

# TWO MODELS FOR A STATIONARY BUNCH IN LONGITUDINAL PHASE SPACE

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## TWO MODELS FOR A STATIONARY BUNCH IN LONGITUDINAL PHASE SPACE

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### Abstract

A stationary bunch is the simplest accelerator concept for testing mathematical models of particle dynamics in longitudinal phase space.

There are two mathematical models for particle dynamics : continuous and discrete. The continuous model is based on differential equations and Hamiltonian formalism, while the discrete model is based on difference equations and recurrence theory. Each of the models is approximate and is judged by comparison with experiment. Both models advantages come from their analytical and computational simplicity and effectiveness. It is a combination of these two models which makes modeling successful. For example, we almost always use the Hamiltonian from the continuous model to calculate the bunch shape (boundary of particles' stable region) and the bunch parameters, such as length and height; then we track particle trajectories using the discrete model.

In this report we construct the bunch shape twice, without mixing the two models; first from the continuous model, then from the discrete model. A comparison of the two bunches reveals that the second bunch is tilted with respect to the first.

The angle between axes of the two bunches depends on the ratio of the synchrotron tune to the number of cavities. The bigger the ratio, the larger the angle.

In this report we review both models and compare some of the results coming independently from each model. The main conclusion is that both results are very close. *The reason for such closeness is that the ratio of synchrotron tune to the number of (equally spaced) accelerating cavities is very small for the vast majority of existing accelerators.*

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## 1. CONTINUOUS MODEL

### 1.1 Basic equations and parameters.

In this section we will review the continuous model describing longitudinal motion of charged particles. This model assumes that one accelerating cavity acts continuously along the whole circumference of the machine. Then a synchrotron motion of non-interacting particles within the **stationary** bucket is governed by the equations:

$$\Delta \dot{E} = \omega \frac{qV}{2\pi} \sin \phi, \quad \dot{\phi} = \frac{\omega}{e_s} \Delta E \quad (1)$$

with

$$\left\{ \begin{array}{ll} \Delta E = E - E_s, & h\omega = 2\pi f, \\ e_s = \beta_s^2 E_s / h\eta, & \eta = 1/\gamma_t^2 - 1/\gamma_s^2, \\ \gamma_s = E_s / E_r, & \beta_s^2 = 1 - 1/\gamma_s^2, \\ q = Qe, & E_r = I_r \cdot A, \end{array} \right. \quad (2)$$

where  $E, \phi$  are particle's energy and phase, subscripts  $r, s, t$  refer to the rest, synchronous and transition energy,  $e$  is proton's charge,  $q$  is charge of the ion subject to synchrotron motion,  $Q$  is the ion charge state (number of stripped electrons),  $A$  is mass number,  $I_r$  is ionic rest energy per nucleon,  $V$  is (total) peak voltage of RF system, driven with the frequency  $f$  which is synchronized with the particle's revolution frequency  $\omega$  as it is shown by (2). Note that  $e_s < 0$  below transition ( $\gamma_s < \gamma_t$ ) and  $e_s > 0$  above transition ( $\gamma_s > \gamma_t$ ). Definitions and details on ion parameters are presented in Appendix A.

For the  $i$ -th particle, the nonlinear system (1) is solved starting from initial position  $\delta E_0^{(i)}, \phi_0^{(i)}$ .

## 1.2 Hamiltonian and Bunch.

Equations (1) lead to or can be derived from the Hamiltonian

$$H(\varphi, \Delta E) = \omega \left[ \frac{\Delta E^2}{2e_s} + \frac{qV}{2\pi} \cos \varphi \right] + \text{arbitrary constant.} \quad (3)$$

This represents a conservation law: the sum of particle kinetic and potential energy (in appropriate canonical variables) is constant during particle motion. (See Appendix B for basic terminology.)

The Hamiltonian (3) has a dimension of energy (Fig.1).

If necessary the Hamiltonian can be normalized by an appropriate choice of constant. One way to do it is to make  $H$  as a positive-definitive form:  $H \geq 0$  (or negative-definitive  $H \leq 0$ ) and to make  $\min H = 0$  ( $\max H = 0$ ).

For example, by adding to (3) the constant  $qV/2\pi$  we get

$$H^+(\varphi, \Delta E) = \omega \left[ \frac{\Delta E^2}{2e_s} + \frac{qV}{\pi} \cos \frac{2\varphi}{2} \right]. \quad (4)$$

This form is convenient for studying a motion after transition when  $e_s > 0$ ,  $H^+ \geq 0$  (Fig.2).

On the other hand, by subtracting  $qV/2\pi$  from (3) we get

$$H^-(\varphi, \Delta E) = \omega \left[ \frac{\Delta E^2}{2e_s} - \frac{qV}{\pi} \sin^2 \frac{\varphi}{2} \right]. \quad (5)$$

This Hamiltonian is more convenient before transition, when  $e_s < 0$ ,  $H^- \leq 0$  (Fig.3). In both cases Hamiltonian equations

$$\frac{d\Delta E}{dt} = -\frac{\partial H}{\partial \varphi}, \quad \frac{d\varphi}{dt} = \frac{\partial H}{\partial \Delta E} \quad (6)$$

lead to (1). If the initial conditions for a particle are  $\Delta E_0, \varphi_0$  then the Hamiltonian for such a particle has a constant value of  $H^\pm(\Delta E_0, \varphi_0) = H_0^\pm$ .

The trajectory for that particle can be expressed either implicitly

$$H^\pm(\varphi, \Delta E) = H_0^\pm \quad (7)$$

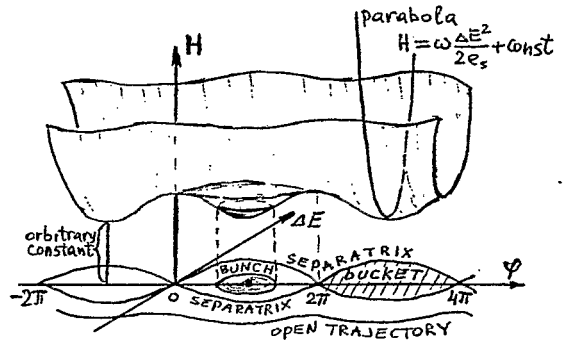


Fig.1. Hamiltonian surface and its projection on the phase plane.

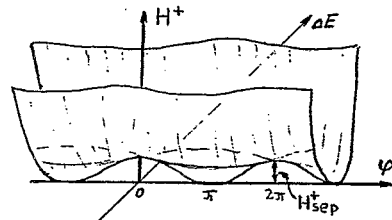


Fig.2. Hamiltonian surface of energy above transition.

or explicitly:

$$\Delta E^{\pm} = \Delta E^{\pm}(\varphi, H_o^{\pm}), \quad (8)$$

$$\Delta E^{+} = \pm \sqrt{2e_s \left( H_o^{+}/\omega - \frac{qV}{\pi} \cos^2 \frac{\varphi}{2} \right)}, \quad (8^{+})$$

$$\Delta E^{-} = \pm \sqrt{2e_s \left( H_o^{-}/\omega + \frac{qV}{\pi} \sin^2 \frac{\varphi}{2} \right)}. \quad (8^{-})$$

Fixed points of Hamiltonian surface  $H=H(\varphi, \Delta E)$  are found from

$$\frac{\partial H}{\partial \Delta E} = 0, \quad \frac{\partial H}{\partial \varphi} = 0. \quad (9)$$

These points are pairs  $(\varphi = \pi n, \Delta E = 0)$ ,  $n = 0, \pm 1, \pm 2, \dots$ .

If a particle's trajectory is closed in the vicinity of the fixed point, then such a point is called *stable*. Otherwise, the fixed point is called *unstable*. It's easy to see from the periodicity conditions

$$H^{\pm}(\varphi, \Delta E) = H^{\pm}(\Delta E, \varphi \pm 2\pi), \quad (10)$$

that the stable fixed points  $\{\varphi = (2n+1)\pi, \Delta E = 0\}$  for  $H^{+}$  are unstable for  $H^{-}$  and the stable fixed points  $\{\varphi = 2n\pi, \Delta E = 0\}$  for  $H^{-}$  are unstable for  $H^{+}$ . The value of the Hamiltonian calculated at unstable fixed point is a *separator*. It is positive when  $e_s > 0$ , negative when  $e_s < 0$ :

$$H_{sep}^{\pm} = \pm \omega \frac{qV}{\pi} \quad (11)$$

It separates trajectories of all particles with different initial conditions into two classes - open and closed. According to (7) each trajectory can be uniquely characterized by the value of its Hamiltonian. Those trajectories for which

$$|H_o^{\pm}| < |H_{sep}^{\pm}| \quad (12)$$

are closed and those for which

$$|H_o^{\pm}| > |H_{sep}^{\pm}| \quad (13)$$

are open.

A trajectory with

$$H_o^{\pm} = H_{sep}^{\pm} \quad (14)$$

is called a *separatrix*.

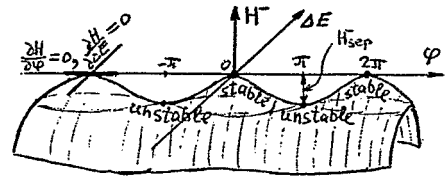


Fig.3. Hamiltonian surface of energy below transition.

It consists of two branches intersecting each other. Two branches of a separatrix between consecutive unstable fixed points compose a closed trajectory called a *bucket*. The length of the stationary bucket is  $2\pi$ . There are  $h$  buckets in the phase space, the configuration of buckets periodically repeating itself. All trajectories within the buckets are closed, while all trajectories outside the buckets are open. At the center of each bucket there is exactly one stable fixed point.

The set of particles within a bucket is called a *bunch*. The precise definition is more flexible; it says a bunch comprises 95% or 100% or 99% (or whatever) percentage of the total bucket particles.

The actual number of buckets in an accelerator is determined by the harmonic number  $h$ . In the AGS we have  $h=12$  buckets lying along the 807.075m circumference of machine. So, we have  $807/12= 67.25m$  as the length of each bucket.

In this report we will deal exclusively with energies below transition ( $e_s < 0$ ) and with the Hamiltonian  $\bar{H}$  on the interval  $(-\pi, \pi)$ . For these reasons we will drop the superscript  $\bar{}$  for the Hamiltonian and will write  $H$  instead of  $\bar{H}$  and  $-|e_s|$  instead of  $e_s$ .

### 1.3 Synchrotron Frequency and Synchrotron Tune.

In the phase space  $(\varphi, \delta E)$  a particle's trajectory is a closed curve and for that reason the particle's motion is periodic i.e. it performs *synchrotron oscillations*. The particle's *synchrotron frequency*  $\Omega$ , depends on the value of that particle's Hamiltonian:

$$H(\varphi, \Delta E) = -\omega \left[ \frac{\Delta E^2}{2|e_s|} + \frac{qV}{\pi} \sin^2 \frac{\varphi}{2} \right], \tag{15}$$

which is a constant,  $H_0$ , determined by particle's initial position:

$$H_0 = H(\Delta E_0, \varphi_0) \tag{16}$$

The larger the absolute value of  $H$ , the smaller is the synchrotron frequency  $\Omega = \Omega(H)$ .

While a particle moves along a trajectory, its coordinates  $\Delta E = \Delta E(t)$ ,  $\varphi = \varphi(t)$  are changing with time according to (15). However, the Hamiltonian doesn't change; its value  $H_0$  remains the same as at initial position  $\Delta E = \Delta E_0$ ,  $\varphi = \varphi_0$  as well as at other positions, say,  $\Delta E = 0, \varphi = \pi > 0$ , or  $\Delta E = \Delta E_0 > 0, \varphi = 0$ ,



for which

$$H_0 = -\omega \frac{qV}{\pi} \sin^2 \frac{r}{2} = -\frac{\omega \cdot \Delta E_b^2}{2|e_s|} \quad (17)$$

In the same way as we characterize a trajectory by its Hamiltonian, we can do it using, instead, either  $r$  or  $\Delta E_b$ . The advantage of this is that we can get rid of the initial conditions for the given particle, and can characterize it by one invariant of motion: either  $H_0$  or  $r$  or  $\Delta E_b$ .

Particles within a bunch have different energies. The most energetic particles move along the trajectory which contains all the others. Then for such a trajectory, parameters  $r, \Delta E_b, H_0$  are called *bunch half-length*, *bunch half-height*, and *bunch Hamiltonian* (Fig.4).

We will apply subscript  $b$  to all bunch parameters. Sometimes  $r_b$  is called *bunch (synchrotron) amplitude*, which should not be confused with particle amplitude  $\sqrt{\Delta E^2 + \varphi^2}$  which is not a constant of motion, while  $r_b$  is. The **bunch half-length  $r_b$  is a useful parameter because it can be easily measured experimentally.**

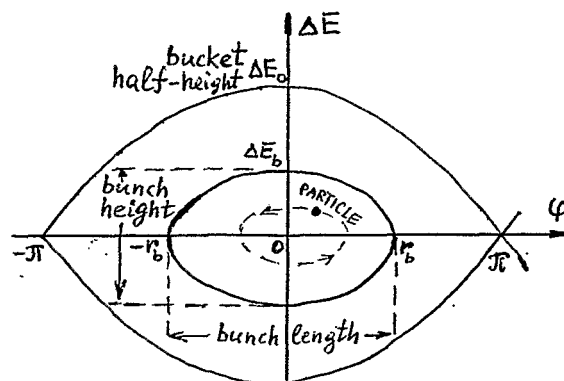


Fig.4. Stationary bucket & bunch.

The bunch half-length or half-height can be expressed by the counterpart to (17):

$$\Delta E_b = \sqrt{\frac{2qV}{\pi} |e_s|} \cdot \sin \frac{r_b}{2}, \quad r_b = 2 \arcsin \left( \frac{\Delta E_b}{\sqrt{\frac{2qV}{\pi} |e_s|}} \right) \quad (18)$$

From this the half-height of the bucket  $\Delta E_0$  can be found by taking  $r_b = \pi$ :

$$\Delta E_0 = \sqrt{\frac{2qV}{\pi} |e_s|} \quad (19)$$

The most important parameter of the motion is the *central synchrotron frequency*  $\Omega_0$ , which is often referred to simply as the *synchrotron frequency* (of a small amplitude). Actually it is the maximal synchrotron frequency. We can find it by linearizing the equations of motion (1): setting  $\sin \varphi \approx \varphi$ , which is a good approximation for all the particles moving within a short bunch  $r_b \ll 1$ .

Thus for a linear system we have instead of (1),

$$\Delta \dot{E} = \omega \frac{qV}{2\pi} \cdot \varphi, \quad \dot{\varphi} = -\frac{\omega}{|e_s|} \cdot \Delta E, \quad (20)$$

and we reduce it to one equation of harmonic oscillations ( $e_s < 0$ ):

$$\ddot{\varphi} = -\omega^2 \frac{qV}{2\pi |e_s|} \cdot \varphi, \quad (21)$$

whose frequency is what we call the (central) synchrotron frequency (for small amplitude):

$$\Omega_0 = \omega \cdot \sqrt{\frac{qV}{2\pi |e_s|}}. \quad (22)$$

If the synchrotron amplitude  $r$  is not small, then the synchrotron frequency depends on amplitude  $\Omega = \Omega(r)$  (see Ref. 1 for more details).

*Synchrotron (longitudinal) tune*  $\nu = \nu(r)$  is the ratio of frequencies

$$\nu = \Omega(r) / \omega. \quad (23)$$

Most applications use a *central synchrotron tune*

$$\nu_0 = \Omega_0 / \omega = \sqrt{\frac{qV}{2\pi |e_s|}}, \quad (24)$$

which is often referred to simply as *synchrotron tune*. We'll follow this tradition. The bucket half-height and tune have a simple relation

$$\Delta E_0 = 2\nu_0 |e_s|. \quad (25)$$

With the same accuracy as  $\sin \varphi \approx \varphi$ , one can rewrite (18) for short bunches as

$$\Delta E_b = r_b \sqrt{\frac{qV |e_s|}{2\pi}}, \quad r_b = \frac{\Delta E_b}{\sqrt{\frac{qV |e_s|}{2\pi}}}. \quad (26)$$

If we can measure the bunch length experimentally, then with (18) or (26) we can calculate the bunch height or bunch energy spread.

### 1.4. Dimensionless Equations.

Let us introduce a dimensionless time measured in units of synchrotron period and a dimensionless energy measured in units of bucket half-height:

$$\begin{cases} \tau = \Omega_0 t, & T_0 = 2\pi/\Omega_0, \\ \delta E = \Delta E/\Delta E_0, & \dot{\phantom{x}} \equiv \frac{d}{d\tau} = \frac{1}{\Omega_0} \frac{d}{dt}. \end{cases} \quad (27)$$

Then the dynamic equations (1) will take the dimensionless form:

$$-2 \frac{\partial H}{\partial \varphi} = 2\delta \dot{E} = \text{Sin} \varphi, \quad |\delta E| \leq 1, \quad (28)$$

$$\frac{\partial H}{\partial \delta E} = \dot{\varphi} = -2\delta E, \quad |\varphi| \leq \pi, \quad (29)$$

with the corresponding new Hamiltonian

$$H(\delta E, \varphi) = -\delta E^2 - \text{Sin}^2 \frac{\varphi}{2}, \quad (30)$$

which is scaled to the old separator:

$$H = H/H_{\text{sep}}. \quad (31)$$

The separatrix (bucket) equation in phase space (Fig.5) is just

$$\delta E = \pm \text{Cos} \frac{\varphi}{2}. \quad (32)$$

Bucket half-height is

$$\delta E_0 = \text{Cos} 0 = 1. \quad (33)$$

Bucket emittance is

$$\epsilon_0 = 4 \int_0^\pi \text{Cos} \frac{\varphi}{2} d\varphi = 8. \quad (34)$$

The trajectory equation of the particle with synchrotron amplitude  $r$  can be written as

$$\delta E = \pm \sqrt{\text{Sin}^2 \frac{r}{2} - \text{Sin}^2 \frac{\varphi}{2}}. \quad (35)$$

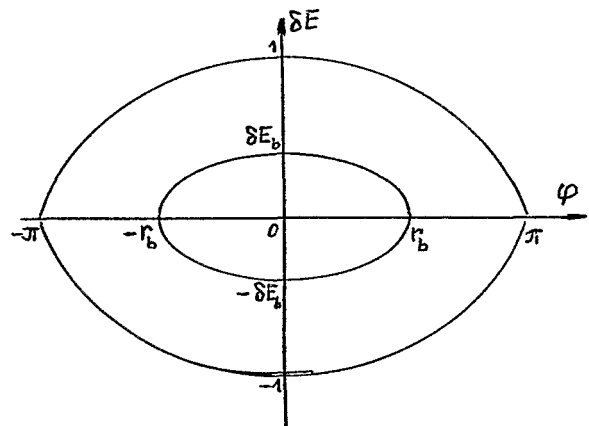


Fig.5. Dimensionless bucket and bunch.

If (35) is an equation of the bunch boundary then the half-height of the bunch is

$$\delta E_b = r_b = \text{Sin} \frac{r_b}{2}. \quad (36)$$

If the bunch half-length is  $r_b$ , then the bunch (longitudinal) emittance is

$$\varepsilon = \varepsilon(r_b) = 4 \int_0^{r_b} \sqrt{\sin^2 \frac{r}{2} - \sin^2 \frac{\varphi}{2}} d\varphi = 8 \left[ E(m) - (1-m)K(m) \right], \quad (37)$$

where  $K(m)$  and  $E(m)$  are complete elliptic integrals of the 1st and 2nd kind (see Appendix C for more details). For a short bunch ( $\sin \frac{r_b}{2} \cong \frac{r_b}{2}$ ) any particle within the bunch is governed by Hamiltonian

$$H(\delta E, \varphi) = -\delta E^2 - \frac{\varphi^2}{4} = -\frac{r^2}{4}, \quad (38)$$

which means that the particle trajectory is an ellipse

$$\left( \frac{\delta E}{r/2} \right)^2 + \left( \frac{\varphi}{r} \right)^2 = 1. \quad (39)$$

The bunch boundary is the same type of ellipse with  $r=r_b$ , and the area of this ellipse is the short bunch emittance:

$$\varepsilon = \frac{\pi r_b^2}{2}. \quad (40)$$

The dimensional bunch half-height, half-length, and emittance are then

$$\Delta E_b = \Delta E_0 \delta E_b, \quad r_b, \quad A = \Delta E_0 \varepsilon, \quad (41)$$

where  $\Delta E_0$  is a bucket half-height according to (19) and (26).

In this section we have considered a continuous model of particle dynamics. This model is good because it is simple and productive. It gives us all the bunch parameters we need. But how accurate is the continuous model? How good is our approximation which reduces the effect of one or many localized cavities to one continuous cavity? Isn't it extreme?

To examine this extreme, we will go in the next section to another extreme and then we'll compare the results from both.

## 2. DISCRETE MODEL

In this section we will review a discrete model describing longitudinal motion of charged particles. This model assumes that all accelerating cavities are localized at one point of the machine ring [2]. The case of many point-like cavities [3] will be discussed at the end of this section. Below, subscript  $n$  will denote for any particle its number of cavity traversals.

Then synchrotron motion of non-interacting particles for stationary conditions is governed by the mapping equations:

$$\begin{cases} \delta E_{n+1} = \delta E_n + \pi \nu_0 \sin \varphi_{n+1}, \\ \varphi_{n+1} = \varphi_n - 4\pi \nu_0 \delta E_n, \end{cases} \quad n=0,1,2,\dots \quad (1)$$

This system can be reduced to the system (1.28)-(1.29) if we replace time differences by time derivatives:

$$\begin{cases} \dot{\delta E} = \frac{\delta E_{n+1} - \delta E_n}{\Delta \tau}, & \dot{\varphi} = \frac{\varphi_{n+1} - \varphi_n}{\Delta \tau}, \\ \Delta \tau = \Omega_0 \Delta t = \Omega_0 \frac{2\pi}{\omega} = 2\pi \nu_0, \\ \tau_{n+1} = \tau_n + \Delta \tau, \quad \tau_0 = t_0 = 0. \end{cases} \quad (2)$$

For the  $l$ -th particle, system (1),(2) is solved by iterations starting from initial position  $\delta E_0^{(i)}, \varphi_0^{(i)}$ .

### 2.1. Stability.

To simplify our analysis, we suppose that the bunch is short enough to linearize a non-linear part of (1) by putting  $\sin \varphi = \varphi$  for any particle in the bunch. Now we are dealing with linear system

$$\begin{cases} \delta E_{n+1} = \delta E_n + \pi \nu_0 \varphi_{n+1}, \\ \varphi_{n+1} = \varphi_n - 4\pi \nu_0 \delta E_n, \end{cases} \quad n=0,1,2,\dots \quad (3)$$

with given initial values  $\varphi_0, \delta E_0$ .

First let's see how stable this system is. Start with decoupling of the system (3).

$$\left\{ \begin{array}{l} \delta E_{n+1} - \delta E_n = \pi \nu_0 \varphi_{n+1}, \\ \delta E_n - \delta E_{n-1} = \pi \nu_0 \varphi_n, \\ \hline \delta E_{n+1} - 2\delta E_n + \delta E_{n-1} = \pi \nu_0 (\varphi_{n+1} - \varphi_n) \\ \qquad \qquad \qquad = -4\pi^2 \nu_0^2 \delta E_n, \end{array} \right. \quad \left\{ \begin{array}{l} \varphi_{n+1} - \varphi_n = -4\pi \nu_0 \delta E_n, \\ \varphi_n - \varphi_{n-1} = -4\pi \nu_0 \delta E_{n-1}, \\ \hline \varphi_{n+1} - 2\varphi_n + \varphi_{n-1} = -4\pi \nu_0 (\delta E_n - \delta E_{n-1}) \\ \qquad \qquad \qquad = -4\pi^2 \nu_0^2 \varphi_n. \end{array} \right.$$

Now let us introduce a complex variable ( $i = \sqrt{-1}$ ):

$$z_n = \varphi_n + i \delta E_n, \quad n=0,1,2,\dots \quad (4)$$

Then we have a single equation

$$z_{n+1} - (2 - \mu^2) z_n + z_{n-1} = 0, \quad n=0,1,2,\dots \quad (5)$$

where

$$\mu = 2\pi \nu_0. \quad (6)$$

Because the 2nd order difference equation (5) is similar to the 2nd order differential equation, we seek a general solution as a linear combination of two particular solutions:

$$z_n = A e^{i\theta n} + B e^{-i\theta n} = \varphi_n + i \delta E_n, \quad n=0,1,2,\dots \quad (7)$$

Here A, B are complex constants (amplitudes), which are determined by

$$\begin{cases} A+B = \varphi_0 + i \delta E_0 = z_0, \\ A e^{i\theta} + B e^{-i\theta} = z_1 = \varphi_1 + i \delta E_1, \end{cases} \quad (8)$$

where  $\varphi_0, \delta E_0$  are initial values,  $\theta$  is characteristic constant (step angle), and  $\varphi_1, \delta E_1$  are calculated from (3):

$$\varphi_1 = \varphi_0 - 2\mu \delta E_0, \quad \delta E_1 = \delta E_0 + \frac{\mu}{2} \varphi_1 = \frac{\mu}{2} \varphi_0 + (1 - \mu^2) \delta E_0. \quad (9)$$

For  $z_n$  to be a solution of (5) it is necessary that step angle  $\theta$  satisfies a characteristic equation when any particular solution from (7) is being used for (5):

$$e^{i\theta} - (2 - \mu^2) + e^{-i\theta} = 0, \quad (10)$$

$$\cos \theta = 1 - \frac{\mu^2}{2}. \quad (11)$$

Step angle  $\theta$  is real if

$$|\mu| \leq 2 \quad (12)$$

or, as it follows from (6),

$$\nu_0 \leq 1/\pi . \quad (13)$$

This is the stability condition. If it is satisfied, then solutions of (7) and (3) are bounded, because  $|z_n| \leq |A| + |B|$  for any  $n$ .

If  $|\mu|$  is small, so then is  $\theta$ , and there is the approximation  $\theta \cong \mu$ .

## 2.2. Discrete conservation law

By substituting the second equation of (3) into the first one, we can rewrite this system as

$$\begin{cases} \varphi_{n+1} = \varphi_n - 2\mu \cdot \delta E_n , \\ \delta E_{n+1} = \frac{\mu}{2} \varphi_n + (1 - \mu^2) \delta E_n , \end{cases} \quad n=0,1,2,\dots \quad (14)$$

Introducing vectors  $X_n$  and the matrix  $M$

$$X_n = \begin{pmatrix} \varphi_n \\ \delta E_n \end{pmatrix}, \quad M = \begin{pmatrix} 1 & -2\mu \\ \frac{\mu}{2} & 1 - \mu^2 \end{pmatrix}, \quad (15)$$

we have instead of (14) a matrix recurrence

$$X_{n+1} = M \cdot X_n \quad (16)$$

or

$$X_n = M^n \cdot X_0 . \quad (17)$$

Mappings (16) and (17) are area preserving because

$$\det(M) = 1. \quad (18)$$

The matrix  $M$  cannot be diagonalized, because in the sense of matrix theory  $M$  is not normal:

$$M^T M \neq M M^T . \quad (19)$$

Following *Courant and Snyder* [4], we'll represent

$$M = I \cdot \cos \theta + J \cdot \sin \theta , \quad (20)$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \frac{\mu}{2\sqrt{1-\frac{\mu^2}{4}}} & -\frac{4}{2\sqrt{1-\frac{\mu^2}{4}}} \\ 1 & \mu \\ 2\sqrt{1-\frac{\mu^2}{4}} & 2\sqrt{1-\frac{\mu^2}{4}} \end{pmatrix} \quad (21)$$

and

$$J^2 = -I, \quad (22)$$

$$M^n = I \cdot \text{Cos}n\theta + J \cdot \text{Sinn}\theta. \quad (23)$$

Then

$$X_n = IX_0 \text{Cos}n\theta + JX_0 \text{Sinn}\theta. \quad (24)$$

Multiplying (24) by the matrix J we have, using (22):

$$JX_n = JX_0 \text{Cos}n\theta - IX_0 \text{Sinn}\theta. \quad (25)$$

Constructing scalar products we get from (24), (25):

$$(X_n, X_n) + (JX_n, JX_n) = (X_0, X_0) + (JX_0, JX_0). \quad (26)$$

This is a conservation law. It says that during the synchrotron motion a quadratic form on the left-hand side is the same as it was at the initial position.

This quadratic form we denote as Q for any vector  $X = (\varphi, \delta E)$  :

$$\begin{aligned} Q(X) &= (X, X) + (JX, JX) = \quad (27) \\ &= \varphi^2 + \delta E^2 + (\alpha\varphi + \beta\delta E)^2 + (\gamma\varphi - \alpha\delta E)^2 = \\ &= (\alpha^2 + \gamma^2 + 1)\varphi^2 + 2(\alpha\beta - \alpha\gamma)\varphi\delta E + (\alpha^2 + \beta^2 + 1)\delta E^2 = \\ &= a_{11}\varphi^2 + 2a_{12}\varphi\delta E + a_{22}\delta E^2, \end{aligned}$$

where

$$a_{11} = \alpha^2 + \gamma^2 + 1 = \frac{5}{4 - \mu^2}, \quad a_{12} = \alpha\beta - \alpha\gamma = \frac{-5\mu}{4 - \mu^2}, \quad a_{22} = \alpha^2 + \beta^2 + 1 = \frac{20}{4 - \mu^2}, \quad (28)$$

and

$$-a_{33} = Q(X_0) = a_{11}\varphi_0^2 + 2a_{12}\varphi_0\delta E_0 + a_{22}\delta E_0^2. \quad (29)$$

In the phase plane  $(\varphi, \delta E)$  the sequence of points  $(\varphi_n, \delta E_n)$ , determined by the mapping (17), lies along the closed trajectory

$$Q(\varphi, \delta E) + a_{33} = 0. \quad (30)$$

A conservation law (26) or (30) tells us that trajectory (30) is an invariant of motion, so are the parameters of the trajectory, like its semi-axes and area (emittance).



### 2.3. Bunch parameters and orientation

As we will see, trajectory (30) is an ellipse whose axes are tilted with respect to co-ordinate axes (Fig.6). We wish to find parameters of this ellipse. If the ellipse confines all the particles, then the ellipse's parameters are bunch parameters.

In order to get the equation of the ellipse in the canonical form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (31)$$

let us introduce new (tilted) axes  $x, y$  with the same origin as old axes and with the rotation angle  $\Phi$  :

$$\begin{cases} x = \varphi \cdot \cos\Phi + \delta E \cdot \sin\Phi, \\ y = -\varphi \cdot \sin\Phi + \delta E \cdot \cos\Phi, \end{cases} \quad (32)$$

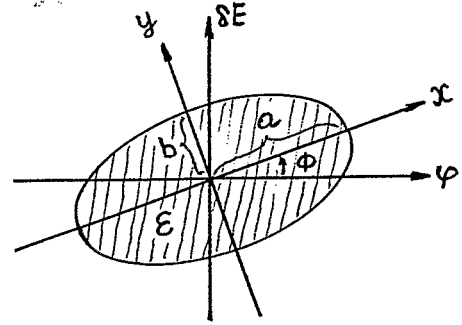


Fig.6. Tilted bunch on the phase space.

$$\tan 2\Phi = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2}{3}\mu. \quad (33)$$

Then the semi axes  $a$  and  $b$  can be expressed as

$$a^2 = -a_{33}/\lambda_2, \quad b^2 = -a_{33}/\lambda_1, \quad (34)$$

where  $\lambda_1, \lambda_2$  are the roots of the characteristic equation:

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}^2) = 0. \quad (35)$$

$$\lambda_{1,2} = \frac{a_{11} + a_{22}}{2} \pm \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2} = \frac{2.5}{4 - \mu^2} (5 \pm \sqrt{9 + 4\mu^2}), \quad (36)$$

$$\lambda_1 \lambda_2 = a_{11}a_{22} - a_{12}^2 = \frac{25}{4 - \mu^2}. \quad (37)$$

The ellipse area (emittance) is

$$\varepsilon = \pi ab = \frac{\pi |a_{33}|}{\sqrt{a_{11}a_{22} - a_{12}^2}} = \frac{\pi |a_{33}| \sqrt{4 - \mu^2}}{5}. \quad (38)$$

## 2.4. Discrete model with N cavities

When N cavities are equally spaced along the machine ring, then the total voltage V is equally divided between all N cavities. Then (1) becomes

$$\begin{cases} \delta E_{n+1} = \delta E_n + \frac{\pi v_0}{N} \sin \varphi_{n+1}, \\ \varphi_{n+1} = \varphi_n - 4 \frac{\pi v_0}{N} \delta E_n, \end{cases} \quad n=0,1,2,\dots,N,N+1,\dots \quad (39)$$

These equations describe changes in energy  $\delta E$  and phase  $\varphi$  for particle going from cavity to cavity N times and repeating this process again and again. The only difference between (39) and (1) is that tune  $\nu_0$  in (1) is replaced by  $\nu_0/N$  in (39). This means that all the facts following from (1) which depended on tune will be the same for the case of N cavities after we replace  $\nu_0$  by  $\nu_0/N$ . In particular

$$\mu = 2\pi\nu_0/N, \quad (40)$$

and the angle  $\Phi$  between ellipse axes and coordinate axes is

$$\tan 2\Phi = \frac{2}{3}\mu = 4\pi\nu_0/3N. \quad (41)$$

If  $\mu < 3/2$  which is always the case then

$$\Phi = \mu/3 - \Delta\Phi \cong \mu/3, \quad (42)$$

where 
$$0 < \Delta\Phi < 0.05\mu^3. \quad (43)$$

### 3. INTERPRETATION OF BUNCH EXPERIMENTAL MEASUREMENTS

When we are dealing with a stationary bunch in an experiment, then the first thing we look for is a "mountain range" or bunch line density distribution, which allows us to measure a longitudinal projection of the bunch length. Having a half-length we then can calculate all the other bunch parameters such as half-height (energy spread) and emittance. Of course, we know from the stationary conditions all the other numbers of interest-- total peak voltage  $V$ , synchronous energy  $E_s$ , RF frequency  $f$ , charge  $q$ , and so on. If the bunch would be an ellipse with the axes exactly aligned along coordinate axes in phase space, then the measured (projected) length would be the exact bunch length (Fig.7).

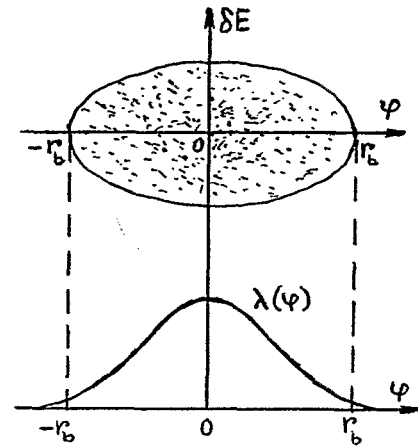


Fig.7. Horizontal bunch and its line density.

Now imagine an experimenter, let's call him Boss, who has gotten from a mountain range the half-length  $r_b$  (radians) of a short bunch. He asks two theorists, let's call them Tom and Mary, to calculate the bunch parameters: emittance, energy spread and half-height.

#### 3.1. Interpretation based on continuous model

Suppose Tom believes in the continuous model, while Mary believes in the discrete model. Tom immediately gets results. According to (1.36) the bunch half-height (dimensionless), which in his case just the energy spread, is

$$\delta E_b = \sin \frac{r_b}{2} . \tag{1}$$

This formula is not approximate, it's exact for short as well as for long bunches. For the short bunch (1) it can be written as

$$\delta E_b = r_b / 2 . \tag{2}$$

Emittance with the same accuracy is due to (2.39),

$$\epsilon = \frac{\pi r_b^2}{2} . \tag{3}$$

### 3.2. Interpretation based on discrete model

Mary considers the bunch as a tilted ellipse (Fig.8).

She says that the measured  $r_b$  is not the real bunch half-length, rather  $r_b$  is the bunch right-most point  $\varphi=r_b$ ,  $\delta E=\delta E_r$ . In the vicinity of this point the particle's trajectory  $\varphi=\varphi(\delta E)$  doesn't change its  $\varphi$ -coordinate:

$$\left. \frac{\partial \varphi}{\partial \delta E} \right|_{\delta E = \delta E_r} = 0. \quad (4)$$

Mary says the real bunch parameters should be calculated from (2.34) and (2.38).

For this she need to calculate  $a_{33}$ , which is part of trajectory equation (2.30):

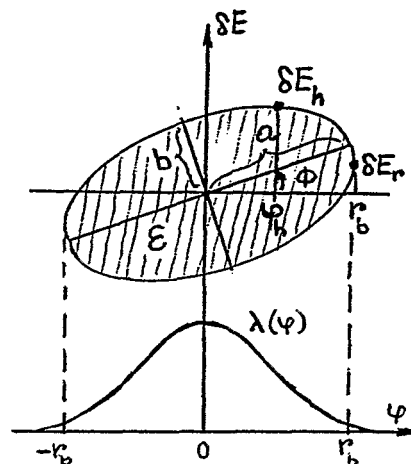


Fig.8. Tilted bunch and its line density.

$$Q(\varphi, \delta E) + a_{33} = 0, \quad (5)$$

$$Q(\varphi, \delta E) = a_{11}\varphi^2 + 2a_{12}\varphi\delta E + a_{22}\delta E^2,$$

where

$$a_{11} = \frac{5}{4-\mu^2}, \quad a_{12} = \frac{-5\mu}{4-\mu^2}, \quad a_{22} = \frac{20}{4-\mu^2}, \quad (6)$$

and

$$-a_{33} = a_{11}\varphi_0^2 + 2a_{12}\varphi_0\delta E_0 + a_{22}\delta E_0^2. \quad (7)$$

To calculate  $a_{33}$  it is necessary to know one point  $(\varphi_0, \delta E_0)$  lying on the trajectory (5). In Mary's case such a point is  $\varphi_0=r_b$ ,  $\delta E_0=\delta E_r$ . Mary knows  $r_b$  from experiment, but not  $\delta E_r$ . The last can be found by solving both (4) and (5) simultaneously.

Indeed, after differentiating (5) with respect to  $\delta E$  one can get (taking into account (4)),

$$\delta E_r = -\frac{a_{12}}{a_{22}} \cdot r_b = \frac{r_b}{4-\mu^2}, \quad (8)$$

from which  $(\varphi_0=r_b, \delta E_0=\delta E_r)$ ,

$$-a_{33} = \frac{a_{11}a_{22} - a_{12}^2}{a_{22}} \cdot r_b^2 = \frac{5}{4} \cdot r_b^2. \quad (9)$$

Now the bunch parameters can be found. The bunch half-length is the major semi-axis  $a$  of the ellipse.

According to (2.34), (2.38) and (9)

$$a = \sqrt{\frac{|a_{33}|}{\lambda_2}} = r_b \sqrt{\frac{5 + \sqrt{9 + 4\mu^2}}{8}} \cong r_b \left(1 + \frac{1}{24}\mu^2\right). \quad (10)$$

Bunch half-height is

$$b = \sqrt{\frac{|a_{33}|}{\lambda_1}} = r_b \sqrt{\frac{5 - \sqrt{9 + 4\mu^2}}{8}} \cong \frac{r_b}{2} \left(1 - \frac{1}{8}\mu^2\right). \quad (11)$$

Bunch emittance is

$$\varepsilon = \pi ab = \frac{\pi r_b^2}{4} \sqrt{4 - \mu^2}. \quad (12)$$

The highest point  $(\delta E_h, \varphi_h)$  of the bunch determines the energy spread  $\delta E_h$ . In the vicinity of this point the particle's trajectory  $\delta E = \delta E(\varphi)$  doesn't change its  $\delta E$ -coordinate:

$$\left. \frac{\partial \delta E}{\partial \varphi} \right|_{\varphi = \varphi_h} = 0. \quad (13)$$

Then

$$\left. \frac{\partial Q}{\partial \varphi} \right|_{\varphi = \varphi_h} = 2a_{11}\varphi_h + 2a_{12}\delta E_h = 0,$$

or

$$\varphi_h = -\frac{a_{12}}{a_{11}} \delta E_h. \quad (14)$$

After taking  $a_{33}$  from (9) and substituting (14) into  $Q + a_{33} = 0$  she gets

$$\delta E_h = \frac{r_b}{2}, \quad \varphi_h = \frac{r_b}{2}\mu. \quad (15)$$

### 3.3. Comparison of the two interpretations

Tom and Mary submitted their results to experimenter Boss, and reminded him to multiply results by  $\Delta E_0$  from (1.19) in order to convert half-height, half-spread and emittance to dimensional form. Boss put Tom and Mary's results along with his own  $r_b$  in the following table.

Table 1.

| bunch parameter          | experimental value | continuous model      | discrete model                         | ratio discrete/continuous   |
|--------------------------|--------------------|-----------------------|--|-----------------------------|
| phase half-spread        | $r_b$              |                       |  |                             |
| bunch half-length        |                    | $r_b$                 | $r_b (1 + \frac{1}{24}\mu^2)$          | $1 + \frac{1}{24}\mu^2 > 1$ |
| bunch half-height        |                    | $\frac{r_b}{2}$       | $\frac{r_b}{2} (1 - \frac{1}{8}\mu^2)$ | $1 - \frac{1}{8}\mu^2 < 1$  |
| bunch energy half-spread |                    | $\frac{r_b}{2}$       | $\frac{r_b}{2}$                        | 1 = 1                       |
| bunch emittance          |                    | $\frac{\pi r_b^2}{2}$ | $\frac{\pi r_b^2}{4} \sqrt{4 - \mu^2}$ | $1 - \frac{1}{8}\mu^2 < 1$  |
| bunch tilt angle $\phi$  |                    | 0                     | $\frac{\mu}{3}$                        | —                           |

-Look guys, Boss says, the only common thing coming from your models is the energy spread (Fig.9)! All the other parameters differ from 4% up to 17% if  $\mu$  would be equal to 1. How can I trust you or your theories?

-Don't worry Boss, Mary says, according to (2.40)  $\mu = 2\pi\nu_0/N$  is small number, because  $\nu_0$  is the synchrotron tune and N is the number of accelerating cavities. For any machine, my estimate is  $\mu < .1$ , so your worst estimate between our two models will be 0.17% not 17%. See for example the following table, where I put for every machine its largest possible tune.

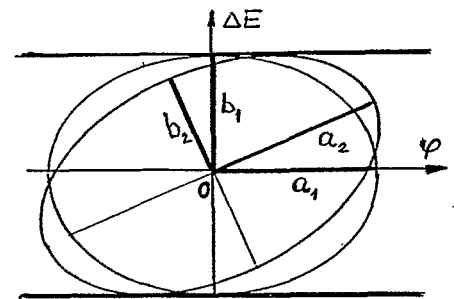


Fig.9. Energy spread is the only common parameter for two models.

Table 2.

| species:                             | protons     |              | heavy ions  |             | electrons         |             |
|--------------------------------------|-------------|--------------|-------------|-------------|-------------------|-------------|
| accelerator:                         | AGS_Booster | AGS          | AGS_Booster | AGS         | RHIC              | LEP         |
| the largest synchrotron tune $\nu_0$ | .003        | .03          | .01         | .01         | $5 \cdot 10^{-6}$ | .1          |
| number of cavities N                 | 2           | 10           | 2           | 1           | 2                 | 128         |
| step angle (radians)                 | .009        | .02          | .03         | .06         | $2 \cdot 10^{-5}$ | .005        |
| $\mu = 2\pi\nu_0 / N$ (degrees)      | $.54^\circ$ | $1.08^\circ$ | $1.8^\circ$ | $3.6^\circ$ | $0^\circ$         | $.28^\circ$ |
| bunch tilt angle $\Phi = \mu/3$      | $.18^\circ$ | $.36^\circ$  | $.6^\circ$  | $1.2^\circ$ | $0^\circ$         | $.09^\circ$ |
| bunch tilt angle for N=1             | $.36^\circ$ | $3.6^\circ$  | $1.2^\circ$ | $1.2^\circ$ | $0^\circ$         | $12^\circ$  |

-Now I see, Boss says, that your two models provide very close results. So, I guess, we have no problems anymore.

-Mary does, Tom said.

-What's that?

-You see, sometimes being impatient, Mary combines all the machine cavities into one effective cavity and does longitudinal tracking in her computer for such a simplified discrete model. This means N=1 and for a number of cases where the synchrotron tune is large, there will be a large exaggeration in the bunch parameters coming from a such a discrete model. For example, the proton bunch tilt angle for the AGS will be  $3.6^\circ$  and for electrons in LEP it will be  $12^\circ$  (see table last row).

### 3.4.Final experiment

-Now you are confusing me again, says Boss. You have too many tricks with your models and computers, and I have too little time to check or study them. Moreover, I'm not sure that you know all the differences between your models today. Maybe you'll find some new one tomorrow. So, my point is to check the problem experimentally. Give me a hint how I can see in the control room whether a stationary bunch is horizontal in phase space or a bunch is tilted?

-Switch the voltage off,-simultaneously said Tom and Mary,- and see how the mountain range will change its shape during 3-4 revolutions while debunching. We have already done this in our computers.

-If the bunch was horizontal before debunching, then during debunching the mountain range will change like in Fig.10, - said Tom.

-On the other hand,-said Mary,- debunching will look like that in Fig.11 if the bunch was originally tilted.

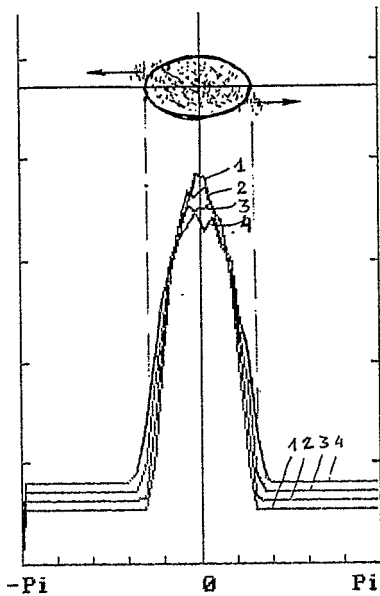


Fig.10. Peak is decreasing monotonically for the horizontal bunch.

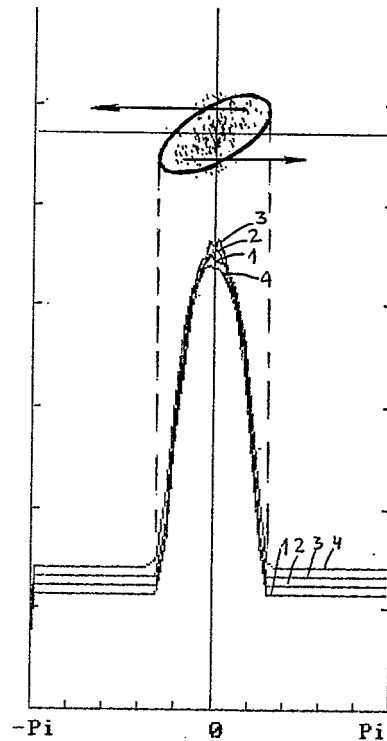


Fig.11. Peak is increasing then falling for the tilted bunch.

-Good bye, said Boss, I'm going to the Control Room to debunch a stationary bunch.

-Good luck, said Tom and Mary, we are going to our computers...

#### ACKNOWLEDGMENTS

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APPENDIX A

*Atomic terminology for the pedestrian.*

ATOMIC UNITS.

- Electron charge  $e=1.6 \cdot 10^{-19}$  C
- Electron mass  $=9.1 \cdot 10^{-28}$  g
- Proton mass  $= 1.67 \cdot 10^{-24}$  g
- Atomic mass unit (amu)  $=1.66 \cdot 10^{-24}$  g
- Hydrogen mass  $= 1.008$ amu

Most of the mass of the atom is in its central particle called the *nucleus*. Surrounding the nucleus are enough negatively charged *electrons* to make the normal atom neutral. The nucleus of any element is composed of *protons and neutrons*. The word *NUCLEON* is used as a generic term meaning proton or neutron. A neutron has no charge.

*Atomic number Z* is the number of protons in the nucleus. It is also the ordering number of a chemical element in the Mendeleev table. Nuclear charge is

$$q=Ze . \tag{1}$$

*Mass number A* is equal to the sum of protons and neutrons in the nucleus. The number of neutrons in the nucleus is  $N=A-Z$ . Nuclei having the same *Z* but different *N* are called *isotopes*.

*Atomic unit of mass (amu)* is 1/12th of the mass of a carbon isotope with a mass number  $A=12$ .

If *Q* electrons have been removed from the neutral atom, then the latter becomes an *ion* with the *charge state Q*.

Here is a part of the table from the Booster Design Manual (Table 1-i). It shows the atomic characteristics for some ions.

| ION | CHARGE STATE<br>Q | ATOMIC NUMBER<br>Z | MASS NUMBER<br>A | IONIC REST ENERGY<br>GeV/nucleon |
|-----|-------------------|--------------------|------------------|----------------------------------|
| P   | +1                | 1                  | 1                | 0.93828                          |
| d   | +1                | 1                  | 2                | 0.93781                          |
| C   | +6                | 6                  | 12               | 0.93125                          |
| S   | +14               | 16                 | 32               | 0.93047                          |
| Au  | +33               | 79                 | 197              | 0.93126                          |

## APPENDIX B

### *Does the particle possess a potential energy?*

The purpose of this section is to clarify terminology pertaining to the basic concepts of classical mechanics such as potential energy and conservation of energy. Good terminology should be a working tool not the source of misconception. Speaking about "potential energy of a particle" we shouldn't forget that there is no such thing as the potential energy of a particle. So, let's talk about some prime concepts of classical mechanics and the resulting terminology.

Everything that follows is taken from the Bible of theoretical physics -- **Course of Theoretical Physics** by Landau and Lifshitz -- with a minimum of my own comments. Because most of the following text will be a quotation from **Mechanics** [5] I will use instead of quotation marks a standard font like this. My own commentary I'll type with a small font like the two paragraphs you just read.

One of the fundamental concepts of mechanics is that of a *particle*. By this we mean a body whose dimensions may be neglected in describing its motion. Let us consider a system of particles which interact with one another but with no other bodies. This is called a *closed system*. It is found that interaction between the particles can be described by adding to the Lagrangian for non-interacting particles a certain function of coordinates, which depends on the nature of interaction. Denoting this function by  $-U$  we have

$$L = \sum \frac{1}{2} m_i v_i^2 - U(q_1, q_2, \dots), \quad (1)$$

where  $q_i$  is the position of the  $i$ -th particle. This is the general form of the Lagrangian for a closed system. The sum  $T = \sum \frac{1}{2} m_i v_i^2$  is called the *kinetic energy*, and  $U$  the *potential energy*, of the system.

I have to emphasize that potential energy describes the interaction between all the particles of the closed system, not the energy state of the individual particle. The only thing that belongs to the individual  $i$ -th particle is its kinetic energy  $\frac{1}{2} m_i v_i^2$ .

The potential energy is defined only to within an additive constant, which has no effect on the equations of motion. This is a particular case of the non-uniqueness of the Lagrangian, which is defined only to within an additive total time derivative of any function of coordinates and time. The most natural and most usual way of choosing this constant is such that the potential energy tends to zero as the distances between the particles tend to infinity.

Hitherto we have spoken only of closed systems. Let us now consider a system  $A$  which is not closed and interacts with another system  $B$  executing a given motion. In such a case we say that the system  $A$

moves in a given external field (due to system  $B$ ). Since the equations of motion are obtained from the principle of least action by independently varying each of the coordinates (i.e. by proceeding as if remainder were given quantities), we can find the Lagrangian  $L_A$  of the system  $A$  by using the Lagrangian  $L$  of the whole system  $A+B$  and replacing the coordinates  $q_B$  therein by given functions of time.

Assuming that the system  $A+B$  is closed, we have  $L = T_A(q_A, \dot{q}_A) + T_B(q_B, \dot{q}_B) - U(q_A, q_B)$ , where the first two terms are the kinetic energies of the systems  $A$  and  $B$  and the third term is their combined potential energy. Substituting for  $q_B$  the given functions of time and omitting the term  $T[q_B(t), \dot{q}_B(t)]$  which depends on time only, and is therefore the total time derivative of a function of time, we obtain

$$L_A = T_A(q_A, \dot{q}_A) - U[q_A, q_B(t)]. \quad (2)$$

Thus the motion of a system in an external field is described by a Lagrangian of the usual type, the only difference being that the potential energy may depend explicitly on time. For example, when a single particle moves in an external field, the general form of the Lagrangian is

$$L = \frac{1}{2}mv^2 - U(q, t). \quad (3)$$

Again we have here energy  $U$  of interaction between a field changing as a given function of time and particle with coordinate  $q$ .

For any closed system there is a valid expression

$$\frac{d}{dt} \left[ \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right] = 0. \quad (4)$$

The quantity in the brackets is called the *energy of the system*:

$$E = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L. \quad (5)$$

The energy of the system can be written as the sum of two quite different terms: kinetic energy which depends on the coordinates and velocities, and the potential energy, which depends on the coordinates of the particles and on time:

$$E = T(q, \dot{q}) + U(q, t). \quad (6)$$

Let us consider the conservation law resulting from the *homogeneity of time*. By virtue of this homogeneity, the Lagrangian of a closed system does not depend explicitly on time. The total time derivative of the Lagrangian will be zero and energy (5) will be constant.

The law of conservation of energy is valid not only for closed systems, but also for those in a constant external field (i.e. one independent of time): the only property of the Lagrangian used in the above derivation, namely that it does not involve the time explicitly, is still valid. In such a case an energy conservation law is

$$E = T(q, \dot{q}) + U(q) = \text{constant}. \quad (7)$$

Mechanical systems whose energy is conserved are sometimes called *conservative system*.

**EXAMPLE 1.** When a mechanical pendulum moves without resistance in the Earth's gravitational field, then the energy conserves:

$$E = \frac{1}{2} m l^2 \dot{\varphi}^2 - m g l \cos \varphi = \text{constant}.$$

Whose energy is conserved? Pendulum's energy? No.

The system's energy. The system of two interacting particles is pendulum-Earth. For that problem Earth's kinetic energy is zero and the potential energy of interaction between pendulum and Earth is  $U = m g l \cos \varphi$ .

We also can say that the system is a pendulum-field. An Earth field. After all, we can say whatever we want, including that the pendulum energy is conserved. But the real sense of such terminology is that which was just described. If we will not forget it there will never be a misunderstanding.

**EXAMPLE 2.** When proton within a stationary bunch exercises a synchrotron motion then the energy is conserved:

$$H(\delta E, \varphi) = -\delta E^2 - \text{Sin} \frac{2\varphi}{2} = \text{constant}.$$

Whose energy is conserved? Proton's energy? No.

The system's energy. The system of two interacting "particles" is the proton and an RF cavity. In this case it is better to say that the system is the proton and the cavity field.

A potential energy  $-\text{Sin}^2(\varphi/2)$  is not the "proton's potential energy". It is potential energy of interaction between a proton and cavity's electrical field.

If we agree with a such understanding and remember it then we can use any words including "a potential energy of a particle".

I'm sorry to bother you with the obvious truth, but the clear terminology should help our mutual understanding and can save time.

### APPENDIX C

#### Calculation of bunch longitudinal emittance

$$\varepsilon = 4 \int_0^{r_0} \sqrt{\sin^2 \frac{r_0}{2} - \sin^2 \frac{\varphi}{2}} d\varphi = \left. \begin{array}{l} \text{introducing new parameters } m=k^2=1-m_1 \\ \text{and variable } \xi: \\ m = \sin^2 \frac{r_0}{2}, \quad k \cdot \sin \xi = \sin \frac{\varphi}{2}, \\ \sqrt{m - \sin^2 \frac{\varphi}{2}} = k \cdot \cos \xi, \quad d\varphi = \frac{2k \cdot \cos \xi d\xi}{\sqrt{1-m \sin^2 \xi}} \\ \varphi=0 \Rightarrow \xi=0, \quad \varphi=r_0 \Rightarrow \xi=\pi/2. \end{array} \right\}$$

$$= 4 \cdot \int_0^{\pi/2} \frac{2m \cdot \cos^2 \xi d\xi}{\sqrt{1-m \sin^2 \xi}} = 8 \int_0^{\pi/2} \frac{m-1+1-m \sin^2 \xi}{\sqrt{1-m \sin^2 \xi}} d\xi = 8 \left[ E(m) - m_1 K(m) \right]. \quad (1)$$

The power series for E and K are [6]

$$E(m) = \frac{\pi}{2} \left[ 1 - \frac{m}{4} - \frac{3}{64} m^2 - \frac{5}{256} m^3 - \dots \right], \quad |m| < 1, \quad (2)$$

$$K(m) = \frac{\pi}{2} \left[ 1 + \frac{m}{4} + \frac{9}{64} m^2 + \frac{225}{64 \cdot 36} m^3 + \dots \right], \quad |m| < 1, \quad (3)$$

which gives us

$$\varepsilon(m) = E - (1-m^2)K = \frac{\pi m}{4} \left[ 1 + \frac{m}{4} - \frac{m^2}{8} \dots \right]. \quad (4)$$

I have found that the best single parabola fitting to this expression over the interval  $|m| < 1$  is

$$\varepsilon(m) = E(m) - m_1 K(m) \approx m(.762 + .209m). \quad (5)$$

A relative error for this expression is less than 3% .