppose the total number of these particles is N_k (Fig.5).

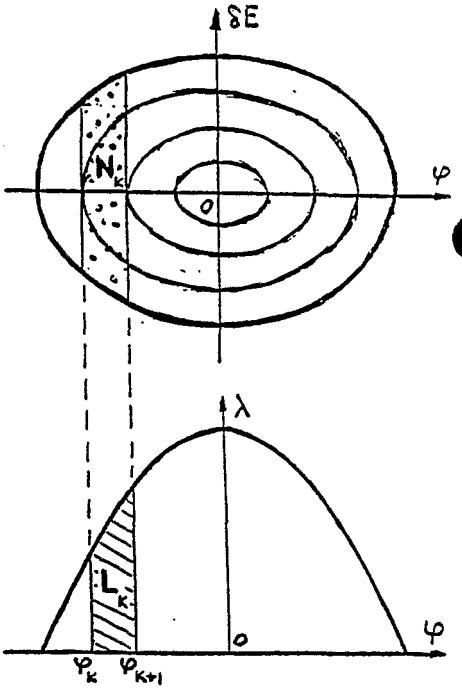

Fig.5. From phase space comes N_k, from line density comes L_k.

If the density distribution which was created within the bunch were exactly the same distribution as that from which the experimental $\lambda(\varphi)$ was taken, then we would have

$$L_k = N_k \ , \qquad (16)$$

where

$$\int_{\delta E(\varphi_{min})}^{\delta E(\varphi_{max})} \int_{\varphi_{k-1}}^{\varphi_k} \rho(\varphi,\delta E)\,d\varphi\,d\delta E = \int_{\varphi_{k-1}}^{\varphi_k} \lambda(\varphi)\,d\varphi = L_k. \qquad (17)$$

The algorithm in question is aimed at generating a particle distribution which makes the integral in (17) as close to N_k as possible:

$$L_k \cong N_k \ . \qquad (18)$$

The purpose of Block 5 is to evaluate separately L_k and N_k for subsequent comparison by another branch of the algorithm.

We will assume that along with the interpolating algorithm for λ we have an integrating algorithm for computing any L_k from the given λ.

Thus, all that is required of Block 5 is to calculate a pair L_k, N_k for any given k. We call this process *a projection.*

3.2 The algorithm B.

Step1: Choose M,N.

Step2: **Block 1.** Define rectangles $\mathcal{R}_{ij}$.

Step3: **Block 2.** Define M rings $\mathcal{E}_k$ ($\mathcal{E}_1$ is the largest, $\mathcal{E}_M$ is the smallest).

Step4: Put k=1.

Step5: **Block 4.** Cover $\mathcal{E}_k$. **Block 5.** Get L_k,N_k.

Step6: If k=1 then put k=2, go to **Step5**.
If $k \neq 1$ then go to **Step7**.

Step7: Check: $C_r \cdot \frac{L_k}{L_{k-1}} \leqslant \frac{N_k}{N_{k-1}}$, where $0.95 \leqslant C_r \leqslant 1.1$ is *a corrector*, which is supposed to partially compensate for errors due to truncated rectangles. The corrector is determined experimentally after 2-3 runs of the algorithm.

Step8: If **Step7** is false, then go to **Step5** (add more particles to k-th ring). If **Step7** is true then go to **Step9**.

Step9: If k=M then go to **Step10**, otherwise set k=k+1 and go to **Step7**.

Step10: STOP:The job is done.

Figures 6a,b,c illustrate three cases for which algorithm B was applied. The solid line on each figure represents a given line density $\lambda=\lambda(\varphi)$. After all the particles were deposited into the bunch (shown above the graph of λ) according to algorithm B, we used the newly-created bunch to reconstruct the line density $\bar{\lambda}=\bar{\lambda}(\varphi)$, shown by the dashed line. This gives us an indication as to the accuracy of the algorithm. The local density, ρ, shown by the dotted line, was computed by direct counting of the particles near the φ-axis. For stability studies, this ρ-distribution needs a smoothing treatment.

The main difference between these three cases is in the behavior of the local density ρ. It is *monotonic* increasing from the left edge of the bunch to its center in first case (a). For the second case (b), ρ is a constant. In the third case (c), ρ is increasing and then is falling. It is the latter (*non-monotonic*) case which makes use of the rings a necessity. For a pure monotonic ρ the rings are not the best tool because, the ring cover bears a significant relative error in its "homogeneous" particle distribution unless the ring section number, N, made is large. This can be done at the price of increasing (as N^2) of the total number of particles.

For the monotonic case, the full ellipse (not the ring) is a better construction for generating a cover as, we will see in the next section.

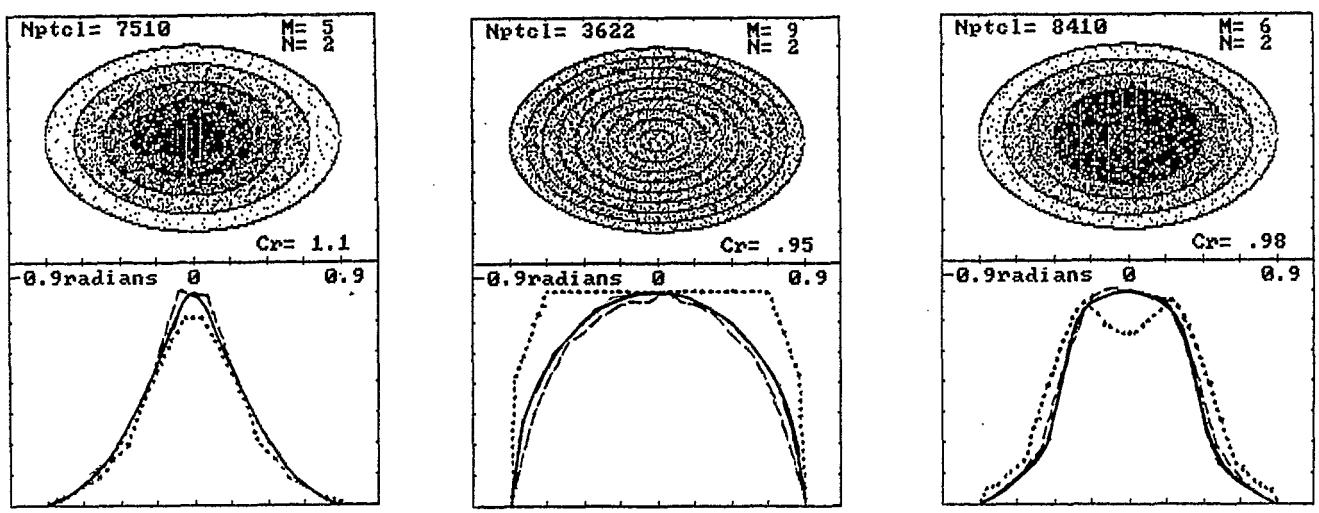

Fig.6. ρ is monotonic in case **a**, *constant in* **b**, *non-monotonic in* **c**.

3.3. The algorithm A.

When ρ is either constant or monotonically increasing, then to recover ρ from λ the algorithm A which follows is more economical and accurate than B. The latter can be easily converted to A by a simple modification of three blocks. In **Block** 1 we will put N=1 and in **Block** 4 we will use only the right side of the inequality:

$$\delta\bar{E}_j^2 + \mathrm{Sin}^2\frac{\bar{\varphi}_i}{2} \leqslant \mathrm{Sin}^2\frac{\varphi_k}{2} \ .$$

This means we have changed a definition in **Block** 2, which should be read now as follows:

Block 2. The disks.

Now we break bunch $\mathbb{B}$ into M ellipse-like disks $\mathcal{D}_k$ bounded by closed trajectories $\delta E_k(\varphi)$, $k=1,2,\ldots,M$.

$$\delta E_k = \pm\sqrt{\mathrm{Sin}^2\frac{\varphi_k}{2} - \mathrm{Sin}^2\frac{\varphi}{2}}\ , \qquad 0 \leqslant \varphi \leqslant \varphi_k \ . \tag{4A}$$

$$\varphi_k = -r_b + h^r \cdot k \ , \qquad k=0,1,\ldots,M. \tag{5A}$$

After these changes, algorithm B become a new algorithm, A, which is simple, economic and accurate when it is applied to the case of a monotonic ρ. Figures 7a,b,c repeat the examples from section 3.2. However, this time the calculations performed using algorithm A.

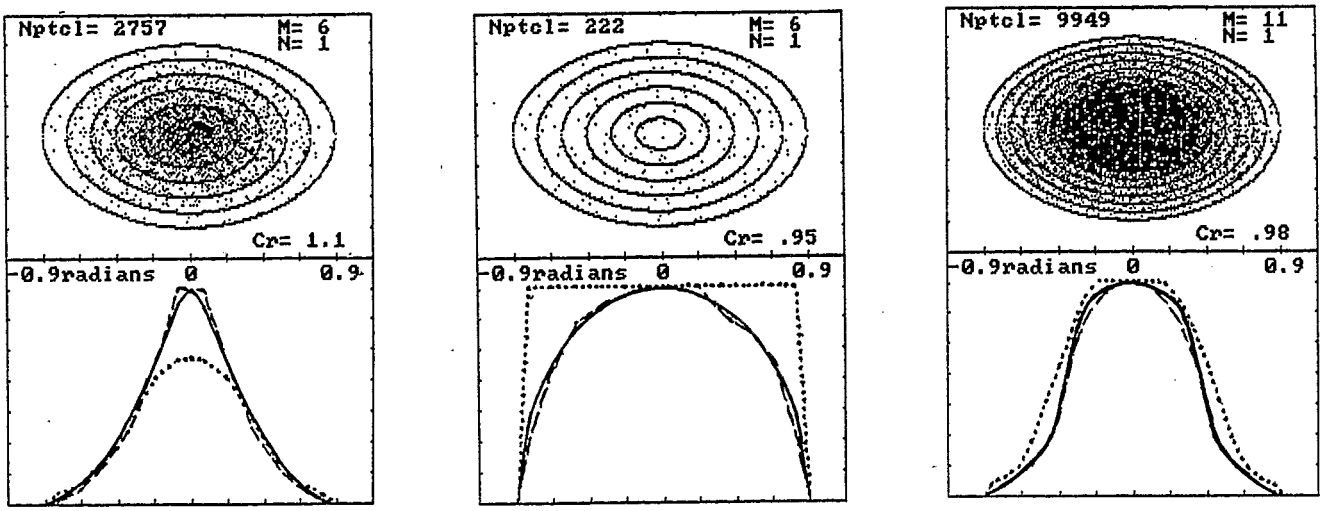

Fig.7. The same as in Fig.6, but computed by algorithm A.

By comparing the results of algorithms A and B we conclude that for a monotonic ρ algorithm A provides better accuracy with a smaller number of particles. This is a significant advantage when the recovered particle distribution will be used for further computer simulations.

However, what we have from experiment is λ , not ρ, and we usually don't know whether ρ is monotonic or not. For this reason, we need to use both algorithms. Algorithm B should give us a hint as to the behavior of ρ. If it is monotonic, we can repeat the calculation using algorithm A, which provides better accuracy while using a smaller number of particles.

ACKNOWLEDGMENTS

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