

BNL-104764-2014-TECH

AGS/AD/Tech Note No. 348;BNL-104764-2014-IR

TWO MODELS FOR A STATIONARY BUNCH IN LONGITUDINAL PHASE SPACE

J. M. Kats

April 1991

Collider Accelerator Department Brookhaven National Laboratory

U.S. Department of Energy

USDOE Office of Science (SC)

Notice: This technical note has been authored by employees of Brookhaven Science Associates, LLC under Contract No.DE-AC02-76CH00016 with the U.S. Department of Energy. The publisher by accepting the technical note for publication acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this technical note, or allow others to do so, for United States Government purposes.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or any third party's use or the results of such use of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof or its contractors or subcontractors. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Accelerator Division

Alternating Gradient Synchrotron Department BROOKHAVEN NATIONAL LABORATORY Associated Universities, Inc. Upton, New York 11973

> Accelerator Division Technical Note

AGS/AD/Tech. Note No. 348

TWO MODELS FOR A STATIONARY BUNCH IN LONGITUDINAL PHASE SPACE

Joseph M. Kats

April 10, 1991

Abstract

A stationary bunch is the simplest accelerator concept for testing mathematical models of particle dynamics in longitudinal phase space. There are two mathematical models for particle dynamics :

continuous and discrete. The continuous model is based on differential equations and Hamiltonian formalism, while the discrete model is based on difference equations and recurrence theory. Each of the models is approximate and is judged by comparison with experiment. Both models advantages come from their analytical and computational simplicity and effectiveness. It is a combination of these two models which makes modeling successful. For example, we almost always use the Hamiltonian from the continuous model to calculate the bunch shape (boundary of particles' stable region) and the bunch parameters, such as length and height; then we track particle trajectories using the discrete model.

In this report we construct the bunch shape twice, without mixing the two models; first from the continuous model, then from the discrete model. A comparison of the two bunches reveals that the second bunch is tilted with respect to the first.

The angle between axes of the two bunches depends on the ratio of the synchrotron tune to the number of cavities. The bigger the ratio, the larger the angle.

In this report we review both models and compare some of the results coming independently from each model. The main conclusion is that both results are very close. The reason for such closeness is that the ratio of synchrotron tune to the number of (equally spaced) accelerating cavities is very small for the vast majority of existing accelerators.

CONTENTS

1. Continuous Model	3
1.1.Basic Equations and Parameters	3
1.2.Hamiltonian and Bunch	4
1.3.Synchrotron Frequency and Synchrotron Tune	6
1.4.Dimensionless Equations	9
2. Discrete Model	11
2.1.Stability	11
2.2.Discrete conservation law	13
2.3.Bunch parameters and orientation	15
2.4.Discrete model with N cavities	16
3. Interpretation of Bunch Experimental Measurements	17
3.1. Interpretation based on continuous model	17
3.2. Interpretation based on discrete model	18
3.3. Comparison of the two interpretations	20
3.4. Final experiment	21
Acknowledgments	22
References	23
Appendix A. Atomic terminology for the pedestrian.	24
Appendix B. Does the particle possess a potential energy?	25
Appendix C. Calculation of bunch longitudinal emittance.	28

•

1. CONTINUOUS MODEL

1.1 Basic equations and parameters.

In this section we will review the continuous model describing longitudinal motion of charged particles. This model assumes that one accelerating cavity acts continuously along the whole circumference of the machine. Then a synchrotron motion of non-interacting particles within the **stationary** bucket is governed by the equations:

$$\Delta \vec{E} = \omega \frac{qV}{2\pi} \sin \varphi , \quad \dot{\varphi} = \frac{\omega}{\varphi} \Delta E \qquad (1)$$

with

$$\Delta E = E - E_{s}, \qquad h \omega = 2\pi f,$$

$$e_{s} = \beta_{s}^{2} E_{s} / h \eta, \qquad \eta = 1 / \gamma_{t}^{2} - 1 / \gamma_{s}^{2},$$

$$\gamma_{s} = E_{s} / E_{r}, \qquad \beta_{s}^{2} = 1 - 1 / \gamma_{s}^{2},$$

$$q = 0e, \qquad E_{r} = I_{r} \cdot A,$$
(2)

where E, φ are particle's energy and phase, subscripts r.s.t refer to the rest, synchronous and transition energy, e is proton's charge, q is charge of the ion subject to synchrotron motion, Q is the ion charge state (number of stripped electrons), A is mass number, I_r is ionic rest energy per nucleon, V is (total) peak voltage of RF system, driven with the frequency f which is synchronized with the particle's revolution frequency ω as it is shown by (2). Note that $e_{\rm g}<0$ below transition $(\gamma_{\rm g}<\gamma_{\rm t})$ and $e_{\rm g}>0$ above transition $(\gamma_{\rm g}>\gamma_{\rm t})$. Definitions and details on ion parameters are presented in Appendix A.

For the *l*-th particle, the nonlinear system (1) is solved starting from initial position $\delta E_{o}^{(i)}, \varphi_{o}^{(i)}$.

1.2 Hamiltonian and Bunch.

Equations (1) lead to or can be derived from the Hamiltonian

$$H(\varphi, \Delta E) = \omega \left(\frac{\Delta E}{2e_s}^2 + \frac{qV}{2\pi} \cos\varphi \right) + arbitrary \ constant.$$
(3)

This represents a conservation law: the sum of particle kinetic and potential energy (in appropriate canonical variables) is constant during particle motion. (See Appendix B for basic terminology.) The Hamiltonian (3) has a dimension of energy (Fig.1).

If necessary the Hamiltonian can be normalized by an appropriate choice of constant. One way to do it is to make H as a positive-definitive form: $H \ge 0$ (or negative-definitive $H \le 0$) and to make minH=0 (maxH=0).

For example, by adding to (3) the constant $q\omega V/2\pi$ we get

$$\mathbf{H}^{+}(\varphi, \Delta \mathbf{E}) = \omega \left(\frac{\Delta \mathbf{E}}{2 \theta_{s}}^{2} + \frac{q \nabla}{\pi} \cos^{2} \frac{\varphi}{2} \right) . \tag{4}$$

This form is convenient for studying a motion after transition when $e_{s}>0$, $H^{+} \ge 0$ (Fig.2).



Fig.1. Hamiltonian surface and its projection on the phase plane.

On the other hand, by subtracting $q\omega V/2\pi$ from (3) we get

$$\mathbf{H}^{-}(\varphi, \Delta \mathbf{E}) = \omega \left(\frac{\Delta \mathbf{E}^{2}}{2e_{s}} - \frac{qV}{\pi} \operatorname{Sin}^{2} \frac{\varphi}{2} \right) .$$
 (5)

This Hamiltonian is more convenient before transition, when $e_s < 0$, $H \leq 0$ (Fig.3). In both cases Hamiltonian equations

$$\frac{d\Delta E}{dt} = -\frac{\partial H}{\partial \varphi} , \qquad \frac{d\varphi}{dt} = \frac{\partial H}{\partial \Delta E}$$
 (6)

lead to (1). If the initial conditions for a particle are $\Delta E_{o}, \phi_{o}$ then the Hamiltonian for such a particle has a constant value of $H^{\pm}(\Delta E_{o}, \phi_{o})=H_{o}^{\pm}$. The trajectory for that particle can be expressed either implicitly



Fig.2. Hamiltonian surface of energy above transition.

$$\mathbf{H}^{\pm}(\boldsymbol{\varphi}, \Delta \mathbf{E}) = \mathbf{H}_{o}^{\pm}$$
(7)

- 4 -

or explicitly:

$$\Delta E^{\pm} = \Delta E^{\pm}(\varphi, \mathbf{H}_{o}^{\pm}), \qquad (8)$$

$$\Delta E^{+} = \pm \sqrt{2e_{s}} \left(H_{o}^{+} / \omega - \frac{qV}{\pi} \cos^{2} \frac{\varphi}{2} \right) , \qquad (8^{+})$$

$$\Delta E^{-} = \pm \sqrt{2e_{s}} \left(H_{o}^{-} / \omega + \frac{qV}{\pi} \sin^{2} \frac{q}{2} \right) . \qquad (8^{-})$$

Fixed points of Hamiltonian surface $H=H(\phi, \Delta E)$ are found from

$$\frac{\partial \mathbf{H}}{\partial \Delta \mathbf{E}} = 0, \quad \frac{\partial \mathbf{H}}{\partial \varphi} = 0. \tag{9}$$

These points are pairs ($\varphi=\pi n, \Delta E=0$), $n=0,\pm 1,\pm 2,\ldots$.

If a particle's trajectory is closed in the vicinity of the fixed point, then such a point is called *stable*. Otherwise, the fixed point is called *unstable*. It's easy to see from the periodicity conditions

 $H^{\pm}(\varphi, \Delta E) = H^{\pm}(\Delta E, \varphi \pm 2\pi)$.



Fig.3. Hamiltonian surface of energy below transition.

that the stable fixed points { $\varphi=(2n+1)\pi, \Delta E=0$ } for H^+ are unstable for H and the stable fixed points{ $\varphi=2n\pi, \Delta E=0$ } for H^- are unstable for H^+ . The value of the Hamiltonian calculated at unstable fixed point is a separator. It is positive when $\varepsilon_{a}>0$, negative when $\varepsilon_{a}<0$:

(10)

$$\mathbf{H}_{sep}^{\pm} = \pm \omega \frac{qV}{\pi} \tag{11}$$

It separates trajectories of all particles with different initial conditions into two classes - open and closed. According to (7) each trajectory can be uniquely characterized by the value of its Hamiltonian. Those trajectories for which

$$\left|\mathbf{H}_{o}^{\pm}\right| < \left|\mathbf{H}_{sep}^{\pm}\right| \tag{12}$$

are closed and those for which

$$\left|\mathbf{H}_{o}^{\pm}\right| > \left|\mathbf{H}_{sep}^{\pm}\right| \tag{13}$$

are open.

A trajectory with

$$H_{o}^{\pm} = H_{sep}^{\pm}$$
(14)

is called a separatrix.

It consists of two branches intersecting each other. Two branches of a separatrix between consecutive unstable fixed points compose a closed trajectory called *a bucket*. The length of the stationary bucket is 2π . There are *h* buckets in the phase space, the configuration of buckets periodically repeating itself. All trajectories within the buckets are closed, while all trajectories outside the buckets are open. At the center of each bucket there is exactly one stable fixed point.

The set of particles within a bucket is called *a bunch*. The precise definition is more flexible; it says a bunch comprises 95% or 100% or 99% (or whatever) percentage of the total bucket particles.

The actual number of buckets in an accelerator is determined by the harmonic number h. In the AGS we have h=12 buckets lying along the 807.075m circumference of machine. So, we have 807/12= 67.25m as the length of each bucket.

In this report we will deal exclusively with energies below transition (e_s <0) and with the Hamiltonian H on the interval ($-\pi,\pi$). For these reasons we will drop the superscript for the Hamiltonian and will write H instead of H and $-|e_s|$ instead of e_s .

1.3 Synchrotron Frequency and Synchrotron Tune.

In the phase space $(\varphi, \delta E)$ a particle's trajectory is a closed curve and for that reason the particle's motion is periodic i.e. it performs synchrotron oscillations. The particle's synchrotron frequency Ω , depends on the value of that particle's Hamiltonian:

$$H(\varphi, \Delta E) = -\omega \left(\frac{\Delta E^2}{2|e_s|} + \frac{qV}{\pi} \sin^2 \frac{\varphi}{2} \right) , \qquad (15)$$

which is a constant, $\mathbf{H}_{\mathbf{x}}$, determined by particle's initial position:

$$\mathbf{H}_{\mathbf{A}} = \mathbf{H}(\Delta \mathbf{E}_{\mathbf{A}}, \boldsymbol{\varphi}_{\mathbf{A}}) \tag{16}$$

The larger the absolute value of H, the smaller is the synchrotron frequency $\Omega=\Omega(H)$.

While a particle moves along a trajectory, its coordinates $\Delta E = \Delta E(t)$, $\varphi = \varphi(t)$ are changing with time according to (15). However, the Hamiltonian doesn't change; its value $H_{_{O}}$ remains the same as at initial position $\Delta E = \Delta E_{_{O}}$, $\varphi = \varphi_{_{O}}$ as well as at other positions, say, $\Delta E = 0$, $\varphi = r > 0$, or $\Delta E = \Delta E_{_{D}} > 0$, $\varphi = 0$,

- 6 -

- 7 -

for which

$$H_{o} = -\omega \frac{qV}{\pi} \sin^{2} \frac{r}{2} = -\frac{\omega \cdot \Delta E_{b}^{2}}{2|e_{s}|} . \qquad (17)$$

In the same way as we characterize a trajectory by its Hamiltonian, we can do it using, instead, either r or ΔE_b . The advantage of this is that we can get rid of the initial conditions for the given particle, and can characterize it by one invariant of motion: either H_o or r or ΔE_b .

Particles within a bunch have different energies. The most energetic particles move along the trajectory which contains all the others. Then for such a trajectory, parameters $r, \Delta E_b, H_o$ are called bunch half-length, bunch half-height, and bunch Hamiltonian (Fig.4).

We will apply subscript b to all bunch parameters. Sometimes r_b is called *bunch* (synchrotron) amplitude, which should not be confused with particle amplitude $\sqrt{\Delta E^2 + \phi^2}$ which is not a constant of motion, while r_b is. The bunch half-length r_b is a useful parameter because it can be easily measured experimentally.



Fig.4.Stationary bucket & bunch.

The bunch half-length or half-height can be expressed by the counterpart to (17):

$$\Delta E_{b} = \sqrt{2\frac{qV}{\pi}|e_{s}|} \cdot \sin\frac{r_{b}}{2}, \quad r_{b} = 2\arcsin\left(\frac{\Delta E_{b}}{\sqrt{2\frac{qV}{\pi}|e_{s}|}}\right). \quad (18)$$

From this the half-height of the bucket ΔE_{a} can be found by taking $r_{b} = \pi$:

$$\Delta E_{o} = \sqrt{2\frac{qV}{\pi}|e_{s}|} .$$
 (17)

The most important parameter of the motion is the central synchrotron frequency Ω_{o} , which is often is referred to simply as the synchrotron frequency (of a small amplitude). Actually it is the maximal synchrotron frequency. We can find it by linearizing the equations of motion (1): setting $\sin \varphi \cong \varphi$, which is a good approximation for all the particles moving within a short bunch $r_{\rm b} \ll 1$.

Thus for a linear system we have instead of (1),

$$\Delta \dot{E} = \omega \frac{qV}{2\pi} \cdot \varphi , \qquad \dot{\varphi} = -\frac{\omega}{|\varphi_{a}|} \cdot \Delta E, \qquad (20)$$

and we reduce it to one equation of harmonic oscillations ($e_{s}<0$):

$$\dot{\varphi}' = -\omega^2 \frac{qV}{2\pi |e_{\rm g}|} \cdot \varphi \quad , \tag{21}$$

whose frequency is what we call the (central) synchrotron frequency (for small amplitude):

$$\Omega_{o} = \omega \cdot \sqrt{\frac{qV}{2\pi \left| e_{s} \right|}} \quad . \tag{22}$$

If the synchrotron amplitude r is not small, then the synchrotron frequency depends on amplitude $\Omega=\Omega(r)$ (see Ref.1 for more details).

Synchrotron (longitudinal) tune
$$v=v(r)$$
 is the ratio of frequencies
 $v=\Omega(r)/\omega$. (23)

Most applications use a central synchrotron tune

. . · ·

$$\nu_{o} = \Omega_{o} / \omega = \sqrt{\frac{qV}{2\pi |e_{s}|}} , \qquad (24)$$

which is often referred to simply as *synchrotron tune*. We'll follow this tradition. The bucket half-height and tune have a simple relation

$$\Delta E_{o} = 2\nu_{o} |e_{s}| . \qquad (25)$$

With the same accuracy as Sinq≅q, one can rewrite (18) for short bunches as

$$\Delta E_{b} = r_{b\gamma} \sqrt{\frac{q \nabla |\theta_{s}|}{2\pi}}, \quad r_{b} = \frac{\Delta E_{b}}{\sqrt{\frac{q \nabla |\theta_{s}|}{2\pi}}}.$$
(26)

If we can measure the bunch length experimentally, then with (18) or (26) we can calculate the bunch height or bunch energy spread.

1.4. Dimensionless Equations.

Let us introduce a dimensionless time measured in units of synchrotron period and a dimensionless energy measured in units of bucket half-height:

$$\begin{cases} \tau = \Omega_{o}t, & T_{o}=2\pi/\Omega_{o}, \\ \delta E = \Delta E/\Delta E_{o}, & = \frac{d}{d\tau} = \frac{1}{\Omega_{o}}\frac{d}{d\tau}. \end{cases}$$
(27)

Then the dynamic equations (1) will take the dimensionless form:

$$-2\frac{\partial \mathbf{H}}{\partial \varphi} = 2\delta \dot{\mathbf{E}} = \sin \varphi , \qquad |\delta \mathbf{E}| \leq 1, \qquad (28)$$

$$\frac{\partial \mathbf{H}}{\partial \delta \mathbf{E}} = \dot{\boldsymbol{\varphi}} = -2\delta \mathbf{E} , \qquad |\boldsymbol{\varphi}| \leq \pi, \qquad (29)$$

with the corresponding new Hamiltonian

$$H(\delta E, \varphi) = -\delta E^2 - \sin^2 \frac{\varphi}{2} , \qquad (30)$$

which is scaled to the old separator:

$$H = H/H_{sep} .$$
(31)

The separatrix (bucket) equation in phase space (Fig.5) is just

 $\delta E = t Cos \frac{\varphi}{2}$. Bucket half-height is

δE_=Cosû=1.

Bucket emittance is

$$\varepsilon_{o} = 4 \int_{0}^{\pi} \cos \frac{\varphi}{2} d\varphi = 8.$$
 (34)

The trajectory equation of the particle with synchrotron amplitude r can be written as

$$\delta E = \pm \sqrt{\sin^2 \frac{r}{2} - \sin^2 \frac{\varphi}{2}} . \quad (35)$$

If (35) is an equation of the bunch boundary then the half-height of the bunch is

$$\delta E_{b} = m = \sin \frac{r_{b}}{2} . \qquad (36)$$



bucket and bunch.



If the bunch half-length is $r_b^{}$, then the bunch (longitudinal) emittance is

$$\varepsilon = \varepsilon(r_{b}) = 4 \int_{0}^{r_{b}} \sqrt{\sin^{2} \frac{r_{b}}{2} - \sin^{2} \frac{\varphi}{2}} \, d\varphi = 8 \left[E(m) - (1 - m) K(m) \right], \quad (37)$$

where K(m) and E(m) are complete elliptic integrals of the 1st and 2nd kind (see Appendix C for more details). For a short bunch $(\sin \frac{r_b}{2} \approx \frac{r_b}{2})$ any particle within the bunch is governed by Hamiltonian

$$H(\delta E, \varphi) = -\delta E^{2} - \frac{\varphi^{2}}{4} = -\frac{r^{2}}{4}, \qquad (38)$$

which means that the particle trajectory is an ellipse

$$\left(\frac{\delta E}{r/2}\right)^2 + \left(\frac{\varphi}{r}\right)^2 = 1 \quad . \tag{39}$$

The bunch boundary is the same type of ellipse with $r=r_{b}$, and the area of this ellipse is the short bunch emittance:

$$\varepsilon = \frac{\pi r_b^2}{2} . \tag{40}$$

The dimensional bunch half-height, half-length, and emittance are then

$$\Delta E_{b} = \Delta E_{o} \delta E_{b}, \ r_{b}, \ A = \Delta E_{o} \varepsilon , \qquad (41)$$

where ΔE_{\perp} is a bucket half-height according to (19) and (26).

In this section we have considered a continuous model of particle dynamics. This model is good because it is simple and productive. It gives us all the bunch parameters we need. But how accurate is the continuous model? How good is our approximation which reduces the effect of one or many localized cavities to one continuous cavity? Isn't it extreme?

To examine this extreme, we will go in the next section to another extreme and then we'll compare the results from both.

2. DISCRETE MODEL

In this section we will review a discrete model describing longitudinal motion of charged particles. This model assumes that all accelerating cavities are localized at one point of the machine ring [2]. The case of many point-like cavities [3] will be discussed at the end of this section. Below, subscript n will denote for any particle its number of cavity traversals.

Then synchrotron motion of non-interacting particles for stationary conditions is governed by the mapping equations:

$$\begin{cases} \delta E_{n+1} = \delta E_n + \pi v_0 \text{Sin } \phi_{n+1}, \\ \phi_{n+1} = \phi_n - 4\pi v_0 \delta E_n, \quad n=0,1,2,\dots \end{cases}$$
(1)

This system can be reduced to the system (1.28)-(1.29) if we replace time differences by time derivatives:

$$\delta \vec{E} = \frac{\delta E_{n+1} - \delta E_n}{\Delta \tau} , \quad \dot{\varphi} = \frac{\varphi_{n+1} - \varphi_n}{\Delta \tau} ,$$

$$\Delta \tau = \Omega_o \Delta t = \Omega_o \frac{2\pi}{\omega} = 2\pi \nu_o ,$$

$$\tau_{n+1} = \tau_n + \Delta \tau , \quad \tau_o = t_o = 0 .$$
(2)

For the *l*-th particle, system (1),(2) is solved by iterations starting from initial position $\delta E_{\phi}^{(i)}, \phi_{\phi}^{(i)}$.

2.1. Stability.

To simplify our analysis, we suppose that the bunch is short enough to linearize a non-linear part of (1) by putting $\sin \phi = \phi$ for any particle in the bunch. Now we are dealing with linear system

$$\begin{cases} \delta E_{n+1} = \delta E_n + \pi \nu_o \phi_{n+1}, \\ \phi_{n+1} = \phi_n - 4\pi \nu_o \delta E_n, \quad n=0,1,2,\dots \end{cases}$$
(3)

with given initial values $\phi_{\rm c}, \delta E_{\rm c}.$

First let's see how stable this system is. Start with decoupling of the system (3).

Now let us introduce a complex variable ($l=\sqrt{-1}$):

• •

$$z_n = \varphi_n + i \delta E_n$$
, n=0,1,2,... (4)

Then we have a single equation

. .

$$z_{n+1}^{-}(2-\mu^2)z_n^{+}z_{n-1}^{=0} = 0, \quad n=0,1,2,...,$$
 (5)

where

$$\mu = 2\pi\nu_{o} \qquad (6)$$

Because the 2nd order difference equation (5) is similar to the 2nd order differential equation, we seek a general solution as a linear combination of two particular solutions:

$$z_n = Ae^{i\theta n} + Be^{-i\theta n} = \varphi_n + i\delta E_n, \quad n = 0, 1, 2, \dots, \qquad (7)$$

Here A,B are complex constants (amplitudes), which are determined by

$$\begin{cases} A+B = \varphi_0 + i\delta E_0 = z_0, \\ Ae^{i\theta} + Be^{-i\theta} = z_1 = \varphi_1 + i\delta E_1, \end{cases}$$
(8)

where $\varphi_0, \delta E_0$ are initial values, θ is characteristic constant (step angle), and $\varphi_1, \delta E_1$ are calculated from (3):

$$\varphi_{i} = \varphi_{o} - 2\mu \delta E_{o}$$
, $\delta E_{i} = \delta E_{o} + \frac{\mu}{2} \varphi_{i} = \frac{\mu}{2} \varphi_{o} + (1 - \mu^{2}) \delta E_{o}$. (9)

For z_n to be a solution of (5) it is necessary that step angle θ satisfies a characteristic equation when any particular solution from (7) is being used for (5):

$$e^{i\theta} - (2-\mu^2) + e^{-i\theta} = 0,$$
 (10)

$$\cos\theta = 1 - \frac{\mu^2}{2}.$$
 (11)

- 13 -

 $|\mu| \leq 2$

Step angle θ is real if

or, as it follows from (6),

$$\nu \leq 1/\pi$$
 (13)

(12)

This is the stability condition. If it is satisfied, then solutions of (7) and (3) are bounded, because $|z_n| \leq |A| + |B|$ for any n.

If $|\mu|$ is small, so then is θ , and there is the approximation $\theta \cong \mu$.

2.2. Discrete conservation law

By substituting the second equation of (3) into the first one, we can rewrite this system as

$$\begin{cases} \varphi_{n+1} = \varphi_n - 2\mu \cdot \delta E_n ,\\ \delta E_{n+1} = \frac{\mu}{2} \varphi_n + (1 - \mu^2) \delta E_n , \quad n=0,1,2,\dots \end{cases}$$
(14)

Introducing vectors \mathbf{X}_{n} and the matrix \mathbf{M}

$$\mathbf{X}_{n} = \begin{pmatrix} \varphi_{n} \\ \delta E_{n} \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} \mathbf{1} & -2\mu \\ \mu & \mathbf{1} - \mu^{2} \\ \mathbf{2} & \mathbf{1} - \mu^{2} \end{pmatrix}, \qquad (15)$$

we have instead of (14) a matrix recurrence

$$\mathbf{X}_{\mathbf{p+1}} = \mathbf{M} \cdot \mathbf{X}_{\mathbf{p}} \tag{16}$$

or

$$\mathbf{X}_{n} = \mathbf{M}^{n} \cdot \mathbf{X}_{o} \quad . \tag{17}$$

Mappings (16) and (17) are area preserving because

The matrix M cannot be diagonalized, because in the sense of matrix theory M is not normal:

r

$$M^{T}M \neq MM^{T}$$
. (19)

4

Following Courant and Snyder [4], we'll represent

$$M = I \cdot Cos\theta + J \cdot Sin\theta , \qquad (20)$$

μ

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{1-\frac{\mu^2}{4}}} & -\frac{1}{2\sqrt{1-\frac{\mu^2}{4}}} \\ \frac{1}{2\sqrt{1-\frac{\mu^2}{4}}} & -\frac{\mu}{2\sqrt{1-\frac{\mu^2}{4}}} \\ \frac{1}{2\sqrt{1-\frac{\mu^2}{4}}} & -\frac{1}{2\sqrt{1-\frac{\mu^2}{4}}} \end{pmatrix}$$
(21)

- 14 -

and

$$J^2 = -I$$
 , (22)

 $\mathbf{M}^{n} = \mathbf{I} \cdot \mathbf{Cos} n\theta + \mathbf{J} \cdot \mathbf{Sin} n\theta .$ (23)

Then

$$\mathbf{X}_{n} = \mathbf{I} \mathbf{X}_{o} \mathbf{C} \mathbf{D} \mathbf{S} \mathbf{n} \theta + \mathbf{J} \mathbf{X}_{o} \mathbf{S} \mathbf{S} \mathbf{n} \mathbf{n} \theta .$$
(24)

Multiplying (24) by the matrix J we have, using (22):

$$\mathbf{JX}_{n} = \mathbf{JX}_{o} \mathbb{C}osn\theta - \mathbf{IX}_{o} \mathbb{S}inn\theta .$$
 (25)

Constructing scalar products we get from (24), (25):

$$(\mathbf{X}_{n}, \mathbf{X}_{n}) + (\mathbf{J}\mathbf{X}_{n}, \mathbf{J}\mathbf{X}_{n}) = (\mathbf{X}_{o}, \mathbf{X}_{o}) + (\mathbf{J}\mathbf{X}_{o}, \mathbf{J}\mathbf{X}_{o})$$
(26)

This is a conservation law. It says that during the synchrotron motion a quadratic form on the left-hand side is the same as it was at the initial position.

This quadratic form we denote as Q for any vector $X=(\phi, \delta E)$:

$$Q(\mathbf{X}) = (\mathbf{X}, \mathbf{X}) + (\mathbf{J}\mathbf{X}, \mathbf{J}\mathbf{X}) =$$

$$= \varphi^{2} + \delta E^{2} + (\alpha \varphi + \beta \delta E)^{2} + (\gamma \varphi - \alpha \delta E)^{2} =$$

$$= (\alpha^{2} + \gamma^{2} + 1) \varphi^{2} + 2(\alpha \beta - \alpha \gamma) \varphi \delta E + (\alpha^{2} + \beta^{2} + 1) \delta E^{2} =$$

$$= a_{11} \varphi^{2} + 2a_{12} \varphi \delta E + a_{22} \delta E^{2} ,$$
(27)

where

$$a_{11} = \alpha^2 + \gamma^2 + 1 = \frac{5}{4 - \mu^2}, \quad a_{12} = \alpha\beta - \alpha\gamma = \frac{-5\mu}{4 - \mu^2}, \quad a_{22} = \alpha^2 + \beta^2 + 1 = \frac{20}{4 - \mu^2}, \quad (28)$$

and

.

$$-a_{33} = Q(X_{o}) = a_{11} \varphi_{o}^{2} + 2a_{12} \varphi_{o} \delta E_{o} + a_{22} \delta E_{o}^{2} .$$
(29)

In the phase plane (φ , δ E) the sequence of points (φ_n , δ E_n), determined by the mapping (17), is lies along the closed trajectory

$$Q(\varphi, \delta E) + a_{33} = 0$$
 (30)

A conservation law (26) or (30) tells us that trajectory (30) is an invariant of motion, so are the parameters of the trajectory, like its semi-axes and area (emittance).

2.3. Bunch parameters and orientation

As we will see, trajectory (30) is an ellipse whose axes are tilted with respect to co-ordinate axes(Fig.6). We wish to find parameters of this ellipse. If the ellipse confines all the particles, then the ellipse's parameters are bunch parameters. In order to get the equation of the ellipse

in the canonical form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
(31)

let us introduce new (tilted) axes x, ywith the same origin as old axes and with the rotation angle Φ :

$$\begin{cases} \mathcal{X} = \varphi \cdot \mathbb{C} \circ \mathbb{S} \Phi + \delta \mathbb{E} \cdot \mathbb{S} \inf \Phi , \\ \mathcal{Y} = -\varphi \cdot \mathbb{S} \inf \Phi + \delta \mathbb{E} \cdot \mathbb{C} \circ \mathbb{S} \Phi , \end{cases}$$
(32)



Fig.6. Tilted bunch on the phase space.

$$\tan 2\Phi = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2}{3}\mu \quad . \tag{33}$$

Then the semi axes a and b can be expressed as

$$a^2 = -a_{33}/\lambda_2$$
, $b^2 = -a_{33}/\lambda_1$, (34)

where λ_1,λ_2 are the roots of the characteristic equation:

$$\lambda^{2} - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}^{2}) = 0.$$
(35)

$$\lambda_{1,2} = \frac{a_{11}^{+}a_{22}}{2} \pm \sqrt{\left(\frac{a_{11}^{-}a_{22}}{2}\right)^2 + a_{12}^2} = \frac{2.5}{4 - \mu^2} (5 \pm \sqrt{9 + 4\mu^2}) , \qquad (36)$$

$$\lambda_{1}\lambda_{2} = a_{11}a_{22}-a_{12}^{2} = \frac{25}{4-\mu^{2}}.$$
 (37)

The ellipse area (emittance) is

$$\varepsilon = \pi a b = \frac{\pi |a_{33}|}{\sqrt{a_{11} a_{22} - a_{12}^2}} = \frac{\pi |a_{33}| \sqrt{4 - \mu^2}}{5} .$$
(38)

2.4. Discrete model with N cavities

When N cavities are equally spaced along the machine ring, then the total voltage V is equally divided between all N cavities. Then (1) becomes

$$\begin{cases} \delta E_{n+1} = \delta E_n + \frac{\pi \nu_o}{N} \sin \varphi_{n+1}, \\ \varphi_{n+1} = \varphi_n - 4 \frac{\pi \nu_o}{N} \delta E_n, \quad n=0,1,2,\dots,N,N+1,\dots \end{cases}$$
(37)

These equations describe changes in energy δE and phase φ for particle going from cavity to cavity N times and repeating this process again and again. The only difference between (39) and (1) is that tune ν_{o} in (1) is replaced by ν_{o}/N in (39). This means that all the facts following from (1) which depended on tune will be the same for the case of N cavities after we replace ν_{o} by ν_{o}/N . In particular

$$\mu = 2\pi \nu / N , \qquad (40)$$

and the angle Φ between ellipse axes and coordinate axes is

$$\tan 2\Phi = \frac{2}{3}\mu = 4\pi\nu_{o}/3N$$
 (41)

If $\mu < 3/2$ which is always the case then

$$\Phi = \mu/3 - \Delta \Phi \cong \mu/3 , \qquad (42)$$

where

3. INTERPRETATION OF BUNCH EXPERIMENTAL MEASUREMENTS

When we are dealing with a stationary bunch in an experiment, then the first thing we look for is a "mountain range" or bunch line density distribution, which allows us to measure a longitudinal

projection of the bunch length. Having a half-length we then can calculate all the other bunch parameters such AS half-height (energy spread) and emittance. Of course, we know from the stationary conditions all the other numbers of interest-- total peak voltage V, synchronous energy E, RF frequency f, charge q, and so on. If the bunch would be an ellipse with the axes exactly aligned along coordinate axes in phase space, then the measured



Fig.7.Horizontal bunch and its line density.

(projected) length would be the exact bunch length (Fig.7).

Now imagine an experimenter, let's call him Boss, who has gotten from a mountain range the half-length $r_{\rm b}$ (radians) of a short bunch. He asks two theorists, let's call them Tom and Mary, to calculate the bunch parameters: emittance, energy spread and half-height.

3.1. Interpretation based on continuous model

Suppose Tom believes in the continuous model, while Mary believes in the discrete model. Tom immediately gets results. According to (1.36) the bunch half-height (dimensionless), which in his case just the energy spread, is

$$\delta E_{b} = \sin \frac{r_{b}}{2} . \tag{1}$$

This formula is not approximate, it's exact for short as well as for long bunches. For the short bunch (1) it can be written as

$$\delta E_{b} = r_{b}/2 \quad . \tag{2}$$

Emittance with the same accuracy is due to (2.39),

$$\varepsilon = \frac{\pi r_b^2}{2} . \tag{3}$$

3.2. Interpretation based on discrete model

Mary considers the bunch as a tilted ellipse (Fig.8). She says that the measured r_b is not the real bunch half-length, rather r_b is the bunch right-most point $\varphi = r_b$, $\delta E = \delta E_r$. In the vicinity of this point the particle's trajectory $\varphi = \varphi(\delta E)$ doesn't change its φ -coordinate:

$$\frac{\partial \varphi}{\partial \delta E} \Big|_{\substack{\delta E = \delta E}} = 0 \quad . \tag{4}$$

Mary says the real bunch parameters should be calculated from (2.34) and (2.38). For this she need to calculate a_{33} , which is part of trajectory equation (2.30):



Fig.8.Tilted bunch and its line density.

$$Q(\varphi, \delta E) + a_{33} = 0, \qquad (5)$$

$$a_{11} = \frac{5}{4-\mu^2}$$
, $a_{12} = \frac{-5\mu}{4-\mu^2}$, $a_{22} = \frac{20}{4-\mu^2}$, (6)

and

$$-a_{33} = a_{11} \varphi_{0}^{2} + 2a_{12} \varphi_{0} \delta E_{0} + a_{22} \delta E_{0}^{2} .$$
 (7)

To calculate a_{33} it is necessary to know one point $(\varphi_o, \delta E_o)$ lying on the trajectory (5). In Mary's case such a point is $\varphi_o = r_b$, $\delta E_o = \delta E_r$. Mary knows r_b from experiment, but not δE_r . The last can be found by solving both (4) and (5) simultaneously.

 $Q(\varphi, \delta E) = a_{11} \varphi^2 + 2a_{12} \varphi \delta E + a_{22} \delta E^2$,

Indeed, after differentiating (5) with respect to δE one can get (taking into account (4)),

$$\delta E_{r} = -\frac{a_{12}}{a_{22}} \cdot r_{b} = \frac{r_{b}}{4} \mu , \qquad (B)$$

from which $(\phi_o = r_b, \delta E_o = \delta E_r)$,

$$-a_{33} = \frac{a_{11}a_{22}-a_{12}}{a_{22}} \cdot r_b^2 = \frac{5}{4} \cdot r_b^2 .$$
 (9)

Now the bunch parameters can be found. The bunch half-length is the major semi-axis a of the ellipse .

According to (2.34), (2.38) and (9)

$$a = \sqrt{\frac{|a_{33}|}{\lambda_2}} = r_b \sqrt{\frac{5 + \sqrt{9 + 4\mu^2}}{8}} \cong r_b (1 + \frac{1}{24}\mu^2).$$
(10)

Bunch half-height is

$$b = \sqrt{\frac{|a_{33}|}{\lambda_{1}}} = r_{b} \sqrt{\frac{5 - \sqrt{9 + 4\mu^{2}}}{8}} \cong \frac{r_{b}}{2} (1 - \frac{1}{6}\mu^{2}).$$
(11)

Bunch emittance is

$$\varepsilon = \pi a b = \frac{\pi r_b^2}{4} \sqrt{4 - \mu^2} . \qquad (12)$$

The highest point $(\delta E_h, \phi_h)$ of the bunch determines the energy spread δE_h . In the vicinity of this point the particle's trajectory $\delta E = \delta E(\phi)$ doesn't change its δE -coordinate:

$$\frac{\partial \partial E}{\partial \varphi} \Big|_{\varphi=r_{\rm b}} = 0 \quad . \tag{13}$$

Then

•

$$\frac{\partial \mathbf{Q}}{\partial \varphi}\Big|_{\varphi=\tau_{\mathbf{h}}} = 2a_{\mathbf{i}\mathbf{i}}\varphi_{\mathbf{h}} + 2a_{\mathbf{i}\mathbf{2}}\delta \mathbf{E}_{\mathbf{h}} = 0 ,$$

or

$$\varphi_{h} = -\frac{a_{12}}{a_{11}} \cdot \delta E_{h}$$
 (14)

After taking $a_{_{93}}$ from (9) and substituting (14) into $Q+a_{_{93}}=0$ she gets

$$\delta E_{\rm h} = \frac{r_{\rm b}}{2} , \quad \varphi_{\rm h} = \frac{r_{\rm b}}{2} \mu .$$
 (15)

3.3. Comparison of the two interpretations

Tom and Mary submitted their results to experimenter Boss, and reminded him to multiply results by ΔE_{o} from (1.19) in order to convert half-height, half-spread and emittance to dimensional form. Boss put Tom and Mary's results along with his own r_{b} in the following table.

T	ao	ιe	7.

bunch parameter	experimental value	continuous model	discrete model	ratio discrete/continuous
phase half-spread	, г ь			
bunch half-length	1	r _b	$r_{b}(1+\frac{1}{24}\mu^{2})$	$1 + \frac{1}{24} \mu^2 > 1$
bunch half-height		$\frac{r_{b}}{2}$.	$\frac{r_{\rm b}}{2}(1-\frac{1}{6}\mu^2)$	$1-\frac{1}{\sigma}\mu^2 \leq 1$
bunch energ half-spread	;у I	$\frac{r_{b}}{2}$	$\frac{r_{b}}{2}$	1 =1
bunch emittance		$\frac{\pi r_b^2}{2}$	$\frac{\pi r_b^2}{4} \sqrt{4-\mu^2}$	$1-\frac{i}{8}\mu^2 < 1$
bunch tilt angle Φ		0	<u>4</u> 3	

-Look guys, Boss says, the only common thing coming from your models is the energy spread (Fig.7)! All the other parameters differ from 4% up to 17% if μ would be equal to 1.

How can I trust you or your theories? -Don't worry Boss, Mary says, according to (2.40) $\mu = 2\pi v_o/N$ is small number, because v_o is the synchrotron tune and N is the number of accelerating cavities. For any machine, my estimate is $\mu < .1$, so your worst estimate between our two models will be 0.17% not 17%. See for example the following table, where I put for every machine its largest possible tune.



Fig.9.Energy spread is the only common parameter for two models.

species:	proto	ons	J	heavy ions		electrons
accelerator:	AGS_Booster	AGS	AGS_Booster	AGS	RHIC	LEP
the largest synchrotron tune v_{a}	.003	.03	.01	.01	5,10-0	^s .1
number of cavities N	2	10	2	1	2	128
step angle (rad $\mu=2\pi\nu_{o}/N$ (deg bunch tilt angle $\Phi=\mu/3$	ians).009 ^{rees)} .54 [°] .18 [°]	.02 1.08° .36°	.03 1.8° .6°	.06 3.6° 1.2°	2.10 ⁻⁵ 0 ⁰ 0 ⁰	• .005 .28° .09°
bunch tilt angle for N=1	.36°	3.6°	1.2°	1.2°	0 °	12°

-Now I see, Boss says, that your two models provide very close results. So, I guess, we have no problems anymore.

-Mary does, Tom said.

-What's that?

-You see, sometimes being impatient, Mary combines all the machine cavities into one effective cavity and does longitudinal tracking in her computer for such a simplified discrete model. This means N=1 and for a number of cases where the synchrotron tune is large, there will be a large exaggeration in the bunch parameters coming from a such a discrete model. For example, the proton bunch tilt angle for the AGS will be 3.6° and for electrons in LEP it will be 12° (see table last row).

3.4.Final experiment

-Now you are confusing me again, says Boss. You have too many tricks with your models and computers, and I have too little time to check or study them. Moreover, I'm not sure that you know all the differences between your models today. Maybe you'll find some new one tomorrow. So, my point is to check the problem experimentally. Give me a hint how I can see in the control room whether a stationary bunch is horizontal in phase space or a bunch is tilted? -Switch the voltage off,-simultaneously said Tom and Mary,- and see how the mountain range will change its shape during 3-4 revolutions while debunching. We have already done this in our computers.

-If the bunch was horizontal before debunching, then during debunching the mountain range will change like in Fig.10, - said Tom.

-On the other hand,-said Mary,- debunching will look like that in Fig.11 if the bunch was originally tilted.



Fig.10.Peak is decreasing monotonically for the horizontal bunch.



Fig.11. Peak is increasing then falling for the tilted bunch.

-Good bye, said Boss, I'm going to the Control Room to debunch a stationary bunch.

-Good luck, said Tom and Mary, we are going to our computers...

ACKNOWLEDGMENTS

I am grateful to Drs. L.Ahrens, M.Goldman, E.Raka, A.Ruggiero for useful discussions. I am thankful to Marion Heimerle for reading and improving the manuscript.

REFERENCES

[1] J.Kats, "A Simple Approximation for Synchrotron Frequency". AGS Tech.Note No.324, Aug.2, 1989, BNL.
[2] G.Dome, "Synchrotron Motion in a Stationary Bucket with RF Noise". Workshop on pp in the SPS, SPS-PP-1, 7 May 1980, pp. 102-117.
[3] G.Dome, "Theory of RF Acceleration and RF Noise". Proceedings of 1983 CERN Accelerator School: Antiprotons for Colliding Beam Facilities. CERN 84-15, pp.215-260.
[4] E.D.Courant and H.S.Snyder, "Theory of the Alternating-Gradient Synchrotron", Annals of Physics, Vol.3, No.1, 1958, pp.1-48.
[5] L.D.Landau and E.M.Lifshitz, Mechanics, 3rd edition, 1982, Pergamon Press.
[4] M.Abramovitz and I.A.Stegun, Handbook of Mathematical

Functions, National Bureau of Standards, 3rd printing,1965.

APPENDIX A

Atomic terminology for the pedestrian.

ATOMIC UNITS.

Electron charge $e=1.6 \cdot 10^{-19}$ C Electron mass =9.1 $\cdot 10^{-28}$ g Proton mass = $1.67 \cdot 10^{-24}$ g Atomic mass unit (amu) = $1.66 \cdot 10^{-24}$ g Hydrogen mass = 1.008amu

Most of the mass of the atom is in its central particle called the *nucleus*. Surrounding the nucleus are enough negatively charged *electrons* to make the normal atom neutral. The nucleus of any element is composed of *protons and neutrons*. The word *NUCleON* is used as a generic term meaning proton or neutron. A neutron has no charge.

Atomic number Z is the number of protons in the nucleus. It is also the ordering number of a chemical element in the Mendeleev table. Nuclear charge is

$$q=Ze$$
 . (1)

Mass number A is equal to the sum of protons and neutrons in the nucleus. The number of neutrons in the nucleus is N=A-Z. Nuclei having the same Z but different N are called *isotopes*.

Atomic unit of mass (amu) is 1/12th of the mass of a carbon isotope with a mass number A=12.

If Q electrons have been removed from the neutral atom, then the latter becomes an *ion* with the *charge state* Q.

Here is a part of the table from the Booster Design Manual (Table 1-1). It shows the atomic characteristics for some ions.

ION	CHARGE STATE Q	ATOMIC NUMBER Z	MASS NUMBER A	IONIC REST ENERGY GeV/nucleon
р	+1	1	1	0.93828
d	+1	1	2	0.93781
C	+6	6	12	0.93125
S ·	+14	16	32	0.93047
Au	+33	79	197	0.93126

APPENDIX B

Does the particle possess a potential energy?

The purpose of this section is to clarify terminology pertaining to the basic concepts of classical mechanics such as potential energy and conservation of energy. Good terminology should be æ vorking tool not the source of misconception. Speaking energy about "potential of a particle" we shouldn't forget that there is no such thing as the energy potential of a particle. So, let's about talk some prime concepts of classical mechanics and the resulting terminology.

Everything that follovs taken is from the Bible of theoretical physics -- Course of Theoretical Physics by Landau and Lifshitz with a minimum of my own comments. Because most of the folloving text vill be a quotation from Mechanics (5) I vill use instead of quotation marks a standard font like this. My own commentary type with a 1-11 small font like the two paragraphs you just read.

One of the fundamental concepts of mechanics is that of a *particle*. By this we mean a body whose dimensions may be neglected in describing its motion. Let us consider a system of particles which interact with one another but with no other bodies. This is called a *closed system*. It is found that interaction between the particles can be described by adding to the Lagrangian for non-interacting particles a certain function of coordinates, which depends on the nature of interaction. Denoting this function by -U we have

$$L = \sum \frac{1}{2} m_i v_i^2 - U(q_1, q_2, ...), \qquad (1)$$

where q_i is the position of the *i*-th particle. This is the general form of the Lagrangian for a closed system. The sum $T = \sum_{2}^{1} m_{i} v_{i}^{2}$ is called the *kinetic energy*, and U the *potential energy*, of the system.

I have to emphasize that potential energy describes the interaction all system, between the particles of the closed not the energy state of the individual particle. The only thing that belongs to the individual i-th particle is its kinetic energy $.5m_iv_i^{-}$.

The potential energy is defined only to within an additive constant, which has no effect on the equations of motion. This is a particular case of the non-uniqueness of the Lagrangian, which is defined only to within an additive total time derivative of any function of coordinates and time. The most natural and most usual way of choosing this constant is such that the potential energy tends to zero as the distances between the particles tend to infinity.

Hitherto we have spoken only of closed systems. Let us now consider a system A which is not closed and interacts with another system Bexecuting a given motion. In such a case we say that the system A

• •

moves in a given external field (due to system B). Since the equations nf motion are obtained from the principle of least action by independently varying each of the coordinates (i.e. by proceeding as if remainder were given quantities), we can find the Lagrangian L_{\star} of the system A by using the Lagrangian L of the whole system A+Band replacing the coordinates q_n therein by given functions of time. Assuming that the system A+B is closed, we have $L=T_A(q_A,\dot{q}_A)+T_B(q_B,\dot{q}_B) -U(q_{_{\rm A}},q_{_{\rm B}})$, where the first two terms are the kinetic energies of the systems A and B and the third term is their combined potential energy. Substituting for $q_{_{\rm B}}$ the given functions of time and omitting the term $T[q_{\mathbf{n}}(t), \dot{q}_{\mathbf{n}}(t)]$ which depends on time only, and is therefore the total time derivative of a function of time, we obtain

$$L_{A} = T_{A}(q_{A}, \dot{q}_{A}) - U[q_{A}, q_{B}(t)].$$
⁽²⁾

Thus the motion of a system in an external field described by ìs a Lagrangian of the usual type, the only difference being that the potential energy may depend explicitly on time. For example, when a single particle moves in an external field, the general form of the Lagrangian is

$$L = \frac{1}{2}mv^2 - U(q, t).$$
 (3)

Again we have here energy $ar{U}$ of interaction between a field changing as a given function of time and particle with coordinate q .

For any closed system there is a valid expression

$$\frac{\mathrm{d}}{\mathrm{dt}} \left(\sum \dot{q}_{i} \frac{\partial L}{\partial \dot{q}_{i}} - L \right) = 0.$$
⁽⁴⁾

The quantity in the brackets is called the energy of the system:

$$E = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \quad . \tag{5}$$

The energy of the system can be written as the sum of two quite different terms: kinetic energy which depends on the coordinates and velocities, and the potential energy, which depends on the coordinates of the particles and on time:

$$E=T(q,\dot{q})+U(q,t)$$
. (6)

Let us consider the conservation law resulting from the homogeneity of time. By virtue of this homogeneity, the Lagrangian of a closed system does not depend explicitly on time. total derivative The time of the Lagrangian will be zero and energy (5) will be constant.

The law of conservation of energy is valid not only for closed systems, but also for those in a constant external field (i.e.one independent of time): the only property of the Lagrangian used in the above derivation, namely that it does not involve the time explicitly, is still valid. In such a case an energy conservation law is

$$E=T(q, \dot{q})+U(q)=constant.$$
(7)

Mechanical systems whose energy is conserved are sometimes called *conservative* system.

EXAMPLE 1. When a mechanical pendulum moves without resistance in the Earth's gravitational field, then the energy conserves:

$$E = \frac{1}{2}m\ell^2 \phi^2 - mg\ell \cos \phi = constant.$$

Whose energy is conserved? Pendulum's energy? No.

The system's energy. The system of two interacting particles is pendulum-Earth. For that problem Earth's kinetic energy is zero and the potential energy of interaction between pendulum and Earth is $U=mgl\cos \phi$.

We also can say that the system is a pendulum-field. An Earth field. After all, we can say whatever we want, including that the pendulum energy is conserved. But the real sense of such terminology is that which was just described. If we will not forget it there will never be a misunderstanding.

EXAMPLE 2. When proton within a stationary bunch exercises a synchrotron motion then the energy is conserved:

$$H(\delta E, \varphi) = -\delta E^2 - \sin^2 \varphi = constant.$$

Whose energy is conserved? Proton's energy? No.

The system's energy. The system of two interacting "particles" is the proton and an RF cavity. In this case it is better to say that the system is the proton and the cavity field.

A potential energy $-\sin^2(\phi/2)$ is not the "proton's potential energy". It is potential energy of interaction between a proton and cavity's electrical field.

If we agree with a such understanding and remember it then we can use any words including "a potential energy of a particle".

I'm sorry to bother you with the obvious truth, but the clear terminology should help our mutual understanding and can save time.

APPENDIX C

Calculation of bunch longitudinal emittance

$$\varepsilon = 4 \int_{0}^{r_{o}} \sqrt{\sin^{2} \frac{r_{o}}{2} - \sin^{2} \frac{\varphi}{2}} \, d\varphi = \begin{vmatrix} \inf r_{o} & \lim r_{o}$$

$$= 4 \cdot \int_{0}^{\pi/2} \frac{2m \cdot \cos^{2} \xi d\xi}{\sqrt{1 - m \sin^{2} \xi}} = 8 \int_{0}^{\pi/2} \frac{m - 1 + 1 - m \sin^{2} \xi}{\sqrt{1 - m \sin^{2} \xi}} d\xi = 8 \left[E(m) - m_{1} K(m) \right].$$
(1)

The power series for E and K are [4]

$$\mathsf{E}(m) = \frac{\pi}{2} \left[1 - \frac{m}{4} - \frac{3}{64} m^2 - \frac{5}{256} m^3 - \dots \right], \qquad |m| < 1, \tag{2}$$

$$K(m) = \frac{\pi}{2} \left[1 + \frac{m}{4} + \frac{9}{64}m^2 + \frac{225}{64 \cdot 36}m^3 + \dots \right], \qquad |m| < 1,$$
(3)

which gives us

÷ ~

$$\varepsilon(m) = E - (1 - m^2) \kappa = \frac{\pi m}{4} \left[1 + \frac{m}{4} - \frac{m^2}{8} \dots \right].$$
 (4)

I have found that the best single parabola fitting to this expression over the interval $|m| \le 1$ is

$$\varepsilon(m) = E(m) - m_1 K(m) \approx m(.762 + .209m).$$
 (5)

A relative error for this expression is less than 3% .

.

.