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MEASURING A BEAM EMITTANCE USING LINEAR LEAST-SQUARE ANALYSIS

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1. Summary

This report describes one method of measuring the beam emittance. Many beam width measurements are fitted to the first order TRANSPORT Equation using a least-squared analysis procedure. The values and the standard deviations are determined. The method was used to measure the Single-Bunch Extraction (SBE) beam emittance to the 'D' line. Test programs and Fortran Source programs are given illustrating the use of least-square analysis. The necessary measurements to find a real emittance are also given.

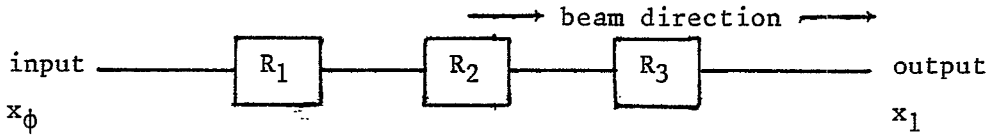
2. Introduction

Four steps are necessary to find the characteristics of a beam in a transport line. The necessary equation must be developed that expresses the beam width as a function of the emittance or characteristics of the beam. The variables of this equation, which are transport matrix elements, must be found. The horizontal or vertical width of the beam must be measured using flags, swics (segmented wire ion chambers), multi-wire devices or other instrumentation. The best emittance that uses these matrix elements and beam width measurements must be determined using some form of least-square fitting.

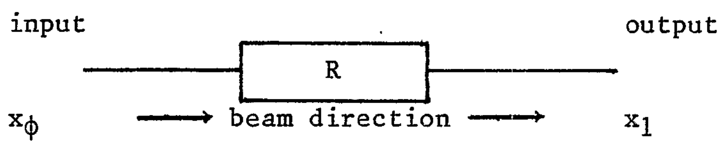
3. The "Transport Equation"

This report will use the notation of the QTUNE program...AGS Tech. Note 181. This notation is the same as the TRANSPORT Program except that a 5 x 5 matrix is used for the magnetic elements instead of a 6 x 6 matrix which TRANSPORT uses. Only first order theory is considered.

The beam is considered a collection of particles traveling down a beam line with the magnetic elements described with a matrix R_i.



or



$$(R) = (R_3) \times (R_2) \times (R_1) \tag{1}$$

$$DET(R) = 1$$

The characteristics of the beam particles at the output can be determined from the following matrix equation:

$$(X_1) = (R) \times (X_\phi) \tag{2}$$

For a 5 x 5 order matrix for the magnetic elements (R), Equation 2 can be expanded to:

$$\begin{bmatrix} X_1 \\ \theta_1 \\ Y_1 \\ \phi_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix} \times \begin{bmatrix} X_\phi \\ \theta_\phi \\ Y_\phi \\ \phi_\phi \\ \delta_\phi \end{bmatrix} \tag{3}$$

where standard TRANSPORT definitions apply to the particle characteristics:

X_o ---- horizontal displacement of input ray, in inches, with respect to assumed central trajectory.

θ_o ---- the angle (mr) that this input ray makes in horizontal plane with respect to central trajectory.

Y_o ---- vertical displacement of input ray (inches) with respect to central trajectory.

ϕ_o ---- the angle (mr) that this input ray makes in vertical plane with respect to central trajectory.

δ_o ---- $\Delta P/P$ = fractional momentum deviation (%) of this input ray and the assumed central trajectory.

One set of units for the (R) matrix is:

$$
\begin{array}{lllll}
R_{11}\left(\frac{\text{In X}}{\text{In X}}\right) & R_{12}\left(\frac{\text{In X}}{\text{mr X}}\right) & R_{13}\left(\frac{\text{In X}}{\text{In Y}}\right) & R_{14}\left(\frac{\text{In X}}{\text{mr Y}}\right) & R_{15}\left(\frac{\text{In X}}{\%}\right) \\[2ex]
R_{21}\left(\frac{\text{mr X}}{\text{In X}}\right) & R_{22}\left(\frac{\text{mr X}}{\text{mr X}}\right) & R_{23}\left(\frac{\text{mr X}}{\text{In Y}}\right) & R_{24}\left(\frac{\text{mr X}}{\text{mr Y}}\right) & R_{25}\left(\frac{\text{mr X}}{\%}\right) \\[2ex]
R_{31}\left(\frac{\text{In Y}}{\text{In X}}\right) & R_{32}\left(\frac{\text{In Y}}{\text{mr X}}\right) & R_{33}\left(\frac{\text{In Y}}{\text{In Y}}\right) & R_{34}\left(\frac{\text{In Y}}{\text{mr Y}}\right) & R_{35}\left(\frac{\text{In Y}}{\%}\right) \\[2ex]
R_{41}\left(\frac{\text{mr Y}}{\text{In X}}\right) & R_{42}\left(\frac{\text{mr Y}}{\text{mr X}}\right) & R_{43}\left(\frac{\text{mr Y}}{\text{In Y}}\right) & R_{44}\left(\frac{\text{mr Y}}{\text{mr Y}}\right) & R_{45}\left(\frac{\text{mr Y}}{\%}\right) \\[2ex]
R_{51}\left(\frac{\%}{\text{In X}}\right) & R_{52}\left(\frac{\%}{\text{mr X}}\right) & R_{53}\left(\frac{\%}{\text{In Y}}\right) & R_{54}\left(\frac{\%}{\text{mr Y}}\right) & R_{55}\left(\frac{\%}{\%}\right)
\end{array}
$$

The beam is considered an array of particles that is described with a 5th order symmetrical sigma ellipsoid. The symmetric SIGMA matrix at the beam line input is:

$$
(\sigma_o) = \begin{bmatrix}
\sigma_{11} & \sigma_{21} & \sigma_{31} & \sigma_{41} & \sigma_{51} \\
\sigma_{21} & \sigma_{22} & \sigma_{32} & \sigma_{42} & \sigma_{52} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{43} & \sigma_{53} \\
\sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{54} \\
\sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55}
\end{bmatrix} \qquad (4)
$$

$\sqrt{\sigma_{11}}$ = X_{max} = maximum (half) width of the beam envelop in the X (bend) plane at the given point (inches).

$\sqrt{\sigma_{22}}$ = θ_{max} = maximum (half) angular divergence of the beam envelope in the X-bend plane.

$\sqrt{\sigma_{33}}$ = Y_{max} = maximum (half) height of the beam envelope.

$\sqrt{\sigma_{44}}$ = ϕ_{max} = maximum (half) angular divergence of the beam envelope in the Y (non-bend) plane.

$\sqrt{\sigma_{55}}$ = δ_{max} = half-width (1/2 $\Delta P/P$) of the momentum interval being transmitted by the system.

The input SIGMA matrix, (σ_o) can be obtained from the horizontal and vertical Twiss parameters of the beam if no x-y coupling is assumed. For the horizontal plane:

$$\begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \varepsilon_H \begin{bmatrix} \beta_H & -\alpha_H \\ -\alpha_H & \delta_H \end{bmatrix} \qquad (5)$$

$$\sigma_{11} = \varepsilon_H \beta_H \qquad \varepsilon = \mathrm{DET} \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$\sigma_{21} = -\varepsilon_H \alpha_H$$

$$\sigma_{22} = \varepsilon_H \gamma_H = \varepsilon_H \left(\frac{1+\alpha_H^2}{\beta_H}\right) \qquad \varepsilon_H = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{21}^2}$$

and for the vertical plane:

$$\sigma_{33} = \varepsilon_V \beta_V \qquad (6)$$

$$\sigma_{43} = -\varepsilon_V \alpha_V \qquad \varepsilon_V = \sqrt{\sigma_{33}\sigma_{44} - \sigma_{43}^2}$$

$$\sigma_{44} = \varepsilon_V \gamma_V = \varepsilon_V \left(\frac{1+\alpha_V^2}{\beta_V}\right)$$

where one set of dimensions for α, β, ε are:

α --- dimensionless

β --- kilo inch

ε --- emittance -- inch-mrad

At any point in the beam line, the SIGMA matrix (σ_1) can be found from the input matrix (σ_ϕ) and the total (R) matrix to this point:

$$(\sigma_1) = (R) \times (\sigma_o) \times (R^T) \tag{7}$$

where (R^T) is the transpose of (R). Equation 7 is the general TRANSPORT equation in matrix form.

Equation 4 shows that the maximum (half) width of the beam envelope in the X plane at point 1 is X_{max} and the maximum (half) height of the beam envelope is Y_{max}:

$$(X_{max})^2 = (\sigma_{11})_1 \tag{8a}$$

$$(Y_{max})^2 = (\sigma_{33})_1 \tag{8b}$$

where the notation indicates evaluating these components at point 1 in the beam line.

Expanding (7) and combining with (8) gives:

$$\begin{aligned}(X_{max})^2 = (\sigma_{11})_1 = {} & \sigma_{11}R_{11}^2 + 2\sigma_{21}R_{11}R_{12} + 2\sigma_{31}R_{11}R_{13} + 2\sigma_{41}R_{11}R_{14} \\ & + 2\sigma_{51}R_{11}R_{15} + \sigma_{22}R_{12}^2 + 2\sigma_{32}R_{12}R_{13} + 2\sigma_{42}R_{12}R_{14} + 2\sigma_{52}R_{12}R_{15} \\ & + \sigma_{33}R_{13}^2 + 2\sigma_{43}R_{13}R_{14} + 2\sigma_{53}R_{13}R_{15} \\ & + \sigma_{44}R_{14}^2 + 2\sigma_{54}R_{14}R_{15} \\ & + \sigma_{55}R_{15}^2 \qquad = 15 \text{ terms.}\end{aligned} \tag{9a}$$

$$(Y_{max})^2 = (\sigma_{33})_1 = \sigma_{11}R_{31}^2 + 2\sigma_{21}R_{32}R_{31} + 2\sigma_{31}R_{33}R_{31} + 2\sigma_{41}R_{34}R_{31}$$

$$+ 2\sigma_{51}R_{35}R_{31} + \sigma_{22}R_{32}^2 + 2\sigma_{32}R_{33}R_{32} + 2\sigma_{42}R_{34}R_{32} + 2\sigma_{52}R_{35}R_{32}$$

$$+ \sigma_{33}R_{33}^2 + 2\sigma_{43}R_{34}R_{33} + 2\sigma_{53}R_{35}R_{33}$$

$$+ \sigma_{44}R_{34}^2 + 2\sigma_{54}R_{35}R_{34} \tag{9b}$$

$$+ \sigma_{55}R_{35}^2 \qquad = 15 \text{ terms.}$$

Equation 9 is the general TRANSPORT Equation.

Some simplifications can be made:

1. Assume that the input beam has no coupling between the horizontal and vertical components...i.e., $\sigma_{31} = \sigma_{41} = \sigma_{51} = \sigma_{32} = \sigma_{42} = \sigma_{52} = \sigma_{53} = \sigma_{54} = \phi$ or:

$$(\sigma_\phi) = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \phi & \phi & \phi \\ \sigma_{21} & \sigma_{22} & \phi & \phi & \phi \\ \phi & \phi & \sigma_{33} & \sigma_{43} & \phi \\ \phi & \phi & \sigma_{43} & \sigma_{44} & \phi \\ \phi & \phi & \phi & \phi & \sigma_{55} \end{bmatrix}$$

then Equation 9 reduces to:

$$(X_{max})^2 = \sigma_{11}R_{11}^2 + 2\sigma_{21}R_{11}R_{12} + \sigma_{22}R_{12}^2 + \sigma_{33}R_{13}^2 + 2\sigma_{43}R_{13}R_{14}$$

$$+ \sigma_{44}R_{14}^2 + \sigma_{55}R_{15}^2 \tag{10a}$$

$$(Y_{max})^2 = \sigma_{11}R_{31}^2 + 2\sigma_{21}R_{32}R_{31} + \sigma_{22}R_{32}^2 + \sigma_{33}R_{33}^2 + 2\sigma_{43}R_{34}R_{33}$$

$$+ \sigma_{44}R_{34}^2 + \sigma_{55}R_{35}^2 \tag{10b}$$

2. Assume further that the above beam is passing through no skew magnetic elements that couple the horizontal and vertical...i.e., $R_{13} = R_{14} = R_{23} = R_{24} = R_{31} = R_{32} = R_{41} = R_{42} = \phi$. Assume also that the fractional momentum deviation is constant...i.e., $R_{51} = R_{52} = R_{53} = R_{54} = \phi$ or

$$(R) = \begin{bmatrix} R_{11} & R_{12} & \phi & \phi & R_{15} \\ R_{21} & R_{22} & \phi & \phi & R_{25} \\ \phi & \phi & R_{33} & R_{34} & R_{35} \\ \phi & \phi & R_{43} & R_{44} & R_{45} \\ \phi & \phi & \phi & \phi & R_{55} \end{bmatrix}$$

$$(X_{max})^2 = \sigma_{11}R_{11}^2 + 2\sigma_{21}R_{11}R_{12} + \sigma_{22}R_{12}^2 + \sigma_{55}R_{15}^2 \qquad (11a)$$

$$(Y_{max})^2 = \sigma_{33}R_{33}^2 + 2\sigma_{43}R_{33}R_{34} + \sigma_{44}R_{34}^2 + \sigma_{55}R_{35}^2 \qquad (11b)$$

Equation 11 is the simplified TRANSPORT Equation which will be used to solve for the emittance parameters. Note that if R_{35} is zero then Y_{max} is independent of fractional momentum deviation.

Rewriting 11a

$$(X_{max})^2 = \sigma_{11}(R_{11}^2) + \sigma_{21}(2R_{11}R_{12}) + \sigma_{22}(R_{12}^2) + \sigma_{55}(R_{15}^2) \qquad (11a)$$

This linear equation with constant coefficients is very similar to the following power series equation:

$$Y = a_{11}X + a_{21}X^2 + a_{22}X^3 + a_{55}X^4$$

where $Y = (X_{max})^2$

X corresponds to R_{11}^2

X^2 corresponds to $2R_{11}R_{12}$

X^3 corresponds to R_{12}^2

X^4 corresponds to R_{15}^2

One would expect that the same least square analysis program for the power series expansion could be used for the TRANSPORT equation if the X's were replaced with the above expressions. The analysis would find the values of σ_{11}, σ_{21}, $\sigma_{22}\cdots$.

4. The TRANSPORT Matrix (R)

To solve Equation 11 it is necessary to find the transport matrix from point ϕ, the input, to point 1, the location where the beam width is measured. For each beam width measured, the (R) matrix needs to be known. The (R) matrix, for each measurement, is determined from point ϕ or the input. The least square fitting will find the beam parameters at point ϕ.

The beam widths can be obtained at several points along the beam line or at one point for different quad settings or a combination of both.

The QTUNE program has an option that will print out the total matrix (R) from the input of the beam line to the end of all the beam line elements and flags. Another option will print out the total matrix from the input to only one point in the beam line. This latter is used if quads are varied and the beam widths are measured at one point.

The TRANSPORT Program or TURTLE Program or other programs can be used to find the total matrix (R). An assumed input beam emittance is needed. The TRANSPORT Equations 9, 10, or 11 can be used to calculate the beam width from the input beam and the (R) matrix printed and compared to the printed beam width. This checks that the correct (R) matrix will be used in the least-square analysis.

The total matrix (R) may also be found by calculating the individual matrices for each of the beam line elements and multiplying. The BASIC program is a convenient program to multiply and print matrices.

Figure 1 shows part of a typical matrix printout from QTUNE. The TRANSPORT printout is similar.

5. The Beam Size

It is necessary to measure the beam size and the standard deviations of the beam size to use the least square fitting procedure. The beam size can be determined from foil irradiations, multiwire or single wire monitors, segmented

```
                    27-Sep-83 11:03

RMS PARAMETERS:
ALPHA, BETA, EPSILON (H,V) AT F13: -5.6700  2.2620  0.0077  0.9870  0.1457  0.0077

CD1  (-1898) CQ1  (-1729) CQ2  (  800) CQ3  ( -454) CQ4  ( 1901) CD283( 2779) CD283( 2779) CQ588( -866) CQ687(  753) CQ687( [illegible])
CQ588( -866) CD4  (-2000) CQ9  (-1000) CQ10 ( 2658) CQ11 (-3502) CQ12 ( 2390) CD3T (   -1)

ELEMENT      Z(INCHES)              ELEMENT OR TOTAL MATRIX FROM START                        TRANSPORT BEAM MATRIX

                     1.00000   0.00000   0.00000   0.00000  -1.16500
                     0.00000   1.00000   0.00000   0.00000  -2.95000
 START ===           0.00000   0.00000   1.00000   0.00000   0.00000
                     0.00000   0.00000   0.00000   1.00000   0.00000
                     0.00000   0.00000   0.00000   0.00000   1.00000
              0.000 INCHES
                          1.00000   0.00000   0.00000   0.00000  -1.16500     0.424 IN
                          0.00000   1.00000   0.00000   0.00000  -2.95000     1.079 MR   0.986
     TOTAL MATRIX ===     0.00000   0.00000   1.00000   0.00000   0.00000     0.102 IN   0.000  0.000
                          0.00000   0.00000   0.00000   1.00000   0.00000     0.980 MR   0.000  0.000 -0.702
                          0.00000   0.00000   0.00000   0.00000   1.00000     0.120 PC  -0.330 -0.328  0.000  0.000
              0.000 INCHES

                     1.00000   0.01200   0.00000   0.00000   0.00000
                     0.00000   1.00000   0.00000   0.00000   0.00000
       ===           0.00000   0.00000   1.00000   0.01200   0.00000
                     0.00000   0.00000   0.00000   1.00000   0.00000
                     0.00000   0.00000   0.00000   0.00000   1.00000
            465.751 INCHES
                         -0.39107   0.28018   0.00000   0.00000  -0.41196     0.143 IN
                         -5.11960   1.11078   0.00000   0.00000   2.45613     1.000 MR  -0.849
     TOTAL MATRIX ===     0.00000   0.00000   1.36932   0.45352   0.00000     0.361 IN   0.000  0.000
                          0.00000   0.00000  -4.37843  -0.71986   0.00000     0.505 MR   0.000  0.000 -0.921
                          0.00000   0.00000   0.00000   0.00000   1.00000     0.120 PC  -0.346  0.295  0.000  0.000
            465.751 INCHES

                     1.00000   0.00000   0.00000   0.00000   0.00000
                     0.00000   1.00000   0.00000   0.00000   0.00000
 CF039 ===           0.00000   0.00000   1.00000   0.00000   0.00000
                     0.00000   0.00000   0.00000   1.00000   0.00000
                     0.00000   0.00000   0.00000   0.00000   1.00000
            465.752 INCHES
                         -0.39107   0.28018   0.00000   0.00000  -0.41196     0.143 IN
                         -5.11960   1.11078   0.00000   0.00000   2.45613     1.000 MR  -0.849
     TOTAL MATRIX ===     0.00000   0.00000   1.36932   0.45352   0.00000     0.361 IN   0.000  0.000
                          0.00000   0.00000  -4.37843  -0.71986   0.00000     0.505 MR   0.000  0.000 -0.921
                          0.00000   0.00000   0.00000   0.00000   1.00000     0.120 PC  -0.346  0.295  0.000  0.000
            465.752 INCHES
```

Figure 1. R Matrix Printout

wire ion chambers (swics) or flags. Sizes from different types of devices can be combined in one analysis by weighting the measurements according to their standard deviations.

Foil measurements are obtained by irradiating an aluminum foil in a beam line and then cutting the foil into narrow vertical strips for a horizontal profile. The radiation on these strips, typically greater than .040 inch wide, is measured and normalized to the strip weight. Plotting the strip width versus radiation counts gives a profile display.

Multiwire devices can be inserted into a transport beam. These return a current on each of approximately 30 wires which can be digitized and read into a computer. Plotting these digitized signals versus the wire location gives a profile display.

Swics are similar to the multiwire devices and produce a profile display. Single wire devices can be stepped through the beam to produce a profile. These single wire devices do not obtain a total profile for one pulse but for several pulses.

Flags can be inserted into a beam and then observed with a TV camera. The flags do not produce a profile but a single spot size of the beam.

All devices, except the flags, produce a profile display similar to Figure 2. From this display it is necessary to obtain the beam width. This will be defined in this report as the half width that includes 99% of the beam. This 99% half width can be obtained in different ways:

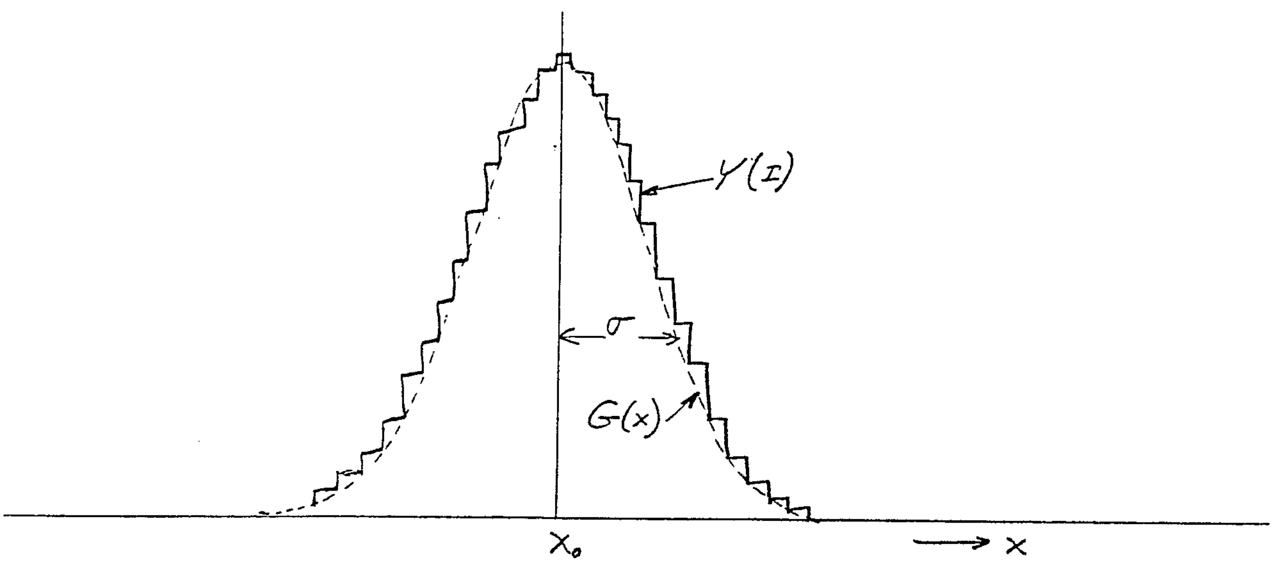


Figure 2. Typical Profile Display

The beam can be assumed to have a Gaussian distribution. At the AGS this is probably true for the fast extracted beam (FEB), the single bunch extracted beam (SBE), and the vertical plane of the slow extracted beam (SEB). It is not true for the horizontal plane of the slow extracted beam.

The Gaussian description is:

$$G(x) = \frac{A}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-x_o}{\sigma}\right)^2} \tag{12}$$

where:

x_o --- mean or center of the beam

σ --- standard deviation or sigma width of the beam

A --- area under the function

x --- independent position variable

The 99% half width is 2.57 σ.

The most accurate means to find the three parameters of the Gaussian is to apply a least-square analysis to the profile data Y(I). The linear least square analysis subroutine LSQFIT of Appendix A can be used if Equation 12 is changed to a linear equation. Taking the natural log of both sides of Equation 12 will produce a modified linear equation with constant coefficients. The Fortran program TEST3 in Appendix A shows how values of Y(I) are used to find the constants A, σ, and x_o and their standard deviations. The 99% half width is 2.57 σ and the standard deviation of the 99% half width is 2.57 times the standard deviation of σ.

Another means of determining the beam width is to use the general equation for the centroid of a beam and the standard deviation of the beam as Weng and Weisberg do:

$$x_o = \frac{\sum_i x_i Y_i}{\sum_i Y_i} \tag{13}$$

$$\text{STD DEV} = \sqrt{\frac{\sum_i (x_i - x_o)^2 Y_i}{\sum_i Y_i}} \tag{14}$$

The 99% half width is 2.57 (STD DEV)

For a true Gaussian function Equation 14 will find a smaller beam width then Equation 12. For a profile with a standard deviation of 5 wires, the calculation of Equation 14 will give a 4.5% smaller width if 25 wires are used and 33% smaller width if 13 wires are used.

It is also necessary to determine the standard deviation of the 99% half width. Since a least squared analysis was not used to find the 99% half width, this standard deviation can be estimated as 2.57 times the half-spacing of one wire or less.

A less accurate means of finding the beam width can be used if oscilloscope photos are available of the profiles. If a Gaussian shaped beam is assumed, then one can easily show from Equation 12 that the half width of the beam at half magnitude is 1.177 σ. Once σ is obtained the 99% half width is then calculated. The standard deviation can be estimated from the shape of the profile and is usually about 50% to 100% of the wire spacing. This procedure was used to find the beam widths for the SBE emittance calculation.

One should note that the profile data of Figure 2 is not directly received from the instrumentation. The instrumentation data must be modified before analyzing. Due to noise or inaccuracies one usually disregards data in the tails of the profile that is below 5 percent of the peak signal for multiwire devices, foil irradiations, and single wire devices. Weisberg has shown that additional modifications must be made for swics. A no beam signal must be subtracted from the swic data and then a background must be removed. For the CW039 swic used for the SBE measurements, this background consists of a Gaussian signal with a σ of 5 wires that is matched to the swic signal in the tails. After the offset and background are removed, the above procedures can be used.

Profile information is not readily available from flag information. Some profile information can be obtained by digitizing the TV scan signals. Normally one can estimate the total width and height of the beam and this can be halved to find the 99% half sizes. The standard deviation for flag measurements is usually large...about 30 or 40% of the 99% half size.

6. The Least Square Analysis

As described in Appendix A the LSQFIT subroutine will do a least square fit to linear equations with constant coefficients.

$$Y = a_1X_1(z) + a_2X_2(z) + a_3X_3(z) + \ldots \tag{15}$$

where z is the independent variable; $X_1(z) \ldots X_i(z)$ are known functions of the independent variable z but independent of the coefficients a_i; $a_1 \ldots a_n$ are constant coefficients to be found and Y is the dependent variable.

A typical equation that fits this criteria is the power series equation:

$$Y = a_1X + a_2X^2 + a_3X^3 \tag{16}$$

The LSQFIT routine can be used directly for this power series equation to find the coefficients a_1, a_2, a_3 given many sets of (x,y) data points. The standard deviations of the Y points can be used as input to the subroutine and the a_i values with their standard deviations will be obtained. The analysis can be performed if no weighting is used for the points...i.e., the accuracy of all the points is known equally or if each point is known to a different accuracy.

If the fitting is good and the chi square is small, the standard deviation of a_i will be small and one will know a_i accurately. If the fitting is poor because the data was measured poorly or because the equation does not fit the data accurately, a large chi square will result with large standard deviations. Using different modes of weighting the input data, the standard deviations of the results can be made proportional to the accuracy of the data or to the accuracy of the fitting. Appendix A describes the different modes possible.

From Equation 11a and considering only the horizontal plane for simplicity, one must do a least square analysis on Equation 17 to find the beam parameters:

$$X_{max} = \left(\sigma_{11}R_{11}^2 + 2\sigma_{21}R_{11}R_{12} + \sigma_{22}R_{12}^2 + \sigma_{55}R_{15}^2\right)^{\frac{1}{2}} \tag{17}$$

This equation is not a linear equation with constant coefficients but can be modified to a linear equation:

$$(X_{max})^2 = \sigma_{11}R_{11}^2 + 2\sigma_{21}R_{11}R_{12} + \sigma_{22}R_{12}^2 + \sigma_{55}R_{15}^2 \qquad (11a)$$

A problem arises for this modified equation because the LSQFIT routine returns the standard deviations of the σ parameters and not the standard deviations of the Twiss parameters that are calculated from these using Equation 5. The standard deviation of X_{max} is known but the standard deviations of $(X_{max})^2$ must be input to LSQFIT.

Bevington shows that if the fluctuations in the observations of σ_{11}, σ_{12}, σ_{22} are correlated and if:

$$A = f(\sigma_{11}, \sigma_{12}, \sigma_{22}) \qquad (18)$$

then:

the standard deviation of A can be found from the correlation or error matrix and the partial derivatives as:

$$\begin{aligned}(\text{SIG A})^2 = {} & (\text{SIG }\sigma_{11})^2\left(\frac{\partial f}{\partial \sigma_{11}}\right)^2 + (\text{SIG }\sigma_{12})^2\left(\frac{\partial f}{\partial \sigma_{12}}\right)^2 \\ & + (\text{SIG }\sigma_{22})^2\left(\frac{\partial f}{\partial \sigma_{22}}\right)^2 \\ & + 2\,(\text{SIG }\sigma_{11}\sigma_{12})^2\left(\frac{\partial f}{\partial \sigma_{11}}\right)\left(\frac{\partial f}{\partial \sigma_{12}}\right) + 2(\text{SIG }\sigma_{11}\sigma_{22})^2\left(\frac{\partial f}{\partial \sigma_{11}}\right)\left(\frac{\partial f}{\partial \sigma_{22}}\right) \\ & + 2\,(\text{SIG }\sigma_{12}\sigma_{22})^2\left(\frac{\partial f}{\partial \sigma_{12}}\right)\left(\frac{\partial f}{\partial \sigma_{22}}\right)\end{aligned} \qquad (19)$$

Commonly $(\text{SIG A})^2$ is known as the variance of A and $(\text{SIG }\sigma_{11})^2$ is the variance of σ_{11}. The standard deviation of A is SIG A and the standard deviation of σ_{11} is SIG σ_{11}. The coupling elements or covariances are terms as SIG $\sigma_{12}\sigma_{22}$. This error matrix is available from the LSQFIT subroutine in the common/LSQ/ statement and is called SIGUV2. The elements of the matrix, for 3 unknowns, are:

$$\text{SIGUV2} = \begin{bmatrix} (\text{SIG } \sigma_{11})^2 & (\text{SIG } \sigma_{11}\ \sigma_{21})^2 & (\text{SIG } \sigma_{11}\sigma_{22})^2 \\ (\text{SIG } \sigma_{11}\sigma_{21})^2 & (\text{SIG } \sigma_{21})^2 & (\text{SIG } \sigma_{21}\sigma_{22})^2 \\ (\text{SIG } \sigma_{11}\sigma_{22})^2 & (\text{SIG } \sigma_{21}\sigma_{22})^2 & (\text{SIG } \sigma_{22})^2 \end{bmatrix} \quad (20)$$

As one can see, the diagonal elements of the matrix are the variances of σ_{11}, σ_{21}, σ_{22} and the off diagonal elements are the covariances. The matrix is a symmetrical matrix. The partial derivatives must be evaluated exactly for each function or can be approximated as:

$$\frac{\partial f}{\partial \sigma_{11}} = \frac{\Delta f}{\Delta \sigma_{11}} \text{ with } \sigma_{12} \text{ and } \sigma_{22} \text{ constant.}$$

The derivatives are found for the Gaussian function and for the Transport equation in Appendix A.

Barton has shown that Equation (19) can also be expressed as a matrix equation:

$$(\text{SIG A})^2 = (\frac{\partial f}{\partial})^T (\text{SIGUV2})(\frac{\partial f}{\partial}) \quad (21)$$

where the matrix $(\partial f/\partial)$ is the vector: $\begin{bmatrix} \frac{\partial f}{\partial \sigma_{11}} \\ \frac{\partial f}{\partial \sigma_{21}} \\ \frac{\partial f}{\partial \sigma_{22}} \end{bmatrix}$

and the transpose of this vector is:

$$(\frac{\partial f}{\partial})^T = \begin{bmatrix} \frac{\partial f}{\partial \sigma_{11}} & \frac{\partial f}{\partial \sigma_{21}} & \frac{\partial f}{\partial \sigma_{22}} \end{bmatrix} \quad (21a)$$

The advantage of the matrix equation is in the computer calculation of the standard deviations. A function subroutine.

Function STDEV (DERIV)

as described in Appendix A, is used to calculate the standard deviation of any function if the partial derivatives are given.

If one knows that the fluctuations in the data are uncorrelated, then the covariances are zero. For example, if the standard deviation of X_{max} is SIGXMAX, and XMAX is input data, then the standard deviation of $(XMAX)^2$ is:

$$(XMAX)^2 = f(XMAX) \tag{22}$$

$$\begin{aligned} \left(SIG(XMAX)^2\right)^2 &= (SIGXMAX)^2 \left(\frac{\partial f}{\partial XMAX}\right)^2 \\ &= \left(2(XMAX)\ SIGXMAX\right)^2 \end{aligned} \tag{23}$$

or

$$SIG(XMAX)^2 = 2(XMAX)\ SIGXMAX \tag{24}$$

The standard deviation of $(XMAX)^2$ is used as input to the LSQFIT subroutine.

After measureing the beam widths for several different quad settings or locations in the beamline, the TEST4 or TEST5 programs in Appendix A can be used to find the Twiss parameters of the beam. The matrix elements must be determined with a TRANSPORT or QTUNE type program.

7. The Necessary Measurements

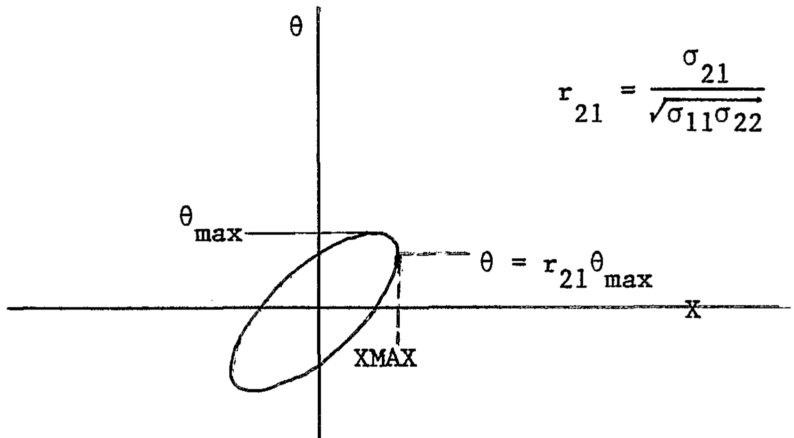
As described in Appendix A, the TEST4, or TEST5 programs can be used to find the beam parameters if several beam width measurements are made. Also input to these programs are certain matrix elements of the Transport matrix from the starting point to the measurement point. This is not sufficient information to guarantee that an emittance will be found. Experimentally it has been shown that if beam sizes can be measured on both sides of a waist, then an emittance can usually be found. It sometimes happens in practice that due to power supply limitations or device location limitations that beam sizes cannot be measured on both sides of a waist at one location. This section will show what are the necessary measurements.

Equation 4 gives the description of the 5th order symmetrical beam ellipsoid. The projection of this ellipsoid on the x/θ plane is described with the σ_{11}, σ_{21}, σ_{22} small square matrix. This is normally called the horizontal emittance and the Twiss parameters can be calculated from these using Equation 5. This projection is an ellipse, as given in the Transport Manual or SLAC 91 as:

$$\sigma_{22}X^2 - 2\sigma_{21}X\theta + \sigma_{11}\,\theta^2 = \varepsilon^2 \tag{25}$$

or, in terms of the Twiss parameters:

$$\gamma X^2 + 2\alpha X\theta + \beta\theta^2 = \varepsilon \tag{26}$$



The necessary conditions are that sufficient points be found on this ellipse so that the ellipse can be constructed. The points should be located on widely spaced portions of the ellipse. The problem, therefore, reduces to finding where on this input ellipse are the points located that are measured at the measurement point. Equation 2 is used to solve this problem.

$$(X)_1 = (R) \times (X)_\phi \tag{2}$$

$(X)_1$ is the vector that gives the value of X, θ, Y, ϕ, and δ at the measurement point. $(X)_o$ is the vector at the input and the values of the X_ϕ and θ_ϕ components of $(X)_\phi$ give a point on the input ellipse.

$$(X)_\phi = (R)^{-1} \times (X)_1 \tag{27}$$

Experimentally one knows the X value at the measurement point, but does not know the values of θ, Y, ϕ, and δ. To solve this problem one must assume a given beam input emittance and, then, propagate this beam through the R matrix using TRANSPORT or QTUNE. Figure 1 gives a typical QTUNE or TRANSPORT printout with the total R matrix and the beam matrix. As is standard, the maximum sizes or $\sqrt{\sigma_{ii}}$ and the correlation components are printed. The correlation components are defined as, for example:

$$r_{21} = \frac{\sigma_{21}}{\sqrt{\sigma_{11}\sigma_{22}}} \tag{28}$$

One can easily show from the ellipse equation or the SLAC 91 report, that the value of θ when X = XMAX is

$$\theta = r_{21}\,\theta_{max}. \tag{29}$$

One now knows the X and θ components of the $(X)_1$ vector at the measurement point. The beam ellipsoid has many projections on many planes... each described by a small square part of the 5 x 5 order sigma matrix. For example;

a) The X/θ plane ellipse is defined with σ_{11}, σ_{22}, σ_{21} or σ_{11}, σ_{22}, r_{21}
b) The X/Y plane ellipse is defined with σ_{11}, σ_{33}, σ_{31} or σ_{11}, σ_{33}, r_{31}
c) The X/ϕ plane ellipse is defined with σ_{11}, σ_{44}, σ_{41} or σ_{11}, σ_{44}, r_{41}
d) The X/δ plane ellipse is defined with σ_{11}, σ_{55}, σ_{51} or σ_{11}, σ_{55}, r_{51}

Therefore, from Equation 29, one can write that the 5 components of the $(X)_1$, vector are:

$$\begin{aligned} X_1 &= X_{max} &&= \sqrt{\sigma_{11}} \\ \theta_1 &= r_{21}\theta_{max} &&= r_{21}\sqrt{\sigma_{22}} \\ Y_1 &= r_{31}Y_{max} &&= r_{31}\sqrt{\sigma_{33}} \\ \phi_1 &= r_{41}\phi_{max} &&= r_{41}\sqrt{\sigma_{44}} \\ \delta_1 &= r_{51}\delta_{max} &&= r_{51}\sqrt{\sigma_{55}} \end{aligned} \tag{30}$$

If no skew elements exist in the transport matrix or the beam matrix, r_{31} and r_{41} are zero. For the SBE, however, r_{51} is not zero.

Once the $(X)_1$, vector is found, then the $(X)_0$ vector can be found from Equation 27 and a point is located on the X/θ ellipse. Actually, two points are known since both XMAX and -XMAX are measured. The other point is diagonally opposite the calculated point. This procedure can be used to determine if enough widely spaced points are found to define the ellipse. Mathematically, if one finds that all the points are crowded together in one portion of the ellipse, then the TEST4 program may not be able to find an emmitance or the emittance found may have a large standard deviation. Thus if one finds that power supply or other limitations prevent measuring beam sizes on both sides of a waist at one location, the solution may be to make one or more measurments in another location.

This same procedure can be used if one tries to find the fractional momentum deviation, δ, of the input beam, from beam width measurements. For this case one wants to find the ellipse in the X/δ plane and one needs to make sufficient measurements so that points are found widely spaced on the X/δ or σ_{11}, σ_{55}, σ_{51} ellipse.

One can show using this procedure that if one measures beam sizes at one location on both sides of a waist that one will measure points spread out around the ellipse.

8. Results for the SBE

The beam widths were measured from the swic at CW039 for the SBE beam with an intensity of 9.4TP in the AGS on October 19, 1983. This swic is located 39 feet from F13 after the CQ1, CQ2, CQ3, and CQ4 quads. The horizontally focussing quad CQ1 was varied over a wide range and the hori-zontal profiles were photographed from an oscilloscope. The beam widths were determined from the photographs by removing a background and then determining the 99% half width from the half maximum points. The assumed standard deviations were about 10% of the measured widths. Using a program similiar to TEST5 in Appendix A, Figure 3 shows the results of the horizontal and vertical fitting. For the horizontal plane $(XMAX)^2$ is plotted against the reciprocal of the equivalent horizontal focal length of the group of four quadrupoles. This is the $-R_{21}$ element of the total R matrix at the swic location and is approximately the reciprocal of the focal length of CQ1 since CQ1 was the strongest horizontal focussing guad. It is also approximately proportional to the current in CQ1. For the vertical plane $(Y_{max})^2$ is plotted agaisnt the equivalent vertical focal length element, $-R_{43}$. Also shown on the graphs are the fitted points with one standard deviation error bars. This graph is not necessary for the computer analysis. The vertical focussing quad CQ2 was varied for the vertical measurements. The R matrices were calculated from calibrated computer readbacks of the currents from the power supplies. The horizontal widths passed through a waist but the vertical widths could only approach a waist.

The results of the least square analysis are shown in Figures 4a-5b. Also included in the analysis is the beam size measured at the CF011 flag. The results for mode +1 and the unweighted mode +2 are given at F13.

The results for mode 1 or instrumental weighting using the measured standard deviations for the horizontal plane are:

Alpha = -3.69 , STD. DEV. = 0.737
Beta = 0.963 , STD. DEV. = 0.188 kilo inch
Epsilon = 0.0762, STD. DEV. = 0.0112 in-mrad

The results for mode 2 or no weighting for the horizontal plane are:

Alpha = -1.07 , STD. DEV. = 0.501
Beta = 0.314 , STD. DEV. = 0.113 kilo inch
Epsilon = 0.0709, STD. DEV. = 0.0193 in-mrad

The assumed values for the horizontal plane were:

Alpha = -5.67
Beta = 2.26 kilo inch
Epsilon = 0.0755 in-mrad

The mode 1 results for the vertical plane are:

Alpha = 2.609 , STD. DEV. = 1.472
Beta = 0.444 , STD. DEV. = 0.199 kilo inch
Epsilon = 0.0495, STD. DEV. = 0.024 in-mrad

The mode 2 results for the vertical plane are:

Alpha = 2.516 , STD. DEV. = 0.595

Beta = 0.437 , STD. DEV. = 0.0877 kilo inch

Epsilon = 0.0532, STD. DEV. = 0.0102 in-mrad

The assumed values for the vertical plane were:

Alpha = 0.987

Beta = 0.1457 kilo inch

Epsilon = 0.0755 in-mrad.

From the results one can see that the results for the horizontal plane are good. The mode 1 results should be used in both planes since they include the measured inaccuracies. The mode 2 results are added for comparison only. The horizontal emittance is close to the assumed emittance and the standard deviation is only 15% of the magnitude. The orientation of the horizontal emittance ellipse is different from the assumed theoretical values. Figure 3 shows that the beam size did pass through a horizontal waist.

The data for the vertical plane is less accurate because the beam sizes did not pass through a waist. Because of power supply limitations, it was not possible to measure beam sizes on both sides of a waist. Also included in the analysis is a point from the flag at CF011 but with a large standard deviation. The standard deviation for the vertical emittance size is about 50% of the magnitude but the measured size is almost within one standard deviation of the theoretical size. The fact that the measured data is within 1 standard deviation for about one-third of the points on Figure 3 indicates that the standard deviations are correct. Ninety-nine percent of the points should be within 2.57 standard deviations.

One is usually concerned with the beam sizes down a beam line if the Twiss parameters are known. The TRANSPORT or QTUNE program or Equations 5-11 can be used. If one knows the standard deviations of the Twiss parameters, one can find the standard deviation of the beam size down the transport line. Since Equations 9-11 use the sigma elements to find the beam size, it is easier to calculate the standard deviation of the beam sizes from the standard deviations of the sigma elements. These are also found in TEST4 and TEST5. The TEST4 program shows how to calculate these standard deviations using the Function STDEV.

9. Conclusions

This report gives the first measurements of the SBE emitance. The results are for 2.57 sigma or a 99% beam. They are, for the horizontal plane:

Alpha	= -3.69	, STD. DEV. =	0.737	
Beta	= 0.963	, STD. DEV. =	0.188	kilo inch
Epsilon	= 0.0762	, STD. DEV. =	0.0112	in-mrad

The results for the vertical plane are:

Alpha	= 2.609	, STD. DEV. =	1.472	
Beta	= 0.444	, STD. DEV. =	0.199	kilo inch
Epsilon	= 0.0495	, STD. DEV. =	0.024	in-mrad

The measurements were made with 9.4 TP in the AGS ring.

Besides providing the above data, this report shows in detail how one can measure the beam parameters using linear least square analysis. As Witkover and others have shown, it is a necessity when measuring the emittance that beam size measurements be made at suitable locations in a beam line. Ideally, to prevent small errors in size measurements from causing large standard deviations in the Twiss parameters, the beam sizes should be measured on both sides of a waist. If one or more quads are varied, the beam size at the measurement point should go thru a waist in both planes or sufficient other points should be taken to construct an ellipse in the proper plane at the input point. This conclusion is emphasized by comparing the results for the SBE in the horizontal and vertical planes. If one knows the Twiss parameters and their standard deviations, one has a method to find the beam size and the standard deviation of the size anywhere in the beam line.

Another major conclusion of this report is a knowledge of how to use least square fitting for any problem that can be described with a linear equation with constant coefficients. This report is particularly useful for modified equations. The standard deviations are found for both standard equations and modified equations.

Sufficient test programs are given so that one can immediately find the beam horizontal and vertical Twiss parameters. Fractional momentum deviation can also be included in the analysis.

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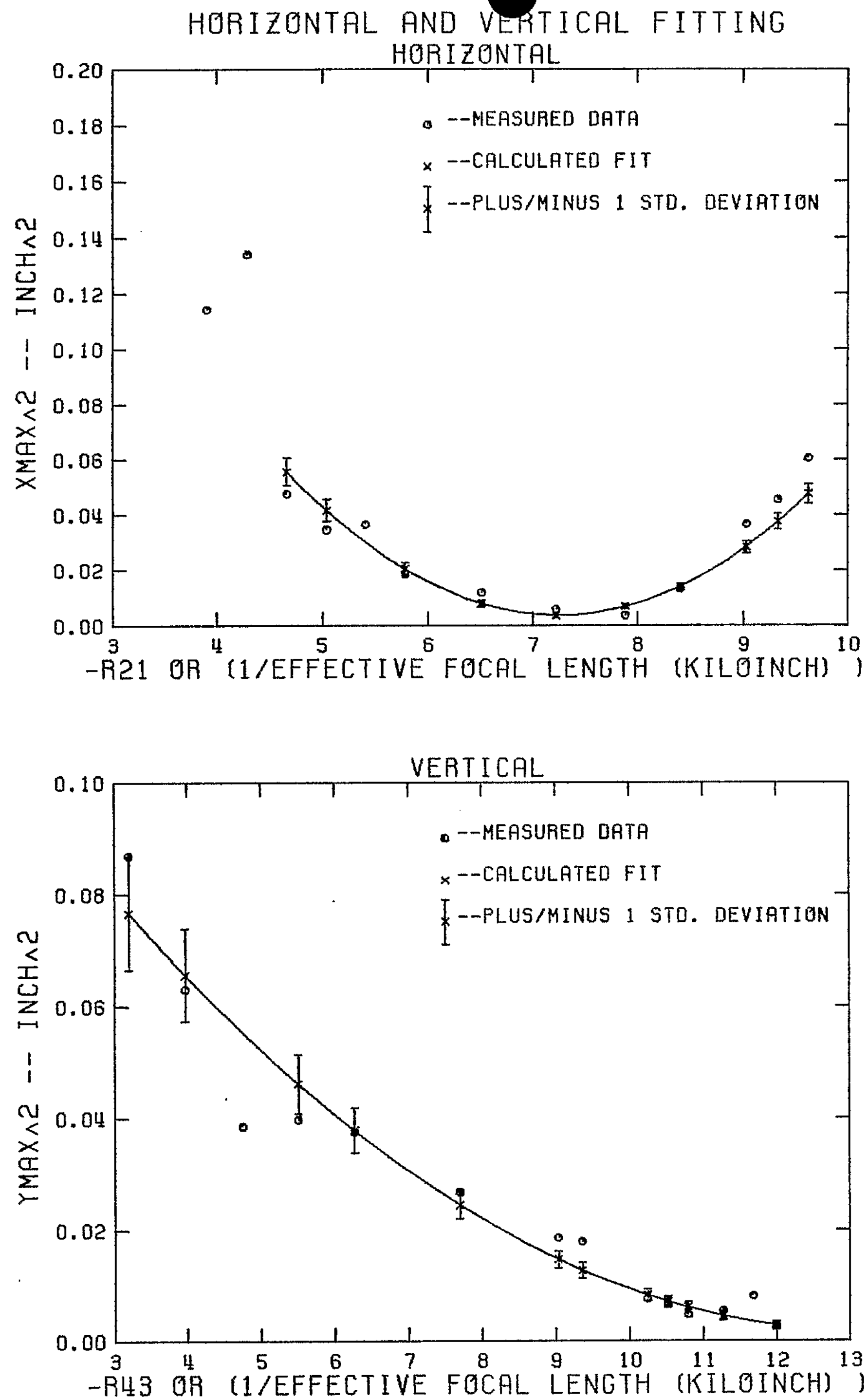

Figure 3. Horizontal and Vertical Fitting

```
HORIZONTAL DATA

INPUT DATA

J=  1  XMAX= 0.2500 SIGXMAX= 0.1000 R11=  1.00000 R12=  0.13400 R15= -1.56030 DMOM= 0.12 MODE=  1
J=  2  XMAX= 0.1360 SIGXMAX= 0.0162 R11=  0.46317 R12=  0.23968 R15= -0.20453 DMOM= 0.12 MODE=  1
J=  3  XMAX= 0.1090 SIGXMAX= 0.0100 R11=  0.58472 R12=  0.21474 R15=  0.01127 DMOM= 0.12 MODE=  1
J=  4  XMAX= 0.0763 SIGXMAX= 0.0100 R11=  0.70177 R12=  0.19069 R15=  0.21921 DMOM= 0.12 MODE=  1
J=  5  XMAX= 0.0601 SIGXMAX= 0.0100 R11=  0.81073 R12=  0.16832 R15=  0.41269 DMOM= 0.12 MODE=  1
J=  6  XMAX= 0.1150 SIGXMAX= 0.0100 R11=  0.89671 R12=  0.15067 R15=  0.56539 DMOM= 0.12 MODE=  1
J=  7  XMAX= 0.1910 SIGXMAX= 0.0128 R11=  1.00169 R12=  0.12915 R15=  0.75173 DMOM= 0.12 MODE=  1
J=  8  XMAX= 0.2460 SIGXMAX= 0.0128 R11=  1.09991 R12=  0.10897 R15=  0.92806 DMOM= 0.12 MODE=  1
J=  9  XMAX= 0.2130 SIGXMAX= 0.0206 R11=  1.05140 R12=  0.11894 R15=  0.84191 DMOM= 0.12 MODE=  1
J= 10  XMAX= 0.1860 SIGXMAX= 0.0218 R11=  0.33992 R12=  0.26497 R15= -0.42139 DMOM= 0.12 MODE=  1
J= 11  XMAX= 0.2180 SIGXMAX= 0.0218 R11=  0.27794 R12=  0.27768 R15= -0.53140 DMOM= 0.12 MODE=  1

  INSTRUMENTAL WEIGHTING -- WEIGHT = 1/SIGMAY^2

J=  1  A=  .073451  SIGMAA=  .007398  FVALU=  0.0000E+00  CHISQR=   0.373337E+01
J=  2  A=  .281488  SIGMAA=  .029048  FVALU=  0.8924E-04  CHISQR=   0.373337E+01
J=  3  A= 1.157928  SIGMAA=  .112203  FVALU=  0.2853E+02  CHISQR=   0.373337E+01

    COVARIANCE MATRIX

SIG(1,1)= 5.472829E-05
SIG(2,1)= 2.055007E-04  SIG(2,2)= 8.437607E-04
SIG(3,1)= 6.754356E-04  SIG(3,2)= 3.056460E-03  SIG(3,3)= 1.258960E-02

 ALPHA= -3.69119   STD. DEVIATION =  0.73704
 BETA=  0.96318   STD. DEVIATION =  0.18784
 EPSILON=  .07625948  STD. DEVIATION =  .01119430
 BEAM FIT CHISQR =   0.556669E-02

     (    ) = SQUARE OF WIDTH OR WIDTH FIT OR STD. DEV. OF WIDTH FIT SQUARED.
J = 1 WIDTH=  0.25000 ( .06250) WIDTH FIT=  0.45248 ( .20474) STD. DEVIATION= 0.0186 ( 0.0168) ERROR%= 80.99
J = 2 WIDTH=  0.13600 ( .01850) WIDTH FIT=  0.14276 ( .02038) STD. DEVIATION= 0.0076 ( 0.0022) ERROR%=  4.97
J = 3 WIDTH=  0.10900 ( .01188) WIDTH FIT=  0.08844 ( .00782) STD. DEVIATION= 0.0072 ( 0.0013) ERROR%=-18.86
J = 4 WIDTH=  0.07630 ( .00582) WIDTH FIT=  0.06027 ( .00363) STD. DEVIATION= 0.0078 ( 0.0009) ERROR%=-21.01
J = 5 WIDTH=  0.06010 ( .00361) WIDTH FIT=  0.08193 ( .00671) STD. DEVIATION= 0.0053 ( 0.0009) ERROR%= 36.32
J = 6 WIDTH=  0.11500 ( .01322) WIDTH FIT=  0.11785 ( .01389) STD. DEVIATION= 0.0049 ( 0.0012) ERROR%=  2.48
J = 7 WIDTH=  0.19100 ( .03648) WIDTH FIT=  0.16828 ( .02832) STD. DEVIATION= 0.0064 ( 0.0021) ERROR%=-11.89
J = 8 WIDTH=  0.24600 ( .06052) WIDTH FIT=  0.21803 ( .04754) STD. DEVIATION= 0.0083 ( 0.0036) ERROR%=-11.37
J = 9 WIDTH=  0.21300 ( .04537) WIDTH FIT=  0.19334 ( .03738) STD. DEVIATION= 0.0073 ( 0.0028) ERROR%= -9.23
J =10 WIDTH=  0.18600 ( .03460) WIDTH FIT=  0.20405 ( .04163) STD. DEVIATION= 0.0096 ( 0.0039) ERROR%=  9.70
J =11 WIDTH=  0.21800 ( .04752) WIDTH FIT=  0.23574 ( .05557) STD. DEVIATION= 0.0107 ( 0.0051) ERROR%=  8.14
```

Figure 4a. Horizontal Data - Mode 1

```
HORIZONTAL DATA
  INPUT DATA

J=  1  XMAX= 0.2500 SIGXMAX= 0.0100 R11=  1.00000 R12=  0.13400 R15= -1.56030 DMOM= 0.12 MODE=  2
J=  2  XMAX= 0.1360 SIGXMAX= 0.0100 R11= +0.46317 R12=  0.23968 R15= -0.20453 DMOM= 0.12 MODE=  2
J=  3  XMAX= 0.1090 SIGXMAX= 0.0100 R11= +0.58472 R12=  0.21474 R15=  0.01127 DMOM= 0.12 MODE=  2
J=  4  XMAX= 0.0763 SIGXMAX= 0.0100 R11= +0.70177 R12=  0.19069 R15=  0.21921 DMOM= 0.12 MODE=  2
J=  5  XMAX= 0.0601 SIGXMAX= 0.0100 R11= +0.81073 R12=  0.16832 R15=  0.41269 DMOM= 0.12 MODE=  2
J=  6  XMAX= 0.1150 SIGXMAX= 0.0100 R11= +0.89671 R12=  0.15067 R15=  0.56539 DMOM= 0.12 MODE=  2
J=  7  XMAX= 0.1910 SIGXMAX= 0.0100 R11= +1.00169 R12=  0.12915 R15=  0.75173 DMOM= 0.12 MODE=  2
J=  8  XMAX= 0.2460 SIGXMAX= 0.0100 R11= +1.09991 R12=  0.10897 R15=  0.92806 DMOM= 0.12 MODE=  2
J=  9  XMAX= 0.2130 SIGXMAX= 0.0100 R11= +1.05140 R12=  0.11894 R15=  0.84191 DMOM= 0.12 MODE=  2
J= 10  XMAX= 0.1860 SIGXMAX= 0.0100 R11= +0.33992 R12=  0.26497 R15= -0.42139 DMOM= 0.12 MODE=  2
J= 11  XMAX= 0.2180 SIGXMAX= 0.0100 R11= +0.27794 R12=  0.27768 R15= -0.53140 DMOM= 0.12 MODE=  2

NO WEIGHTING FOR MODIFIED EQUATIONS

J=  1  A=  .022252   SIGMAA=  .010838  FVALU=   0.0000E+00   CHISQR=   0.215956E+02
J=  2  A=  .075676   SIGMAA=  .037828  FVALU=   0.1193E-02   CHISQR=   0.215956E+02
J=  3  A=  .483179   SIGMAA=  .178978  FVALU=   0.7288E+01   CHISQR=   0.215956E+02

COVARIANCE MATRIX

SIG(1,1)= 1.174567E-04
SIG(2,1)= 3.027271E-04  SIG(2,2)= 1.430957E-03
SIG(3,1)= 4.182739E-04  SIG(3,2)= 4.939827E-03  SIG(3,3)= 3.203309E-02

 ALPHA= -1.06758   STD. DEVIATION =  0.50076
 BETA=  0.31391   STD. DEVIATION =  0.11340
 EPSILON=  .07088539   STD. DEVIATION =  .01932522
 BEAM FIT CHISQR =   0.247447E-02

      (    ) = SQUARE OF WIDTH OR WIDTH FIT OR STD. DEV. OF WIDTH FIT SQUARED.
J = 1 WIDTH=  0.25000 ( .06250) WIDTH FIT=  0.29371 ( .08627) STD. DEVIATION= 0.0363 ( 0.0213) ERROR%= 17.48
J = 2 WIDTH=  0.13600 ( .01850) WIDTH FIT=  0.12779 ( .01633) STD. DEVIATION= 0.0238 ( 0.0061) ERROR%= -6.03
J = 3 WIDTH=  0.10900 ( .01188) WIDTH FIT=  0.10434 ( .01089) STD. DEVIATION= 0.0208 ( 0.0043) ERROR%= -4.28
J = 4 WIDTH=  0.07630 ( .00582) WIDTH FIT=  0.09469 ( .00897) STD. DEVIATION= 0.0186 ( 0.0035) ERROR%= 24.10
J = 5 WIDTH=  0.06010 ( .00361) WIDTH FIT=  0.10057 ( .01011) STD. DEVIATION= 0.0183 ( 0.0037) ERROR%= 67.33
J = 6 WIDTH=  0.11500 ( .01322) WIDTH FIT=  0.11409 ( .01302) STD. DEVIATION= 0.0196 ( 0.0045) ERROR%= -0.79
J = 7 WIDTH=  0.19100 ( .03648) WIDTH FIT=  0.13764 ( .01894) STD. DEVIATION= 0.0221 ( 0.0061) ERROR%=-27.94
J = 8 WIDTH=  0.24600 ( .06052) WIDTH FIT=  0.16407 ( .02692) STD. DEVIATION= 0.0249 ( 0.0082) ERROR%=-33.30
J = 9 WIDTH=  0.21300 ( .04537) WIDTH FIT=  0.15071 ( .02271) STD. DEVIATION= 0.0235 ( 0.0071) ERROR%=-29.24
J =10 WIDTH=  0.18600 ( .03460) WIDTH FIT=  0.15944 ( .02542) STD. DEVIATION= 0.0272 ( 0.0087) ERROR%=-14.28
J =11 WIDTH=  0.21800 ( .04752) WIDTH FIT=  0.17709 ( .03136) STD. DEVIATION= 0.0290 ( 0.0103) ERROR%=-18.77
```

Figure 4b. Horizontal Data - Mode 2

```
VERTICAL DATA

INPUT DATA

J=  1  YMAX= 0.2948 SIGYMAX= 0.0540 R33= 1.66659 R34=  0.52856 R35=  0.00000 DMOM= 0.12 MODE=  1
J=  2  YMAX= 0.2511 SIGYMAX= 0.0272 R33= 1.48376 R34=  0.48135 R35=  0.00000 DMOM= 0.12 MODE=  1
J=  3  YMAX= 0.1992 SIGYMAX= 0.0272 R33= 1.12022 R34=  0.38743 R35=  0.00000 DMOM= 0.12 MODE=  1
J=  4  YMAX= 0.1938 SIGYMAX= 0.0272 R33= 0.94175 R34=  0.34132 R35=  0.00000 DMOM= 0.12 MODE=  1
J=  5  YMAX= 0.1637 SIGYMAX= 0.0218 R33= 0.60082 R34=  0.25322 R35=  0.00000 DMOM= 0.12 MODE=  1
J=  6  YMAX= 0.1365 SIGYMAX= 0.0195 R33= 0.28299 R34=  0.17105 R35=  0.00000 DMOM= 0.12 MODE=  1
J=  7  YMAX= 0.0874 SIGYMAX= 0.0108 R33= -0.00696 R34=  0.09607 R35=  0.00000 DMOM= 0.12 MODE=  1
J=  8  YMAX= 0.0737 SIGYMAX= 0.0218 R33= -0.25233 R34=  0.03259 R35=  0.00000 DMOM= 0.12 MODE=  1
J=  9  YMAX= 0.0519 SIGYMAX= 0.0108 R33= -0.42588 R34= -0.01233 R35=  0.00000 DMOM= 0.12 MODE=  1
J= 10  YMAX= 0.0682 SIGYMAX= 0.0218 R33= -0.13733 R34=  0.06234 R35=  0.00000 DMOM= 0.12 MODE=  1
J= 11  YMAX= 0.0819 SIGYMAX= 0.0131 R33= -0.07327 R34=  0.07891 R35=  0.00000 DMOM= 0.12 MODE=  1
J= 12  YMAX= 0.1336 SIGYMAX= 0.0218 R33= 0.20573 R34=  0.15107 R35=  0.00000 DMOM= 0.12 MODE=  1
J= 13  YMAX= 0.0600 SIGYMAX= 0.0300 R33= 1.00000 R34=  0.13400 R35=  0.00000 DMOM= 0.12 MODE=  1

INSTRUMENTAL WEIGHTING -- WEIGHT = 1/SIGMAY^2

J=  1  A=  .021987  SIGMAA=  .005553  FVALU=  0.0000E+00  CHISQR=  0.234842E+00
J=  2  A= -.129178  SIGMAA=  .025175  FVALU=  0.3803E+01  CHISQR=  0.234842E+00
J=  3  A=  .870417  SIGMAA=  .112428  FVALU=  0.2552E+03  CHISQR=  0.234842E+00

     COVARIANCE MATRIX

SIG(1,1)= 3.083530E-05
SIG(2,1)=-1.103825E-04  SIG(2,2)= 6.337715E-04
SIG(3,1)= 3.409873E-04  SIG(3,2)=-2.577216E-03  SIG(3,3)= 1.264001E-02

 ALPHA=  2.60913   STD. DEVIATION =  1.47192
 BETA=  0.44410   STD. DEVIATION =  0.19929
 EPSILON=  .04951003   STD. DEVIATION =  .02453404
 BEAM FIT CHISQR =   0.147448E-03

      (    ) = SQUARE OF WIDTH OR WIDTH FIT OR STD. DEV. OF WIDTH FIT SQUARED.
J = 1 WIDTH=  0.29480  ( .08691)WIDTH FIT=  0.27687 ( .07666)  STD. DEVIATION=  0.0184 ( 0.0102)  ERROR%= -6.08
J = 2 WIDTH=  0.25110  ( .06305)WIDTH FIT=  0.25605 ( .06556)  STD. DEVIATION=  0.0162 ( 0.0083)  ERROR%=  1.97
J = 3 WIDTH=  0.19920  ( .03968)WIDTH FIT=  0.21474 ( .04611)  STD. DEVIATION=  0.0122 ( 0.0052)  ERROR%=  7.80
J = 4 WIDTH=  0.19380  ( .03756)WIDTH FIT=  0.19457 ( .03786)  STD. DEVIATION=  0.0104 ( 0.0040)  ERROR%=  0.40
J = 5 WIDTH=  0.16370  ( .02680)WIDTH FIT=  0.15634 ( .02444)  STD. DEVIATION=  0.0076 ( 0.0024)  ERROR%= -4.50
J = 6 WIDTH=  0.13650  ( .01863)WIDTH FIT=  0.12133 ( .01472)  STD. DEVIATION=  0.0062 ( 0.0015)  ERROR%=-11.11
J = 7 WIDTH=  0.08738  ( .00764)WIDTH FIT=  0.09059 ( .00821)  STD. DEVIATION=  0.0059 ( 0.0011)  ERROR%=  3.68
J = 8 WIDTH=  0.07368  ( .00543)WIDTH FIT=  0.06670 ( .00445)  STD. DEVIATION=  0.0062 ( 0.0008)  ERROR%= -9.47
J = 9 WIDTH=  0.05186  ( .00269)WIDTH FIT=  0.05257 ( .00276)  STD. DEVIATION=  0.0078 ( 0.0008)  ERROR%=  1.37
J =10 WIDTH=  0.06823  ( .00466)WIDTH FIT=  0.07752 ( .00601)  STD. DEVIATION=  0.0060 ( 0.0009)  ERROR%= 13.61
J =11 WIDTH=  0.08188  ( .00670)WIDTH FIT=  0.08386 ( .00703)  STD. DEVIATION=  0.0059 ( 0.0010)  ERROR%=  2.41
J =12 WIDTH=  0.13360  ( .01785)WIDTH FIT=  0.11299 ( .01277)  STD. DEVIATION=  0.0060 ( 0.0014)  ERROR%=-15.43
J =13 WIDTH=  0.06000  ( .00360)WIDTH FIT=  0.05474 ( .00300)  STD. DEVIATION=  0.0270 ( 0.0030)  ERROR%= -8.76
```

Figure 5a. Vertical Data - Mode 1

VERTICAL DATA

INPUT DATA

```
J=  1  YMAX= 0.2948 SIGYMAX= 0.0100 R33=  1.66659 R34=  0.52856 R35=  0.00000 DMOM= 0.12 MODE=  2
J=  2  YMAX= 0.2511 SIGYMAX= 0.0100 R33=  1.48376 R34=  0.48135 R35=  0.00000 DMOM= 0.12 MODE=  2
J=  3  YMAX= 0.1992 SIGYMAX= 0.0100 R33=  1.12022 R34=  0.38743 R35=  0.00000 DMOM= 0.12 MODE=  2
J=  4  YMAX= 0.1938 SIGYMAX= 0.0100 R33=  0.94175 R34=  0.34132 R35=  0.00000 DMOM= 0.12 MODE=  2
J=  5  YMAX= 0.1637 SIGYMAX= 0.0100 R33=  0.60082 R34=  0.25322 R35=  0.00000 DMOM= 0.12 MODE=  2
J=  6  YMAX= 0.1365 SIGYMAX= 0.0100 R33=  0.28299 R34=  0.17105 R35=  0.00000 DMOM= 0.12 MODE=  2
J=  7  YMAX= 0.0874 SIGYMAX= 0.0100 R33= -0.00696 R34=  0.09607 R35=  0.00000 DMOM= 0.12 MODE=  2
J=  8  YMAX= 0.0737 SIGYMAX= 0.0100 R33= -0.25233 R34=  0.03259 R35=  0.00000 DMOM= 0.12 MODE=  2
J=  9  YMAX= 0.0519 SIGYMAX= 0.0100 R33= -0.42588 R34= -0.01233 R35=  0.00000 DMOM= 0.12 MODE=  2
J= 10  YMAX= 0.0682 SIGYMAX= 0.0100 R33= -0.13733 R34=  0.06234 R35=  0.00000 DMOM= 0.12 MODE=  2
J= 11  YMAX= 0.0819 SIGYMAX= 0.0100 R33= -0.07327 R34=  0.07891 R35=  0.00000 DMOM= 0.12 MODE=  2
J= 12  YMAX= 0.1336 SIGYMAX= 0.0100 R33=  0.20573 R34=  0.15107 R35=  0.00000 DMOM= 0.12 MODE=  2
J= 13  YMAX= 0.0600 SIGYMAX= 0.0100 R33=  1.00000 R34=  0.13400 R35=  0.00000 DMOM= 0.12 MODE=  2
```

NO WEIGHTING FOR MODIFIED EQUATIONS.

```
J=  1  A=  .023277   SIGMAA=  .004067  FVALU=  0.0000E+00  CHISQR=  0.134882E+01
J=  2  A= -.133933   SIGMAA=  .019264  FVALU=  0.1644E+02  CHISQR=  0.134882E+01
J=  3  A=  .892327   SIGMAA=  .082623  FVALU=  0.1166E+03  CHISQR=  0.134882E+01
```

COVARIANCE MATRIX

```
SIG(1,1)= 1.653851E-05
SIG(2,1)=-7.207178E-05  SIG(2,2)= 3.711187E-04
SIG(3,1)= 2.732008E-04  SIG(3,2)=-1.539511E-03  SIG(3,3)= 6.826497E-03
```

```
ALPHA=  2.51643   STD. DEVIATION =  0.59510
BETA=  0.43735   STD. DEVIATION =  0.08772
EPSILON=  .05322340   STD. DEVIATION =  .01026249
BEAM FIT CHISQR =   0.139035E-03
```

```
      (     ) = SQUARE OF WIDTH OR WIDTH FIT OR STD. DEV. OF WIDTH FIT SQUARED.
J = 1 WIDTH=  0.29480  ( .08691) WIDTH FIT=  0.27926 ( .07799) STD. DEVIATION=  0.0070 ( 0.0039) ERROR%=  -5.27
J = 2 WIDTH=  0.25110  ( .06305) WIDTH FIT=  0.25823 ( .06668) STD. DEVIATION=  0.0062 ( 0.0032) ERROR%=   2.84
J = 3 WIDTH=  0.19920  ( .03968) WIDTH FIT=  0.21655 ( .04689) STD. DEVIATION=  0.0047 ( 0.0021) ERROR%=   8.71
J = 4 WIDTH=  0.19380  ( .03756) WIDTH FIT=  0.19621 ( .03850) STD. DEVIATION=  0.0042 ( 0.0016) ERROR%=   1.24
J = 5 WIDTH=  0.16370  ( .02680) WIDTH FIT=  0.15769 ( .02487) STD. DEVIATION=  0.0036 ( 0.0012) ERROR%=  -3.67
J = 6 WIDTH=  0.13650  ( .01863) WIDTH FIT=  0.12250 ( .01501) STD. DEVIATION=  0.0038 ( 0.0009) ERROR%= -10.26
J = 7 WIDTH=  0.08738  ( .00764) WIDTH FIT=  0.09174 ( .00842) STD. DEVIATION=  0.0043 ( 0.0008) ERROR%=   4.99
J = 8 WIDTH=  0.07368  ( .00543) WIDTH FIT=  0.06806 ( .00463) STD. DEVIATION=  0.0047 ( 0.0006) ERROR%=  -7.62
J = 9 WIDTH=  0.05186  ( .00269) WIDTH FIT=  0.05432 ( .00295) STD. DEVIATION=  0.0052 ( 0.0006) ERROR%=   4.75
J =10 WIDTH=  0.06823  ( .00466) WIDTH FIT=  0.07874 ( .00620) STD. DEVIATION=  0.0045 ( 0.0007) ERROR%=  15.40
J =11 WIDTH=  0.08188  ( .00670) WIDTH FIT=  0.08503 ( .00723) STD. DEVIATION=  0.0044 ( 0.0008) ERROR%=   3.85
J =12 WIDTH=  0.13360  ( .01785) WIDTH FIT=  0.11413 ( .01302) STD. DEVIATION=  0.0039 ( 0.0009) ERROR%= -14.58
J =13 WIDTH=  0.06000  ( .00360) WIDTH FIT=  0.05836 ( .00341) STD. DEVIATION=  0.0114 ( 0.0013) ERROR%=  -2.74
```

Figure 5b. Vertical Data - Mode 2

Appendix A -- The LSQFIT Subroutine

A. The LSQFIT subroutine will do a least square fit to linear equations with constant coefficients:

$$Y = a_1X_1(z) + a_2X_2(z) + a_3X_3(z) + \ldots \qquad \text{(A1)}$$

where: z -- independent variable

$X_1(z)$ -- $X_n(z)$ -- known functions of the independent variable z but independent of the coefficients a_i

a_1 -- a_n -- constant coefficients to be found

Y -- dependent variable

This routine is a modification of the LEGFIT routine in Bevington.

Other linear least square fitting subroutines are available but one should use caution with them. Some routines will solve for the coefficients plus a dc or average term. This type of fitting routine cannot be used for emittance analysis.

Some equations that satisfy this criteria are:

1. $Y = a_1X + a_2X^2 + a_3X^3$ (A2)

2. $Y = a_1X + a_2X^3 + a_3X^5$ (A3)

3. $Y = a_1\cos t + a_2\cos 2t + a_3\cos 3t + \ldots$ (A4)

Some equations that do not satisfy this criteria are:

1. $Y = a_1X + a_2X^{a_2} + a_3X^{a_1}$

2. $Y = a_1\ln(a_2X) + a_3X^{a_4}$

Some equations that do not satisfy this criteria but can be modified to meet the criteria are:

1. $G(X) = \frac{A_1}{\sqrt{2\pi A_3{}^2}} e^{-\frac{1}{2}\left(\frac{x-A2}{A_3}\right)^2}$ the Gaussian Function

2. $XMAX = \left(a_{11}R_{11}{}^2 + a_{21}(2R_{11}R_{12}) + a_{22}R_{12}{}^2\right)^{\frac{1}{2}}$ The Transport Equation

The Gaussian function can be modified to a linear equation by taking the natural log of both sides

$$Y = a_1 + a_2X + a_3X^2 \tag{A5}$$

where: $Y = \ell n\ G(x)$

$$a_1 = \ell n \left(\frac{A}{\sqrt{2\pi A_3{}^2}}\right) - \frac{A_2{}^2}{2A_3{}^2} \tag{A6}$$

$$a_2 = \frac{A_2}{A_3{}^2}$$

$$a_3 = \frac{-1}{2A_3{}^2}$$

The Transport equation can be modified by squaring.

$$Y = a_1X_1(z) + a_2X_2(z) + a_3X_3(z) \tag{A1}$$

$$\text{where: } Y = (XMAX)^2 \tag{A7}$$

$$X_1(z) = R_{11}{}^2$$

$$X_2(z) = 2R_{11}R_{12}$$

$$X_3(z) = R_{12}{}^2$$

B. Calling the LSQFIT Routine

To use the LSQFIT subroutine, one calls:

CALL LSQFIT (X, Y, SIGMAY, NPTS, NUMT, IHV, MODE, YFIT, A, SIGMAA, FVALU, CHISQR).

Description of Input Parameters

X -- array of the independent variable

Y -- array of data points for the dependent variable in the same order as X. Y is the measured value for a given value of X for standard equations or Y is a modification of the measured values for modified equations.

SIGMAY -- array of standard deviations of the Y data in the same order as Y. For modified equations the SIGMAY is a modification of the standard deviations of the measured points.

NPTS -- number of pairs of data points, maximum = 40.

NUMT -- number of terms in the function or number of coefficients; maximum = 10.

IHV -- a variable that may or may not be used. For Transport calculations it identifies horizontal or vertical plane.

MODE -- determines the mode of weighting the Y points when doing the least-squares fit.

+1 (instrumental) WEIGHT(I) = 1/SIGMAY(I)**2

ϕ (no weighting) WEIGHT(I) + 1.0

-1 (STATISTICAL) WEIGHT(I) = 1/Y(I)

+2 (no weighting for modified equations)

For standard equations modes -1, ϕ, and +1 are valid. For modes -1 and +1 the data points are weighted and the coefficients A are found. The standard deviations SIGMAA are proportional to the weighting of the Y points. Thus, if mode +1 is used and the analysis is repeated for the same values of the data points but with smaller values of SIGMAY, the same coefficients will be found but with smaller values of SIGMAA. For mode ϕ SIGMAA is proportional to CHISQR or to the fit of the data to the equation.

For modified equations only modes +2 and +1 are valid. For mode +1 SIGMAA is proportional to SIGMAY as for standard equations. For no weighting for modified equations, mode +2 is used. However, when mode +2 is used, the standard deviations of the measured points must all be made equal and then SIGMAA will depend on CHISQR or to the fit of the data to the equation.

Description of Output Parameters

YFIT -- array of calculated values of Y.

A -- array of standard coefficients in Equation A1.

SIGMAA -- array of standard deviations for the coefficients.

CHISQR -- reduced CHI-SQUARE for the fit times the weighting.

$$\text{CHISQR} = \frac{(\text{WEIGHT} * \text{CHI SQUARE})}{(\text{NPTS}-\text{NTERMS})}$$

This is approximately the average weighted CHI SQUARE for a data point.

FVALU -- array of FVALUES for each term

$$\text{FVALU(I)} = \frac{\Delta\text{CHI SQUARE}}{\text{CHISQR}}$$

FVALU is a measure of the effectiveness of each term to the fitting. If FVALU(J) is negative or zero, that term should not be included. FVALU for the first term is always zero. A term should be added only if it reduces CHISQR or makes the fitting better.

An additional function program needs to be added to the program calling LSQFIT. This function routine calculates the $X_i(z)$ value for each term and is different for each equation A1-A5.

It is written as:

Function FCTN (X, I, J, IHV)

X -- the value of X for a data point (may not be used).

I -- index of data points.

J -- index of the term of the function.

IHV -- a variable that can be used.

For Equation A2

$$Y = a_1X + a_2X^2 + a_3X^3$$

Function FCTN (X, I, J, IHV)

```
FCTN -- X**J
RETURN
END
```

Another function program is sometimes needed to calculate standard deviations. This program is always needed for modified equations. The LSQFIT subroutine will return the standard deviations of the A coefficients. However, for modified equations, other values are calculated from these A coefficients and one needs to know the standard deviation of these values. For example, the Twiss parameters are calculated from the σ parameters which are found by LSQFIT. This function program evaluates Equation 19. If:

$$W = f(A_1, A_2, A_3)$$

and A_1, A_2, and A_3 are the coefficients returned by LSQFIT, then the variance of W or the square of the standard deviation of W is:

$$(\mathrm{SIGW})^2 = (\mathrm{SIGA}_1)^2(\frac{\partial f}{\partial A_1})^2 + (\mathrm{SIGA}_2)^2(\frac{\partial f}{\partial A_2})^2 + (\mathrm{SIGA}_3)^2(\frac{\partial f}{\partial A_3})^2$$

$$+ 2(\mathrm{SIGA}_1A_2)^2(\frac{\partial f}{\partial A_1})(\frac{\partial f}{\partial A_2}) + 2(\mathrm{SIGA}_1A_3)^2(\frac{\partial f}{\partial A_1})(\frac{\partial f}{\partial A_3}) +$$

$$2(\mathrm{SIGA}_2A_3)^2(\frac{\partial f}{\partial A_2})(\frac{\partial f}{\partial A_3}) \qquad \text{(A8)}$$

The unknowns of this equation are the partial derivatives -- $\partial f/\partial A$, $\partial f/\partial A_2$, and $\partial f/\partial A_3$. As discussed in Part 6 the variances and covariances of the A coefficients are available from the LSQFIT subroutine through the:

COMMON/LSQ/NTERMS, SIGUV2

statement. The function program is:

Function STDEV (DERIV)

Description of Input Parameters.

DERIV -- array of derivatives, $\partial f/\partial A_1$, $\partial f/\partial A_2$, $\partial f/\partial A_3$. The array must be in the order of the A coefficients. Thus:

$$DERIV(1) = \partial f/\partial A_1$$
$$DERIV(2) = \partial f/\partial A_2$$
$$DERIV(3) = \partial f/\partial A_3$$

Use of the function

SIG = STDEV(DERIV)

If $W = f(A_1, A_2, A_3)$ then SIG is the standard deviation of W. The values of DERIV are calculated in TEST3-5..

A load command that can be used is:

Load TEST1.F4, REL: REGRE/LIB

C. Sample Programs

1. TEST1

The TEST1.F4 program trys to fit the X and Y data to the equation:

$$Y = a_1X + a_2X^2 + a_3X^3$$

The Y data was calculated from:

$$Y = 2X + 3X^2 + 4X^3$$

The MODE used is zero or no weighting is used. The standard deviations of the coefficients will depend on the fitting. The results shown in Figure 1 show that the 3 coefficients are found accurately and the CHISQR is very small. For each term, except the first, the FVALU is positive and large indicating that all terms in the equation are needed to fit the data. The results are:

$$a_1 = 2.000, \text{ STD. DEV.} = 0.0010$$
$$a_2 = 3.000, \text{ STD. DEV.} = 0.0006$$
$$a_3 = 4.000, \text{ STD. DEV.} = 0.0001$$

These compare with the exact values of 2.0, 3.0, and 4.0.

2. TEST2

The TEST2.F4 program trys to fit the same data as TEST1 to:

$$Y = a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4$$

by changing NUMT to 4.

The results shown in Figure A2 show that the CHISQR is less and that the coefficients are not known as precise as TEST1. The FVALU for the 4th term is negative indicating that that term should be omitted since the fitting became worse when it was added. For better accuracy one should repeat TEST2 but with NUMT equal to 3. The results are:

$$a_1 = 2.000, \text{ STD. DEV.} = 0.0333$$
$$a_2 = 2.990, \text{ STD. DEV.} = 0.0349$$
$$a_3 = 4.000, \text{ STD. DEV.} = 0.0114$$
$$a_4 = 0.00\ , \text{ STD. DEV.} = 0.0012$$

3. TEST3

The TEST3.F4 program will fit profile data to the best fitting Gaussian function. As discussed earlier Equation A9 is the Gaussian function equation and cannot be used directly with LSQFIT.

$$G(x) = \frac{A_1}{\sqrt{2\pi A_3{}^2}} e^{-\frac{1}{2}\left(\frac{x-A_2}{A_3}\right)^2} \tag{A9}$$

where:

A_1 -- area under the Gaussian function.

A_2 -- the mean of the function.

A_3 -- the sigma or width or standard deviation of the function.

Taking the natural log of G(x) and rearranging:

$$\ell nG(x) = \left[\ell n(\frac{A_1}{\sqrt{2\pi A_3{}^2}}) - \frac{A_2{}^2}{2A_3{}^2} \right] + \frac{A_2}{A_3{}^2}X + (\frac{-1}{2A_3{}^2})\ X^2$$

or

$$Y = a_1 + a_2X + a_3X^2 \tag{A5}$$

where a_1, a_2, and a_3 are functions of A_1, A_2, A_3 but are independent of X. This is the modified equation that can be solved with LSQFIT. If a_1, a_2, and a_3 are found:

$$A_3 = (\frac{-1}{2a_3})^{\frac{1}{2}}$$

$$A_2 = (\frac{-a_2}{2a_3}) \tag{A10}$$

$$A_1 = \sqrt{2\pi A_3{}^2}\ e^{\left(a_1 + \frac{A_2{}^2}{2A_3{}^2}\right)}$$

The LSQFIT routine can be used to find a_1, a_2, and a_3 if the Y measured points and their standard deviations are modified for LSQFIT;

$$Y = \ell n(\text{measured Y points})$$

$$\tag{A11}$$

$$X = X$$

From Equation 19 the standard deviation of the Y points can be calculated from the standard deviations of the Y measured points:

$$(\mathrm{SIGMAY})^2 = (\mathrm{SIGYMEAS})^2 \left(\frac{\partial f}{\partial(\mathrm{YMEAS})}\right)^2$$

$$= \left(\frac{1}{\mathrm{YMEAS}}\right)^2 (\mathrm{SIGYMEAS})^2 \qquad \text{(A12)}$$

or

$$\mathrm{SIGMAY} = \frac{1}{\mathrm{YMEAS}} (\mathrm{SIGYMEASURED})$$

Y, X, and SIGMAY are used as input to LSQFIT.

The LSQFIT routine returns the fitted values of a_1, a_2, and a_3. The constants A_1, A_2, and A_3 of the Gaussian can be calculated from these using Equation (A10). The routine also returns the standard deviations of a_1, a_2, and a_3.

To find the standard deviations of the Gaussian constants, the function STDEV(DERIV) is used. The partial derivatives are needed:

For A_3 -- the sigma width = $f(a_1, a_2, a_3)$

$$\begin{aligned} \mathrm{DERIV}(1) &= \phi = \partial A_3/\partial a_1 \\ \mathrm{DERIV}(2) &= \phi = \partial A_3/\partial a_2 \\ \mathrm{DERIV}(3) &= (-2a_3)^{-3/2} = \partial A_3/\partial a_3 \end{aligned} \qquad \text{(A13)}$$

For A_2 -- the mean value = $f(a_1, a_2, a_3)$

$$\begin{aligned} \mathrm{DERIV}(1) &= \phi = \partial A_2/\partial a_1 \\ \mathrm{DERIV}(2) &= -1/2a_3 = \partial A_2/\partial a_2 \\ \mathrm{DERIV}(3) &= a_2/2a_3^2 = \partial A_2/\partial a_3 \end{aligned} \qquad \text{(A14)}$$

For A_1 -- the area = $f(a_1, a_2, a_3)$

$$\begin{aligned} \mathrm{DERIV}(1) &= A_1 = \partial f/\partial a_1 \\ \mathrm{DERIV}(2) &= (-a_2/2a_3)A_1 = \partial f/\partial a_2 \\ \mathrm{DERIV}(3) &= A_1(A_2^2 - 1/2a_3) = \partial f/\partial a_3 \end{aligned} \qquad \text{(A15)}$$

The TEST3.F4 program of Figure 3 illustrates using LSQFIT to find the constants in Equation (A9). The data was calculated from a Gaussian with a magnitude (A_1) of 1253.3, mean (A_2) of +2.0, and sigma (A_3) of 5.0 and rounded to the nearest integer. The X data varied from -10 to +10. The standard deviation of all the measured points was 0.5 and the mode was +1. Therefore, the standard deviations found are proportional to these measured standard deviations. The function FCTN for Equation A5 is different from TEST1:

```
Function FCTN (X, I, J, JHV)
FCTN = X**(J-1)
RETURN
END
```

The CHISQR that is returned from LSQFIT is for Equation A5 and not for the Gaussian fit. The reduced CHI SQUARE for the Gaussian function is calculated from:

$$\text{CHI SQUARE} = \sum_{1}^{\text{NPTS}} (\text{calculated Gaussian Y-YMEAS})^2 \qquad \text{(A16)}$$

$$\text{CHISQR} = \frac{\text{CHI SQUARE}}{\text{NPTS-NUMT}} \qquad \text{(A17)}$$

To illustrate the error matrix or covariance matrix, the

COMMON/LSQ/NTERMS, SIGUV2

statement was added to TEST3.F4. This is not needed to calculate the standard deviations. Since the covariance matrix is symmetrical, only the lower half is printed. One can determine the effect of correlation between the A coefficients returned by LSQFIT by comparing the off diagonal elements of the matrix with the diagonal elements. If the off diagonal elements are zero, no correlation exists between any of the A coefficients.

The results of TEST3.F4 are:

A_1 = area = 1260.04 ; STD. DEV. = 0.56
A_2 = mean = 2.000; STD. DEV. = 0.0025
A_3 = sigma = 5.028; STD. DEV. = 0.0027

These compare favorably with the exact values of 1253.3, 2.0, and 5.0.

4. TEST4

The TEST4.F4 program is used to solve the horizontal simplified TRANSPORT Equation, 11a. For the SBE (Single Bunch Extraction) to the "D" line, the fractional momentum deviation for a 99% beam is assumed known ($\delta = \pm 0.12\%$). From Equation 4,

$$\sigma_{55} = \delta^2$$

Rearranging Equation 11a:

$$\mathrm{XMAX} = \left(\sigma_{11}R_{11}^{\ 2} + 2\sigma_{21}R_{11}R_{12} + \sigma_{22}R_{12}^{\ 2} + \sigma_{55}R_{15}^{\ 2}\right)^{\frac{1}{2}} \tag{A18}$$

As discussed earlier this is not suitable for solution with LSQFIT but it can be modified:

$$(\mathrm{XMAX})^2 - \sigma_{55}R_{15}^{\ 2} = \sigma_{11}R_{11}^{\ 2} + 2\sigma_{21}R_{11}R_{12} + \sigma_{22}R_{12}^{\ 2} \tag{A19}$$

This equation is a linear equation of type (A1) and LSQFIT may be used to solve for the coefficients σ_{11}, σ_{21}, and σ_{22}.

$$Y = a_1X_1(z) + a_2X_2(z) + a_3X_3(z) \tag{A1}$$

where a_1, a_2, and a_3 are σ_{11}, σ_{21}, and σ_{22} and the values to input to LSQFIT are:

$$Y = (\text{XMAX measured})^2 - \sigma_{55}R_{15}^2$$

X = beam line distance

$$X_1(z) = R_{11}^2$$

$$X_2(z) = 2R_{11}R_{12}$$

$$X_3(z) = R_{12}^2$$

and from Equation 24

SIGMAY = 2(XMAX measured) (SIGXMAX measured)

If X is given as a beam line distance, the (R) matrix elements can be calculated from a TRANSPORT or QTUNE type program. This would make the necessary FCTN routine complicated. The simpler means is to find the (R) matrices with TRANSPORT or QTUNE and enter them as data into FCTN through a common statement. Thus:

```
Function FCTN(X, I, J, JHV)
COMMON/MATR/R(5, 5, 30)
IF(J. EQ. 1) FCT = R(1, 1, I)**2
IF(J. EQ. 2) FCT = 2*R(1, 1, I)*R(1, 2, I)
IF(J. EQ. 3) FCT = R(1, 2, I)**2
FCTN = FCT
RETURN
```

For this example, X is not used in the FCTN routine. To find the horizontal emittance parameters of equation 11a only the R_{11}, R_{12}, and R_{15} elements of the transport matrices are needed. The R array must be filled in the same order as the Y array - i.e. R(1, 1, 1) must be the R_{11} component of R for the first data point corresponding to (XMAX measured) and R_{15} for the same data point.

The matrix elements for the sample program TEST4.F4 were calculated with QTUNE for elements in the "D" line from F13 up to AD2 at 24.06 GeV/c for a typical running condition. For this area of the "D" line no skew elements are present and Equation 11a describes the transport system. The horizontal sizes used were calculated assuming α, β, ε = -0.938, 0.1904 kiloinch and 0.075525 inch-mrad for a 99% beam. The calculated sizes were rounded to 3 places and the standard deviation for all these points was assumed to be 0.0005.

From the values of σ_{11}, σ_{21}, σ_{22} returned by LSQFIT and Equation 5, the fitted Twiss parameters α, β, and ε can be found

$$\varepsilon = \sqrt{\sigma_{11}\sigma_{22}-\sigma_{21}^{2}}$$

$$\alpha = -\sigma_{21}/\varepsilon \tag{5}$$

$$\beta = \sigma_{11}/\varepsilon$$

It should be pointed out that LSQFIT will find the constants for the modified equation A19 and not Equation 5. This results in sometimes finding that ε^2 is negative or ε is not real. It has been found that if the measured XMAX sizes are on both sides of a waist or that sufficient widely spaced points on the X/θ plane ellipse are used, that a real emittance may be found.

LSQFIT will return the values of σ_{11}, σ_{21}, and σ_{22} and their standard deviations. Using Equations 5 and 19, the standard deviation of the Twiss parameters can be found using the function STDEV.

To use the function STDEV, the partial derivatives are needed.

For ε -- the emittance = $f(\sigma_{11}, \sigma_{21}, \sigma_{22})$

$$\begin{aligned} \text{DERIV(1)} &= \sigma_{22}/2\varepsilon = \partial f/\partial\sigma_{11} \\ \text{DERIV(2)} &= -\sigma_{21}/\varepsilon = \partial f/\partial\sigma_{21} \\ \text{DERIV(3)} &= \sigma_{11}/2\varepsilon = \partial f/\partial\sigma_{22} \end{aligned} \tag{A20}$$

For alpha = $f(\sigma_{11}, \sigma_{21}, \sigma_{22})$

$$\begin{aligned} \text{DERIV(1)} &= \sigma_{22}\sigma_{21}/2\varepsilon^{3} = \partial f/\partial\sigma_{11} \\ \text{DERIV(2)} &= (-1/\varepsilon) - \sigma_{21}^{2}/\varepsilon^{3} = \partial f/\partial\sigma_{21} \\ \text{DERIV(3)} &= \sigma_{11}\sigma_{21}/2\varepsilon^{3} = \partial f/\partial\sigma_{22} \end{aligned} \tag{A21}$$

For beta = $f(\sigma_{11}, \sigma_{21}, \sigma_{22})$

$$\begin{aligned} \text{DERIV(1)} &= (1/\varepsilon) - \sigma_{11}\sigma_{22}/2\varepsilon^{3} = \partial f/\partial\sigma_{11} \\ \text{DERIV(2)} &= \sigma_{11}\sigma_{21}/\varepsilon^{3} = \partial f/\partial\sigma_{21} \\ \text{DERIV(3)} &= -\sigma_{11}^{2}/2\varepsilon^{3} = \partial f/\partial\sigma_{22} \end{aligned} \tag{A22}$$

Of equal concern is the standard deviation of the fitted calculated XMAX points. Rearranging Equation 11A:

$$XMAXFIT = (\sigma_{11}(R_{11}{}^2) + \sigma_{21}(2R_{11}R_{12}) + \sigma_{22}(R_{12}{}^2) + \sigma_{55}(R_{15})^2)^{1/2}$$

or

$$XMAXFIT = f(\sigma_{11}, \sigma_{21}, \sigma_{22}, \sigma_{55}) \tag{A23}$$

This is also the equation that is used to find the beam size anywhere in a beam line knowing the R matrix elements. To find the standard deviation of XMAXFIT, the derivatives are needed for the function STDEV.

$$\begin{aligned} DERIV(1) &= (R_{11}{}^2)/2(XMAXFIT) = \partial f/\partial\sigma_{11} \\ DERIV(2) &= (2R_{11}R_{12})/2(XMAXFIT) = \partial f/\partial\sigma_{21} \\ DERIV(3) &= (R_{12})^2/2(XMAXFIT) = \partial f/\partial\sigma_{22} \\ DERIV(4) &= (R_{15})^2/2(XMAXFIT) = \partial f/\partial\sigma_{55} \end{aligned} \tag{A24}$$

Since the fractional momentum deviation δ or σ_{55} was assumed constant for this example, the variances and covariances for this term are zero. Only three derivatives are needed -- DERIV(1) - DERIV(3). If one was trying to find the value of δ for the beam from many properly chosen beam width measurements, then the standard deviation of the calculated beam size would depend on all four derivatives. The standard deviations of the Twiss parameters would not depend on this fourth derivative.

As shown in TEST4.F4, these derivatives can be more easily expressed using the FCTN function.

The results shown in Figure 4 show that for this ideal case, the Twiss parameters are found accurately. The reduced CHI SQUARE for the fitting, calculated from an equation similar to Equations A15 and A16, is small (1×10^{-7}). For this example all beam width points were assumed known to the same accuracy and mode +1 was used. The results are:

Alpha	= -0.9348	, STD. DEV. = 0.0042
Beta	= 0.1902	, STD. DEV. = 0.00059 kilo inch
Epsilon	= 0.07556	, STD. DEV. = 0.00037 in-mrad

5. TESTS

This example is similar to TEST4 except that both the horizontal and vertical emittance parameters are found.

The simplified TRANSPORT equations (11a) and (11b) are used. No skew components are allowed in the beam line. The data is from the "D" line from F13 up to AD2. The fractional momentum deviation for a 99% beam is assumed known (δ = 0.12%).

Rearranging (11a) and (11b), the modified equations for LSQFIT are:

$$(XMAX)^2 - \sigma_{55}R_{15}{}^2 = \sigma_{11}R_{11}{}^2 + 2\sigma_{21}R_{11}R_{12} + \sigma_{22}R_{11} \tag{11a}$$

$$(YMAX)^2 - \sigma_{55}R_{35}{}^2 = \sigma_{33}R_{33}{}^2 + 2\sigma_{43}R_{33}R_{34} + \sigma_{44}R_{34} \tag{11b}$$

Each equation is solved independently with LSQFIT. Using the IHV parameter to separate horizontal and vertical planes, only one FCTN routine is used.

```
              Function FCTN (X, I, J, JHV)
    c   JHV = 1 for horizontal, JHV = 3 for vertical

      COMMON/MATR/R(5, 5, 30)
      IHV = JHV
      FCT = ϕ
      IF(J.EQ.1) FCT = R(IHV, IHV, I)**2
      IF(J.EQ.2) FCT = 2.*R(IHV, IHV, I)*R(IHV, IHV+1, I)
      IF(J.EQ.3) FCT = R(IHV, IHV+1, I)**2
      FCTN = FCT
      RETURN
      END
```

The assumed values of the Twiss parameters were:

Horizontal α, β, ε = -0.9380, 0.1904 KILOINCH, 0.075525 IN-MRAD
Vertical α, β, ε = 0.9870, 0.1457, 0.075525
Momentum - 24.06 GeV/c, δ = 0.12%

The same horizontal data was used in TEST5 as in TEST4 except that the assumed standard deviations of the horizontal widths in TEST5 was 20 times larger than for TEST4. As a result the mode +1 standard deviations are larger for TEST5 but the unweighted mode +2 standard deviations are the same.

The results are:

	Mode +1	STD. DEV.	Mode +2	STD. DEV.	
Alpha =	-0.9348 ,	0.0842	-0.9348 ,	0.0027	
Beta =	0.1902 ,	0.0119	0.1902 ,	0.00038	kilo inch
Epsilon =	0.07556,	0.0075	0.07556,	0.00024	in-mrad

The results for the vertical plane are:

Alpha =	0.9851 ,	0.130	0.9851 ,	0.0039
Beta =	0.1455 ,	0.008	0.1455 ,	0.0003
Epsilon =	0.07548,	0.0045	0.07548,	0.0001

One should note that the magnitudes of the Twiss parameters are the same for each mode because all weights were assumed equal for Mode 1. If each point was weighted differently, the parameters would have different values for each mode.

The CHISQR returned by LSQFIT is modified by the weighting used as shown in this example. The important CHISQR is the value obtained with the fitted parameters.

TEST5.F4 is the program that can be used to determine horizontal and vertical emittances if no skew elements are in the beam line. XMAX and YMAX are the measured beam sizes with SIGXMA and SIGYMA their standard deviations. The (R) matrix elements RM11 to RM35 are determined with TRANSPORT or QTUNE. It is recommended that the initial values calculated with these programs be inserted into XMAX and YMAX. TEST5 can be run and the Twiss parameters obtained can then be compared with the assumed values in TRANSPORT or QTUNE. This checks that the correct (R) matrices are used. The experimental values of XMAX and YMAX with their standard deviations can then be inserted.

The data for TEST5 was obtained at 12 different points in the beam line. If a quad is varied, similar data could be obtained for each tune and inserted into XMAX, YMAX and the (R) elements.

6. TEST6

Another possibility involves solving a beam line with skew components. At the AGS the "A" and "D" lines beyond AD2&3 have skew quads. Equation (10a) and (10b) are the Transport equations with skew elements in the beam line but without skew components in the input beam. The input fractional momentum deviation for a 99% beam is assumed known.

$$(XMAX)^2 - \sigma_{55}R_{15}^{\ 2} = \sigma_{11}R_{11}^{\ 2} + 2\sigma_{21}R_{11}R_{12} + \sigma_{22}R_{12}^{\ 2} + \sigma_{33}R_{13}^{\ 2} + 2\sigma_{43}R_{13}R_{14} + \sigma_{44}R_{14}^{\ 2} \tag{10a}$$

$$(YMAX)^2 - \sigma_{55}R_{35}^{\ 2} = \sigma_{11}R_{31}^{\ 2} + 2\sigma_{21}R_{32}R_{31} + \sigma_{22}R_{32}^{\ 2} + \sigma_{33}R_{33}^{\ 2} + 2\sigma_{43}R_{34}R_{33} + \sigma_{44}R_{34}^{\ 2} \tag{10b}$$

These equations are linear equations with six constant coefficients. It has not been found possible to use LSQFIT to solve for the six coefficients using ideal calculated values of XMAX and YMAX. The FVALU term becomes negative for the sixth term indicating that CHISQR increases when a sixth coefficient is added. The fault may be that the double precision accuracy is not sufficient on this PDP10 computer. Mathematically it should be possible to solve (10a) or (10b) or solve the sum of (10a) and (10b) for the coefficients.

It is possible to find five constants to satisfy (10a) or (10b) but to find the Twiss parameters in each plane requires six parameters. One can find a good fit (CHISQ < 10^{-5}) to (10a) or (10b) if one disregards the term of the other plane: i.e

A larger beam that is not tilted will fit the data of the beam line with skew components.

The result is that the horizontal and vertical Twiss parameters of the beam cannot be found from horizontal and vertical beam size measurements in a line downstream of skew components if the LSQFIT routine is used.

```
C ********************** TEST1.F4 *****************************:
C  SOLVES:
C   Y = 2.0*X    +  3.0 * X^2    +   4.0* X^3
C
       DIMENSION X(10),Y(10),SIGMAY(10),YFIT(10),A(10),
       1 SIGMAA(10),FVALU(10)
C
       DATA X/0.5,1.0,1.5,2.,2.5,3.,3.5,4.0,4.5,5.0/
       DATA Y/2.25,9.0,23.25,48.0,86.25,141.0,215.25,312.0,434.25,
       1 585.0/
       DATA NPTS,IOUT,MODE,IHV
       1 /  10,   1,   0,   0/
C
10     NUMT = 3
C
       DO 40 K=1,NPTS
40     FVALU(K) = 0.
C
       CALL LSQFIT(X,Y,SIGMAY,NPTS,NUMT,IHV,MODE,YFIT,
       1 A,SIGMAA,FVALU,CHISQR)
C
       DO 50 J=1,NUMT
50     WRITE(IOUT,55)J,A(J),SIGMAA(J),FVALU(J),CHISQR
55     FORMAT(3X,'J= ',I2,2X,'A= ',F8.4,2X,'SIGMAA= ',F8.4,' FVALU= ',
       1 E12.4,2X,'CHISQR= ',E15.6)
       WRITE (IOUT,75)
C
       DO 70 J = 1,NPTS
       ERROR = 100. *( YFIT(J) -Y(J))/Y(J)
70     WRITE(IOUT,60)J,X(J),Y(J),SIGMAY(J),YFIT(J),ERROR
60     FORMAT(3X,'#= ',I2,1X,'X=',F8.3,' Y=',F8.3,' SIGMAY=',
       1 F8.3,' YFIT=',F8.3,2X,' ERROR %= ',F8.4)
C
75     FORMAT(1X,/)
       END

       FUNCTION FCTN( X,I,J,JHV)
       DOUBLE PRECISION XX,F2
       XX = X
       F2 = XX**J
       FCTN = F2
       RETURN
       END
```

```
   J=  1  A=   2.0000  SIGMAA=   0.0010 FVALU=   0.0000E+00  CHISQR=    0.102744E-05
   J=  2  A=   3.0000  SIGMAA=   0.0006 FVALU=   0.3538E+03  CHISQR=    0.102744E-05
   J=  3  A=   4.0000  SIGMAA=   0.0001 FVALU=   0.2046E+10  CHISQR=    0.102744E-05


   #=  1 X=   0.500 Y=   2.250 SIGMAY=   0.000 YFIT=   2.250   ERROR %=  -0.0001
   #=  2 X=   1.000 Y=   9.000 SIGMAY=   0.000 YFIT=   9.000   ERROR %=  -0.0002
   #=  3 X=   1.500 Y=  23.250 SIGMAY=   0.000 YFIT=  23.250   ERROR %=  -0.0002
   #=  4 X=   2.000 Y=  48.000 SIGMAY=   0.000 YFIT=  48.000   ERROR %=  -0.0003
   #=  5 X=   2.500 Y=  86.250 SIGMAY=   0.000 YFIT=  86.250   ERROR %=  -0.0003
   #=  6 X=   3.000 Y= 141.000 SIGMAY=   0.000 YFIT= 141.000   ERROR %=  -0.0003
   #=  7 X=   3.500 Y= 215.250 SIGMAY=   0.000 YFIT= 215.249   ERROR %=  -0.0003
   #=  8 X=   4.000 Y= 312.000 SIGMAY=   0.000 YFIT= 311.999   ERROR %=  -0.0003
   #=  9 X=   4.500 Y= 434.250 SIGMAY=   0.000 YFIT= 434.249   ERROR %=  -0.0003
   #= 10 X=   5.000 Y= 585.000 SIGMAY=   0.000 YFIT= 584.998   ERROR %=  -0.0003
```

Figure A1. TEST1

```
C ********************** TEST2.F4 *****************************:
C SAME AS TEST1.F4 BUT TRYS TO FIT DATA TO:
C  Y =  A1*X  +  A2* X^2  + A3* X^3  + A4* X^4
C
        DIMENSION X(10),Y(10),SIGMAY(10),YFIT(10),A(10),
        1 SIGMAA(10),FVALU(10)
C
        DATA X/0.5,1.0,1.5,2.,2.5,3.,3.5,4.0,4.5,5.0/
        DATA Y/2.25,9.0,23.25,48.0,86.25,141.0,215.25,312.0,434.25,
        1 585.0/
        DATA NPTS,IOUT,MODE,IHV
        1 /  10,   1,   0,   0/
C
10      NUMT = 4
C
        DO 40 K=1,NPTS
40      FVALU(K) = 0.
C
        CALL LSQFIT(X,Y,SIGMAY,NPTS,NUMT,IHV,MODE,YFIT,
        1 A,SIGMAA,FVALU,CHISQR)
C
        DO 50 J=1,NUMT
50      WRITE(IOUT,55)J,A(J),SIGMAA(J),FVALU(J),CHISQR
55      FORMAT(3X,'J= ',I2,2X,'A= ',F8.4,2X,'SIGMAA= ',F8.4,' FVALU= ',
        1 E12.4,2X,'CHISQR= ',E15.6)
        WRITE (IOUT,75)
C
        DO 70 J = 1,NPTS
        ERROR = 100. *( YFIT(J) -Y(J))/Y(J)
70      WRITE(IOUT,60)J,X(J),Y(J),SIGMAY(J),YFIT(J),ERROR
60      FORMAT(3X,'#= ',I2,1X,'X=',F8.3,' Y=',F8.3,' SIGMAY=',
        1 F8.3,' YFIT=',F8.3,2X,' ERROR %= ',F8.4)
C
75      FORMAT(1X,/)
        END

        FUNCTION FCTN( X,I,J,JHV)
        DOUBLE PRECISION XX,F2
        XX = X
        F2 = XX**J
        FCTN = F2
        RETURN
        END
```

```
   J=  1  A=   2.0000  SIGMAA=   0.0333 FVALU=   0.0000E+00  CHISQR=    0.251661E-03
   J=  2  A=   2.9990  SIGMAA=   0.0349 FVALU=   0.3538E+03  CHISQR=    0.251661E-03
   J=  3  A=   4.0000  SIGMAA=   0.0114 FVALU=   0.2046E+10  CHISQR=    0.251661E-03
   J=  4  A=   0.0000  SIGMAA=   0.0012 FVALU=  -0.5971E+01  CHISQR=    0.251661E-03


   #=  1 X=   0.500 Y=   2.250 SIGMAY=   0.000 YFIT=   2.250   ERROR %=  -0.0109
   #=  2 X=   1.000 Y=   9.000 SIGMAY=   0.000 YFIT=   8.999   ERROR %=  -0.0109
   #=  3 X=   1.500 Y=  23.250 SIGMAY=   0.000 YFIT=  23.248   ERROR %=  -0.0095
   #=  4 X=   2.000 Y=  48.000 SIGMAY=   0.000 YFIT=  47.996   ERROR %=  -0.0081
   #=  5 X=   2.500 Y=  86.250 SIGMAY=   0.000 YFIT=  86.244   ERROR %=  -0.0071
   #=  6 X=   3.000 Y= 141.000 SIGMAY=   0.000 YFIT= 140.991   ERROR %=  -0.0062
   #=  7 X=   3.500 Y= 215.250 SIGMAY=   0.000 YFIT= 215.238   ERROR %=  -0.0056
   #=  8 X=   4.000 Y= 312.000 SIGMAY=   0.000 YFIT= 311.984   ERROR %=  -0.0050
   #=  9 X=   4.500 Y= 434.250 SIGMAY=   0.000 YFIT= 434.230   ERROR %=  -0.0046
   #= 10 X=   5.000 Y= 585.000 SIGMAY=   0.000 YFIT= 584.976   ERROR %=  -0.0042
```

Figure A2.   TEST2

```
C ********************** TEST3.F4 ******************************:
C  FITS DATA TO THE BEST GAUSSIAN USING:
C              Y = A0 + A1*X  +  A2* X^2
C
       DIMENSION X(21),Y(21),SIGMAY(21),YFIT(21),A(10),SIGMAA(10),
       1 FVALU(10),YMEAS(21),GASFIT(21),AGAS(3),SIGAGA(3),SIGMEA(21),
       2 SIGFIT(21)
       DIMENSION SIGUV2(10,10),DERIV(3)
       COMMON/LSQ/NTERMS,SIGUV2
C
       DATA X/-10.,-9.,-8.,-7.,-6.,-5.,-4.,-3.,-2.,-1.,0.,1.,
       1 2.,3.,4.,5.,6.,7.,8.,9.,10./
       DATA YMEAS/6.,9.,14.,20.,28.,38.,49.,61.,73.,84.,92.,98.,100.,
       1 98.,92.,84.,73.,61.,49.,38.,28./
       DATA BLK,NPTS,IOUT,MODE,IHV,SIGMEA
       1 /  ' ', 21,   1,   1,   0,  21*0.1/
C
10     NUMT = 3
C
       DO 30 K=1,NPTS
       SIGMAY(K) = SIGMEA(K)
       IF(YMEAS(K) .NE. 0.)SIGMAY(K) = SIGMAY(K)/YMEAS(K)
30     Y(K) = ALOG( YMEAS(K) )
C
       CALL LSQFIT(X,Y,SIGMAY,NPTS,NUMT,IHV,MODE,YFIT,
       1 A,SIGMAA,FVALU,CHISQR)
C
       DO 50 J=1,NUMT
50     WRITE(IOUT,70)J,A(J),SIGMAA(J),FVALU(J),CHISQR
70     FORMAT(3X,'J= ',I2,2X,'A= ',F8.4,2X,'SIGMAA= ',F8.4,' FVALU= ',
       1 E12.4,2X,'CHISQR= ',E15.6)
       WRITE (IOUT,75)
75     FORMAT(10X,'COVARIANCE MATRIX',/)
       DO 80 I=1,NUMT
80     WRITE(IOUT,85)(BLK,I,J,SIGUV2(I,J),J=1,I)
85     FORMAT(2X,9(A1,'SIG(',I1,',',I1,')=',1PE13.6,1X)  )
C
C  FIND GAUSSIAN PARAMETERS
90     AGAS(3) = SQRT( -0.5/A(3) )
       AGAS(2) = -0.5*A(2)/A(3)
       XPI2 = 6.283185
       XEXP = A(1) + 0.5*( AGAS(2)/ AGAS(3) )**2
       CONST = SQRT( XPI2 * AGAS(3)**2)
       AGAS(1) = CONST* EXP(XEXP)
C STD DEV OF AGAS(3), AGAS(2), AGAS(1)
       DERIV(1) = 0.
       DERIV(2) = 0.
       DERIV(3) =  (-2.0*A(3) )**-1.5
       SIGAGA(3) = STDEV(DERIV)
       DERIV(1)  = 0.
       DERIV(2) = -0.5/A(3)
       DERIV(3) = 0.5*A(2)/A(3)**2
       SIGAGA(2) = STDEV(DERIV)
       DERIV(1) = AGAS(1)
       DERIV(2) =  -0.5*A(2)*AGAS(1)/A(3)
       DERIV(3) =  AGAS(1)*(AGAS(2)**2 - 0.5/A(3) )
       SIGAGA(1) = STDEV(DERIV)
C
C  FIND CHISQR FOR GAUSSIAN & STD DEV FOR FITTED VALUES
150    CONST = AGAS(1)/SQRT( XPI2* AGAS(3)**2)
```

Figure A3a. TEST3.F4 A Gaussian Fit

```
      GCHISQ = 0.
      DO 170 J=1,NPTS
      GEXP = ( X(J) - AGAS(2) )/ AGAS(3)
      GEXP = -0.5*GEXP**2
      GASFIT(J) = CONST * EXP(GEXP)
      DERIV(1) = GASFIT(J)
      DERIV(2) = X(J)*GASFIT(J)
      DERIV(3) = GASFIT(J)*( X(J)**2)
      SIGFIT(J) = STDEV(DERIV)
170   GCHISQ = GCHISQ + (GASFIT(J) - YMEAS(J) )**2
      NFREE = NPTS - NUMT
      GCHISR = GCHISQ/FLOAT(NFREE)
C
      DO 190 J=1,3
190   WRITE(IOUT,210)J,AGAS(J),SIGAGA(J),GCHISR
210   FORMAT(3X,'J= ',I2,2X,'GAUSSIAN A =',F9.3,2X,'STD DEV.= ',F9.6,
      1 2X,'GAUSSIAN FIT CHISQR= ',E15.6)
C
      DO 230 J = 1,NPTS,2
      ERROR = 100. *( GASFIT(J) -YMEAS(J))/YMEAS(J)
230   WRITE(IOUT,250)J,X(J),YMEAS(J),GASFIT(J),SIGFIT(J),ERROR
250   FORMAT(3X,'#= ',I2,1X,'X=',F8.3,' YMEAS=',F8.3,' GAUSSIAN FIT=',
      1 F8.3,2X,'STD. DEV. OF FIT=',F7.4,2X,' ERROR %= ',F8.4)
C
      END

      FUNCTION FCTN( X,I,J,JHV)
      DOUBLE PRECISION XX,F2
      XX = X
      F2 = XX**(J-1)
      FCTN = F2
      RETURN
      END

   J=  1  A=    4.5258  SIGMAA=    0.0004 FVALU=    0.0000E+00  CHISQR=     0.406618E+01
   J=  2  A=    0.0791  SIGMAA=    0.0001 FVALU=    0.2714E+00  CHISQR=     0.406618E+01
   J=  3  A=   -0.0198  SIGMAA=    0.0000 FVALU=    0.2121E+06  CHISQR=     0.406618E+01

          COVARIANCE MATRIX

   SIG(1,1)= 1.847967E-07
   SIG(2,1)=-4.242133E-09  SIG(2,2)= 1.491277E-08
   SIG(3,1)=-4.064626E-09  SIG(3,2)=-1.565241E-09  SIG(3,3)= 4.533413E-10

   J=  1  GAUSSIAN A = 1260.041  STD DEV.=  0.556189  GAUSSIAN FIT CHISQR=    0.406252E-01
   J=  2  GAUSSIAN A =    2.000  STD DEV.=  0.002483  GAUSSIAN FIT CHISQR=    0.406252E-01
   J=  3  GAUSSIAN A =    5.028  STD DEV.=  0.002707  GAUSSIAN FIT CHISQR=    0.406252E-01

   #=  1 X= -10.000 YMEAS=   6.000 GAUSSIAN FIT=   5.796  STD. DEV. OF FIT= 0.0170   ERROR %=  -3.3978
   #=  3 X=  -8.000 YMEAS=  14.000 GAUSSIAN FIT=  13.837  STD. DEV. OF FIT= 0.0282   ERROR %=  -1.1674
   #=  5 X=  -6.000 YMEAS=  28.000 GAUSSIAN FIT=  28.198  STD. DEV. OF FIT= 0.0372   ERROR %=   0.7055
   #=  7 X=  -4.000 YMEAS=  49.000 GAUSSIAN FIT=  49.055  STD. DEV. OF FIT= 0.0394   ERROR %=   0.1132
   #=  9 X=  -2.000 YMEAS=  73.000 GAUSSIAN FIT=  72.854  STD. DEV. OF FIT= 0.0372   ERROR %=  -0.1994
   #= 11 X=   0.000 YMEAS=  92.000 GAUSSIAN FIT=  92.367  STD. DEV. OF FIT= 0.0397   ERROR %=   0.3992
   #= 13 X=   2.000 YMEAS= 100.000 GAUSSIAN FIT=  99.971  STD. DEV. OF FIT= 0.0421   ERROR %=  -0.0292
   #= 15 X=   4.000 YMEAS=  92.000 GAUSSIAN FIT=  92.368  STD. DEV. OF FIT= 0.0387   ERROR %=   0.3999
   #= 17 X=   6.000 YMEAS=  73.000 GAUSSIAN FIT=  72.855  STD. DEV. OF FIT= 0.0392   ERROR %=  -0.1981
   #= 19 X=   8.000 YMEAS=  49.000 GAUSSIAN FIT=  49.056  STD. DEV. OF FIT= 0.0440   ERROR %=   0.1151
   #= 21 X=  10.000 YMEAS=  28.000 GAUSSIAN FIT=  28.198  STD. DEV. OF FIT= 0.0416   ERROR %=   0.7081
```

Figure A3b. TEST3.F4 A Gaussian Fit

```
C ************************ TEST4.F4 ***********************
C  SOLVES THE SIMPLE TRANSPORT EQUATION IN HORZ. PLANE.
C
       COMMON /MATR/ R(5,5,30)
       DIMENSION SIGUV2(10,10),DERIV(3)
       COMMON/LSQ/NTERMS,SIGUV2
       DIMENSION X(12),Y(12),SIGMAY(12),YFIT(12),A(10),SIGMAA(10),
       1 FVALU(10),XMAX(12),XMXFIT(12),SIGXMA(12),SIGFIT(12)
       DIMENSION RM11(12),RM12(12),RM15(12)
C
       DATA X/12*1.0/
       DATA XMAX/0.286,0.323,0.318,0.212,0.203,0.204,0.210,0.280,
       1 0.367,0.595,0.695,0.724/
       DATA SIGXMA/12* 0.0005/
       DATA RM11/1.0,1.0,0.86953,0.27873,-0.04124,-0.34625,-0.40928,
       1 -1.07818,-1.78713,-3.50042,-4.22513,-4.43354/
       DATA RM12/0.134,0.17817,0.19780,0.18399,0.22135,0.26032,
       1 0.27304,0.40807,0.55118,0.89704,1.04334,1.08541/
       DATA RM15/-1.5603,-1.69263,-1.60333,-0.88385,-0.63175,
       1 -0.40172,-0.36851,-0.01612,0.35738,1.25999,1.64179,1.75159/
C
       DATA NPTS,IOUT,MODE,IHV,BLK
       1 /   12,   1,  1,   0, ' '/
C
10     NUMT = 3
       DMOM = 0.12
C
       DO 30 K = 1,NPTS
       DO 30 J = 1,5
       DO 30 I = 1,5
30     R(I,J,K) = 0.0
C
       DO 40 K=1,NPTS
       R(1,1,K) = RM11(K)
       R(1,2,K) = RM12(K)
       R(1,5,K) = RM15(K)
       Y(K) = XMAX(K)**2 - (DMOM* R(1,5,K) )**2
40     SIGMAY(K) = 2. * XMAX(K) * SIGXMA(K)
C
       CALL LSQFIT (X,Y,SIGMAY,NPTS,NUMT,IHV,MODE,YFIT,A,SIGMAA,FVALU,
       1 CHISQR)
C
       WRITE (IOUT,270)
       DO 50 J=1,NUMT
50     WRITE(IOUT,70)J,A(J),SIGMAA(J),FVALU(J),CHISQR
70     FORMAT(3X,'J= ',I2,2X,'A= ',F8.6,2X,'SIGMAA= ',F8.6,2X,
       1 'FVALU= ',E12.4,2X,'CHISQR= ',E15.6)
       WRITE(IOUT,270)
C
       WRITE(IOUT,55)
55     FORMAT(10X,'COVARIANCE MATRIX',/)
       DO 58 I=1,NUMT
58     WRITE(IOUT,57)(BLK,I,J,SIGUV2(I,J),J=1,I)
57     FORMAT(2X,9(A1,'SIG(',I1,',',I1,')=',1PE13.6,1X)  )
       WRITE(IOUT,270)
C
C      FIND TWISS PARAMETERS AND STANDARD DEVIATIONS
60     XEPS =  A(1) * A(3) - A(2)**2
       IF(XEPS .GT. 0.)GO TO 75
```

Figure A4a. TEST4.F4 Horizontal Fitting

```
      WRITE(IOUT,65)
65    FORMAT(10X,'*** EMITTANCE NOT REAL ***')
      GO TO 280
75    EPS = SQRT(XEPS)
      ALPHA = -A(2)/EPS
      BETA = A(1)/EPS
      DERIV(1) = 0.5*A(3)/EPS
      DERIV(2) = -A(2)/EPS
      DERIV(3) = 0.5*A(1)/EPS
      SIGEPS = STDEV(DERIV)
      EPS3  = EPS**3
      DERIV(1) = 0.5*A(2)*A(3)/EPS3
      DERIV(2) = (-1.0/EPS) - (A(2)**2)/EPS3
      DERIV(3) = 0.5* A(1) *A(2)/EPS3
      SIGALP = STDEV(DERIV)
      DERIV(1) = (1.0/EPS) - (A(1)*A(3)*0.5)/EPS3
      DERIV(2) = A(1)*A(2)/EPS3
      DERIV(3) = (-0.5)*(A(1)**2)/EPS3
      SIGBET = STDEV(DERIV)
C
C   FIND CHISQR FOR BEAM SIZE & STD. DEVIATIONS FOR FITTED SIZE.
      BCHISQ = 0.
      DO 90 J=1,NPTS
      XMXFIT(J) = SQRT( YFIT(J) + ( DMOM*R(1,5,J)  )**2  )
      XMXFT2 = 2.0 * XMXFIT(J)
      DO 85 IK = 1,3
85    DERIV(IK) = (FCTN(0.,J,IK,IHV)  )/XMXFT2
      SIGFIT(J) = STDEV(DERIV)
90    BCHISQ = BCHISQ + (XMXFIT(J) - XMAX(J)  )**2
      NFREE = NPTS - NUMT
      BCHSQR = BCHISQ/FLOAT(NFREE)
C
100   WRITE(IOUT,110)ALPHA,SIGALP,BETA,SIGBET,EPS,SIGEPS,BCHSQR
110   FORMAT(4X,'ALPHA= ',F8.5,3X,'STD. DEVIATION = ',F8.5,/,
      1 4X,'BETA= ',F8.5,3X,'STD. DEVIATION= ',F8.5,/,
      2 4X,'EPSILON= ',F10.8,3X,'STD. DEVIATION= ',F10.8,/,
      3 4X, 'CHISQR = ',E15.6)
C
120   WRITE(IOUT,270)
C
      DO 130 J=1,NPTS
      ERROR = 100. * (XMXFIT(J) - XMAX(J) )/XMAX(J)
130   WRITE(IOUT,140) J,XMAX(J),XMXFIT(J),SIGFIT(J),ERROR
140   FORMAT(3X,'J =',I2,1X,'XMAX= ',F8.5,2X,'XMAX FIT= ',F8.5,2X,
      1 'STD. DEVIATION= ',F8.5,2X,'ERROR% =',F8.4)
270   FORMAT(1X,/)
280   END

      FUNCTION FCTN(X,I,J,JHV)
      COMMON /MATR/R(5,5,30)
      FCT = 0.
      IF(J.EQ. 1)FCT = R(1,1,I)**2
      IF(J.EQ. 2) FCT = 2. * R(1,1,I) * R(1,2,I)
      IF(J.EQ. 3)FCT = R(1,2,I)**2
      FCTN = FCT
      RETURN
      END
C
```

Figure A4b. TEST4.F4 Horizontal Fitting

```
J=  1  A=  .014373  SIGMAA=  .000114  FVALU=  0.0000E+00  CHISQR=  0.411397E+00
J=  2  A=  .070634  SIGMAA=  .000245  FVALU=  0.2240E+00  CHISQR=  0.411397E+00
J=  3  A=  .744365  SIGMAA=  .001665  FVALU=  0.4856E+06  CHISQR=  0.411397E+00

       COVARIANCE MATRIX

SIG(1,1)= 1.298575E-08
SIG(2,1)= 1.603331E-08  SIG(2,2)= 5.980903E-08
SIG(3,1)=-7.219002E-08  SIG(3,2)= 2.110552E-07  SIG(3,3)= 2.773617E-06

 ALPHA= -0.93478   STD. DEVIATION =  0.00421
 BETA=  0.19022   STD. DEVIATION=  0.00059
 EPSILON=  .07556254   STD. DEVIATION=  .00037361
 CHISQR =      0.102968E-06

J = 1 XMAX=  0.28600  XMAX FIT=  0.28583  STD. DEVIATION=  0.00028  ERROR% = -0.0426
J = 2 XMAX=  0.32300  XMAX FIT=  0.32315  STD. DEVIATION=  0.00029  ERROR% =  0.0476
J = 3 XMAX=  0.31800  XMAX FIT=  0.31828  STD. DEVIATION=  0.00027  ERROR% =  0.0896
J = 4 XMAX=  0.21200  XMAX FIT=  0.21168  STD. DEVIATION=  0.00017  ERROR% = -0.1503
J = 5 XMAX=  0.20300  XMAX FIT=  0.20237  STD. DEVIATION=  0.00020  ERROR% = -0.3113
J = 6 XMAX=  0.20400  XMAX FIT=  0.20434  STD. DEVIATION=  0.00022  ERROR% =  0.1689
J = 7 XMAX=  0.21000  XMAX FIT=  0.20993  STD. DEVIATION=  0.00022  ERROR% = -0.0346
J = 8 XMAX=  0.28000  XMAX FIT=  0.28020  STD. DEVIATION=  0.00024  ERROR% =  0.0704
J = 9 XMAX=  0.36700  XMAX FIT=  0.36705  STD. DEVIATION=  0.00022  ERROR% =  0.0147
J =10 XMAX=  0.59500  XMAX FIT=  0.59529  STD. DEVIATION=  0.00024  ERROR% =  0.0481
J =11 XMAX=  0.69500  XMAX FIT=  0.69494  STD. DEVIATION=  0.00030  ERROR% = -0.0089
J =12 XMAX=  0.72400  XMAX FIT=  0.72377  STD. DEVIATION=  0.00032  ERROR% = -0.0323
```

Figure A4c. TEST4.F4 Horizontal Fitting

```
C ************************ TEST5.F4 ***********************
C  SOLVES THE SIMPLE TRANSPORT EQUATION IN HORZ. & VERT. PLANES.
C
       COMMON /MATR/ R(5,5,30)
       DIMENSION X(12),Y(12),SIGMAY(12),YFIT(12),A(10),SIGMAA(10),
       1 FVALU(10),XMAX(12),XMXFIT(12),SIGXMA(12),YMAX(12),XY(12),
       2 SIGYMA(12),SIGFIT(12),DERIV(3)
       DIMENSION RM11(12),RM12(12),RM15(12),RM33(12),RM34(12),
       1 RM35(12)
C
       DATA X/12*1.0/
       DATA XMAX/0.286,0.323,0.318,0.212,0.203,0.204,0.210,0.280,
       1 0.367,0.595,0.695,0.724/
       DATA SIGXMA/12* 0.01/
       DATA YMAX/0.097,0.130,0.190,0.345,0.371,0.389,0.383,0.328,
       1 0.273,0.171,0.157,0.158/
       DATA SIGYMA/12* 0.01/
       DATA RM11/1.0,1.0,0.86953,0.27873,-0.04124,-0.34625,-0.40928,
       1 -1.07818,-1.78713,-3.50042,-4.22513,-4.43354/
       DATA RM12/0.134,0.17817,0.19789,0.18399,0.22135,0.26032,
       1 0.27304,0.40807,0.55118,0.89704,1.04334,1.08541/
       DATA RM15/-1.5603,-1.69263,-1.60333,-0.88385,-0.63175,
       1 -0.40172,-0.36851,-0.01612,0.35738,1.25999,1.64179,1.75159/
       DATA RM33/1.0,0.99993,1.13612,1.63344,1.56344,1.47543,1.43544,
       1 0.89495,0.33269,-1.02609,-1.60084,-1.76614/
       DATA RM34/0.134,0.17817,0.25080,0.43769,0.46214,0.47586,0.44787,
       1 0.38311,0.29326,0.07615,-0.01569,-0.04211/
       DATA RM35/12 * 0.0/
C
       DATA NPTS,IOUT,BLK
       1 /   12,   1, ' '/
C
10     NUMT = 3
       DMOM = 0.12
C
       DO 30 K = 1,NPTS
       DO 30 J = 1,5
       DO 30 I = 1,5
30     R(I,J,K) = 0.0
C
       DO 50 K=1,NPTS
       R(1,1,K) = RM11(K)
       R(1,2,K) = RM12(K)
       R(1,5,K) = RM15(K)
       R(3,3,K) = RM33(K)
       R(3,4,K) = RM34(K)
50     R(3,5,K) = RM35(K)
C
C
       IHV = 1
C ---------
       DO 410 NPLAN = 1,2
       MODE = 1
65     IF(NPLAN .EQ. 2) GO TO 110
       WRITE(IOUT,70)
70     FORMAT(1X,//,20X,'HORIZONTAL DATA ',/)
       DO 90 K = 1,NPTS
       Y(K) = XMAX(K)**2 - (DMOM* R(1,5,K) )**2
```

Figure A5a. TEST5.F4 Horizontal & Vertical Fitting

030

```
90      SIGMAY(K) = 2. * XMAX(K) * SIGXMA(K)
        GO TO 170
C
110     WRITE(IOUT,130)
130     FORMAT(1X,//,20X,'VERTICAL DATA',/)
        DO 150 K=1,NPTS
        Y(K) = YMAX(K)**2 - (DMOM* R(3,5,K) )**2
150     SIGMAY(K) = 2. * YMAX(K) * SIGYMA(K)
        IHV = 3
170     IF(MODE .EQ. 1)WRITE (IOUT,190)
190     FORMAT(5X,'INSTRUMENTAL WEIGHTING -- WEIGHT = 1/SIGMAY^2',/)
        IF(MODE .EQ. 2)WRITE(IOUT,195)
195     FORMAT(5X,'NO WEIGHTING FOR MODIFIED EQUATIONS',/)
C
C
        CALL LSQFIT (X,Y,SIGMAY,NPTS,NUMT,IHV,MODE,YFIT,A,SIGMAA,FVALU,
        1 CHISQR)
C
        DO 210 J=1,NUMT
210     WRITE(IOUT,230)J,A(J),SIGMAA(J),FVALU(J),CHISQR
230     FORMAT(3X,'J= ',I2,2X,'A= ',F8.6,2X,'SIGMAA= ',F8.6,2X,
        1 'FVALU= ',E12.4,2X,'CHISQR= ',E15.6)
        WRITE(IOUT,430)
C
C       FIND TWISS PARAMETERS AND STANDARD DEVIATIONS
250     XEPS =  A(1) * A(3) - A(2)**2
        IF(XEPS .GT. 0.)GO TO 290
        WRITE(IOUT,270)
270     FORMAT(10X,'*** EMITTANCE NOT REAL ***')
        GO TO 450
290     EPS = SQRT(XEPS)
        ALPHA = -A(2)/EPS
        BETA = A(1)/EPS
        DERIV(1) = 0.5*A(3)/EPS
        DERIV(2) = -A(2)/EPS
        DERIV(3) = 0.5*A(1)/EPS
        SIGEPS = STDEV(DERIV)
        EPS3  = EPS**3
        DERIV(1) = 0.5*A(2)*A(3)/EPS3
        DERIV(2) = (-1.0/EPS) - (A(2)**2)/EPS3
        DERIV(3) = 0.5* A(1) *A(2)/EPS3
        SIGALP = STDEV(DERIV)
        DERIV(1) = (1.0/EPS) - (A(1)*A(3)*0.5)/EPS3
        DERIV(2) = A(1)*A(2)/EPS3
        DERIV(3) = (-0.5)*(A(1)**2)/EPS3
        SIGBET = STDEV(DERIV)
C
C   FIND CHISQR FOR BEAM SIZE & STD. DEVIATIONS FOR FITTED SIZE.
        BCHISQ = 0.
        DO 330 J=1,NPTS
        XY(J) = XMAX(J)
        IF(IHV .EQ. 3)XY(J) = YMAX(J)
        XMXFIT(J) = SQRT( YFIT(J) + ( DMOM*R(IHV,5,J)  )**2  )
        XMXFT2 = 2.0 * XMXFIT(J)
        DO 310 IK = 1,3
310     DERIV(IK) = (FCTN(0.,J,IK,IHV)  )/XMXFT2
        SIGFIT(J) = STDEV(DERIV)
330     BCHISQ = BCHISQ + (XMXFIT(J) - XY(J)  )**2
        NFREE = NPTS - NUMT
```

Figure A5b. TEST5.F4 Horizontal & Vertical Fitting

031

```
      BCHSQR = BCHISQ/FLOAT(NFREE)
C
350   WRITE(IOUT,370)ALPHA,SICALP,BETA,SIGBET,EPS,SIGEPS,BCHSQR
370   FORMAT(4X,'ALPHA= ',F8.5,3X,'STD. DEVIATION = ',F8.5,/,
      1 4X,'BETA= ',F8.5,3X,'STD. DEVIATION= ',F8.5,/,
      2 4X,'EPSILON= ',F10.8,3X,'STD. DEVIATION= ',F10.8,/,
      3 4X, 'CHISQR = ',E15.6)
C
390   WRITE(IOUT,430)
      MODE = MODE + 1
      IF(MODE .EQ. 2)GO TO 65
C
410   CONTINUE
C ---------
430   FORMAT(1X,/)
450   END

      FUNCTION FCTN(X,I,J,IHV)
C  IHV = 1 FOR HORZ; IHV =3 FOR VERT.
      COMMON /MATR/R(5,5,30)
      FCT = 0.
      IF(J.EQ. 1)FCT = R(IHV,IHV,I)**2
      IF(J.EQ. 2) FCT = 2. * R(IHV,IHV,I) * R(IHV,IHV+1,I)
      IF(J.EQ. 3)FCT = R(IHV,IHV+1,I)**2
      FCTN = FCT
      RETURN
      END

                   HORIZONTAL DATA

   INSTRUMENTAL WEIGHTING -- WEIGHT = 1/SIGMAY^2

 J=  1  A=  .014373  SIGMAA=  .002279  FVALU=  0.0000E+00  CHISQR=   0.102849E-02
 J=  2  A=  .070634  SIGMAA=  .004891  FVALU=  0.2240E+00  CHISQR=   0.102849E-02
 J=  3  A=  .744365  SIGMAA=  .033308  FVALU=  0.4856E+06  CHISQR=   0.102849E-02

  ALPHA= -0.93478   STD. DEVIATION =  0.03417
  BETA=  0.19022   STD. DEVIATION=  0.01189
  EPSILON=  .07556253   STD. DEVIATION=  .00747223
  CHISQR =    0.102968E-06

                   HORIZONTAL DATA

   NO WEIGHTING FOR MODIFIED EQUATIONS

 J=  1  A=  .014373  SIGMAA=  .000073  FVALU=  0.0000E+00  CHISQR=   0.102849E-02
 J=  2  A=  .070634  SIGMAA=  .000157  FVALU=  0.2240E+00  CHISQR=   0.102849E-02
 J=  3  A=  .744365  SIGMAA=  .001068  FVALU=  0.4856E+06  CHISQR=   0.102849E-02

  ALPHA= -0.93478   STD. DEVIATION =  0.00270
  BETA=  0.19022   STD. DEVIATION=  0.00038
  EPSILON=  .07556253   STD. DEVIATION=  .00023963
  CHISQR =    0.102968E-06
```

Figure A5c. TEST5.F4 Horizontal & Vertical Fitting

```
                 VERTICAL DATA

 INSTRUMENTAL WEIGHTING -- WEIGHT = 1/SIGMAY^2

J=  1  A=  .010983  SIGMAA=  .000817  FVALU=  0.0000E+00  CHISQR=   0.[illegible]698E-03
J=  2  A= -.074361  SIGMAA=  .007543  FVALU=  0.1730E+02  CHISQR=   0.[illegible]E-03
J=  3  A= 1.022262  SIGMAA=  .047960  FVALU=  0.5016E+06  CHISQR=   0.[illegible]98E-03

 ALPHA=  0.98514   STD. DEVIATION =  0.13052
 BETA=  0.14550   STD. DEVIATION=  0.00850
 EPSILON=  .07548274   STD. DEVIATION=  .00450265
 CHISQR =    0.906142E-07

                 VERTICAL DATA

 NO WEIGHTING FOR MODIFIED EQUATIONS

J=  1  A=  .010983  SIGMAA=  .000025  FVALU=  0.0000E+00  CHISQR=   0.[illegible]5693E-03
J=  2  A= -.074361  SIGMAA=  .000227  FVALU=  0.1739E+02  CHISQR=   0.9[illegible]698E-03
J=  3  A= 1.022262  SIGMAA=  .001443  FVALU=  0.5016E+06  CHISQR=   0.[illegible]E-03

 ALPHA=  0.98514   STD. DEVIATION =  0.00393
 BETA=  0.14550   STD. DEVIATION=  0.00026
 EPSILON=  .07548274   STD. DEVIATION=  .00013551
 CHISQR =    0.906142E-07
```

Figure A5d.. TEST5.F4 Horizontal & Vertical Fitting

033

Appendix B -- The Necessary Fortran Source Routines

The necessary routines are:

1) LSQFIT -- least square fit subroutine
2) FCTN -- a function routine
3) MATINV -- inverts a symmetric two-dimensional matrix
4) STDEV -- calculates standard deviations for modified equations.

The Fortran Source routines are included:

```
C ********************** REGRE.F4 ****************************:
C 1/17/84 -- REMOVED REGRE.F4 ROUTINE & ADDED STDEV FUNCTION.
C
C  THIS IS A LIST OF SUBROUTINES THAT CAN BE USED TO CALCULATE THE
C  EMITTANCE OF A BEAM FROM FLAG SIZES.  BESIDES DOING REGRESSION
C  ANALYSIS, IT CALCULATES STD. DEVIATIONS & CORRELATIONS.
C
C         THE ROUTINES ARE FROM:  BEVINGTON "DATA REDUCTION & ERROR
C                 ANALYSIS FOR THE PHYSICAL SCIENCES".  .
C
C        STDEV -- ADDED
C        LSQFIT -- MODIFIED FROM LEGFIT IN BEVINGTON.
C
C
C       SUBROUTINE LSQFIT
C
C  PURPOSE
C MAKE A LEAST-SQUARES FIT TO DATA WITH A LINEAR EQUATION WITH
C       CONSTANT COEFF.
C
C  USAGE
C       CALL LSQFIT(X,Y,SIGMAY,NPTS,NUMT,IHV,MODE,
C                 YFIT,A,SIGMAA,FVALU,CHISQR)
C
C
C       DESCRIPTION OF INPUT PARAMETERS
C  X -- ARRAY OF Z LOCATIONS OR QUAD VALUES(INDEP. VARIABLE)
C  Y   -- ARRAY OF DATA POINTS FOR DEPENDENT VARIABLE.
C  SIGMAY -- ARRAY OF STANDARD DEVIATIONS FOR Y DATA POINTS.
C  NPTS  - NUMBER OF PAIRS OF DATA POINTS.
C  NUMT --  NUMBER OF TERMS IN THE FUNCTION
C               NUMT=3 TO SOLVE FOR A1,A2,A3
C               NUMT=4 TO SOLVE FOR A1,A2,A3,AND A4
C  IHV -- = 1 FOR HORZ. & = 3 FOR VERTICAL.
C  MODE -- DETERMINES MODE OF WEIGHTING LEAST-SQUARES FIT.
C               +2 (SAME AS INSTRUMENTAL TO CALCULATE COEFF BUT STD.
C                  DEVIATIONS CALCULATED FROM CHISQR(AS MODE= 0)
C               +1 (INSTRUMENTAL) WEIGHT(I) = 1./SIGMAY(I)**2
C                0 (NO WEIGHTING) WEIGHT(I) = 1.
C               -1 (STATISTICAL) WEIGHT(I) = 1./Y(I)
C
C          DESCRIPTION OF OUTPUT PARAMETERS
C  YFIT -- ARRAY OF CALCULATED VALUES OF Y
C  A   -- ARRAY OF COEFFICIENTS OF POLYNOMIAL
C  SIGMAA -- ARRAY OF STANDARD DEVIATIONS FOR COEFFICIENTS.
C  SIGUV2 -- ARRAY OF VARIANCES & COVARIANCES(U,V) -- AVAILABLE ONLY
C               THRU COMMON/LSQ/
C  FVALU -- ARRAY OF FVALUES (NORMALIZED CHANGE OF CHI SQUARE) FOR
C               EACH COEFFICIENT.
C  CHISQR -- REDUCED CHI SQUARE FOR FIT.
C
C
C       SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED.
C         MATINV(ARRAY,NTERMS,DET)
C               --INVERTS A SYMMETRIC TWO-DIMENSIONAL MATRIX OF DEGREE
C                 NTERMS AND CALCULATES ITS DETERMINANT.
C
```

Figure B1. The Fortran Source Routines

```
C   COMMENTS:
C      DIMENSION STATEMENT VALID FOR NPTS UP TO ~~30~~ 40 AND NUMT UP TO 10.
C
        SUBROUTINE LSQFIT(X,Y,SIGMAY,NPTS,NUMT,IHV,MODE,
        1 YFIT,A,SIGMAA,FVALU,CHISQR)
C
        DOUBLE PRECISION P,BETA,ALPHA,CHISQ
        COMMON/LSQ/NTERMS,SIGUV2
        DIMENSION X(1),Y(1),SIGMAY(1),YFIT(1),
        1  A(1),SIGMAA(1),FVALU(1),SIGUV2(10,10)
        DIMENSION WEIGHT(~~30~~ 40),P(~~30~~ 40,10),BETA(10),ALPHA(10,10)
C
C    ACCUMULATE WEIGHTS AND COEFFICIENTS
C
11      NTERMS = 1
        NCOEFF = 1
        JMAX = NUMT
C ---------
20      DO 40 I =1,NPTS
21      IF(MODE)22,27,29
22      IF(Y(I) )25,27,23
23      WEIGHT(I) = 1. / Y(I)
        GO TO 31
25      WEIGHT(I) = 1. /(-Y(I) )
        GO TO 31
27      WEIGHT(I) = 1.
        GO TO 31
29      WEIGHT(I) = 1. / SIGMAY(I)**2
C ---
31      DO 36 L=1,NUMT
        TEMP  = FCTN(X(I),I,L,IHV)
36      P(I,L) = TEMP
C ---
40      CONTINUE
C ---------
C
C   ACCUMULATE MATRICES ALPHA AND BETA
C
C --------
51      DO 54 J=1,NTERMS
        BETA(J) = 0.
        DO 54 K=1,NTERMS
54      ALPHA(J,K) = 0.
C --------
61      DO 66 I=1,NPTS
        DO 66 J=1,NTERMS
        BETA(J) = BETA(J) + P(I,J)*Y(I)*WEIGHT(I)
        DO 66 K = J,NTERMS
        ALPHA(J,K) = ALPHA(J,K) + P(I,J) * P(I,K)*WEIGHT(I)
66      ALPHA(K,J) = ALPHA(J,K)
C ---------
C
C
C       INVERT CURVATURE MATRIX ALPHA
C
C ---------
91      DO 95 J=1,JMAX
        A(J) = 0.
        SIGMAA(J) = 0.
```

Figure B2. The Fortran Source Routines

```
95        CONTINUE
C ---
          DO 97 I =1,NPTS
97        YFIT(I) = 0.
C ---
C
101       CALL MATINV(ALPHA,NTERMS,DET)
C
          IF(DET)111,103,111
103       CHISQR = 0.
          GO TO 170
C
C   CALCULATE COEFFICIENTS,FIT,AND CHI SQUARE
C
C ---------
111       DO 115 J=1,NTERMS
          DO 113 K=1,NTERMS
113       A(J) = A(J) + BETA(K) * ALPHA(J,K)
C -----
          DO 115 I=1,NPTS
115       YFIT(I) = YFIT(I) + A(J) * P(I,J)
C ---------
121       CHISQ = 0.
          DO 123 I =1,NPTS
123       CHISQ = CHISQ + ( Y(I) - YFIT(I) )**2 * WEIGHT(I)
          FREE = NPTS - NCOEFF
          CHISQR = CHISQ/FREE
C
C  TEST FOR END OF FIT
C
132       IF (NCOEFF - 1)137,137,141
137       NTERMS = NTERMS + 1
138       NCOEFF = NCOEFF + 1
          CHISQ1 = CHISQ
          GO TO 51
C
C
141       FVALUE = ( CHISQ1 -CHISQ)/CHISQR
          FVALU(NCOEFF) = FVALUE
          IF (NTERMS -JMAX) 137,151,151
149       GO TO 51
C
C   CALCULATE REMAINDER OF OUTPUT
C
151       IF (MODE)152,154,152
152       VARNCE = 1.
          IF(MODE .EQ. 2)VARNCE = CHISQR
          GO TO 155
154       VARNCE = CHISQR
155       DO 156 J=1,NTERMS
156       SIGMAA(J) = DSQRT( VARNCE*ALPHA(J,J) )
          DO 300 IU = 1,NTERMS
          DO 300 IV = 1,NTERMS
300       SIGUV2(IU,IV) = VARNCE*ALPHA(IU,IV)
161       GO TO 170
170       RETURN
          END
```

Figure B3. The Fortran Source Routines

```
C
C       FUNCTION FCTN (BEAM TRANSPORT)
C
C  PURPOSE-- EVALUATE TERMS OF BEAM SIZE EQUATIONS FOR HORZ. & VERT.
C
C  HORZ:
C (XMAX)^2 -R(1,5)^2*DMOM^2 = R(1,1)^2 *A(1) + 2R(1,1)*R(1,2)*A(2)+
C              + R(1,2)^2 *A(3)
C  WHERE A(1) = SIG(1,1); A(2) = SIG(2,1); A(3)= SIG(2,2)
C
C  VERT.
C (YMAX)^2 = A(1) *R(3,3)^2 + 2R(3,4)*R(3,3)*A(2) + A(3)*R(3,4)^2
C
C  WHERE A(1) = SIG(3,3); A(2)= SIG(4,3); A(3)= SIG(4,4)
C
C   USAGE--    RESULT = FCTN(X,I,J,JHV)
C
C  DESCRIPTION OF PARAMETERS:
C  X -- ARRAY OF DATA POINTS FOR INDEP. VARIABLE( FLAG "Z" LOCATION)
C  I -- INDEX OF DATA POINTS (1-30 FOR 30 FLAGS)
C  J -- INDEX OF TERM IN FUNCTION (1-4)
C  JHV -- =1 FOR HORZ. =3 FOR VERT.
C  R(5,5,30) -- TRANSPORT MATRIX TO EACH OF 30 FLAG LOCATIONS
C
C
        FUNCTION FCTN(X,I, J, JHV)
        COMMON /MATR/ R(5,5,30), DMOM
        DIMENSION X(1)
C
        IHV = JHV
        IF( (J.LT.1) .OR. (J.GT. 4) )GO TO 50
C
        GO TO (10,15,20,25,50)J
 10     FCT = R(IHV,IHV,I)**2
        GO TO 40
 15     FCT = 2.0* R(IHV,IHV,I)* R(IHV,IHV+1,I)
        GO TO 40
 20     FCT = R(IHV,IHV+1,I)**2
        GO TO 40
 25     FCT = R(IHV,5,I)**2
C
 40     FCTN = FCT
 50     RETURN
        END
```


Figure B4. The Fortran Source Routines

```
C    FUNCTION STDEV (DERIVATIVES)
C
C   PURPOSE: CALCULATE THE STD. DEV. OR  SQRT(VARIANCE) FOR MODIFIED
C   EQUATIONS.
C
C   INPUT PARAMETERS:
C   NTERMS -- NUMBER OF TERMS IN THE EQUATION: (FROM LSQFIT)
C
C    Y =  A1X1(Z) + A2X2(Z) + A3X3(Z) ...
C
C    SIGUV2 -- ARRAY OF VARIANCES & COVARIANCES OBTAINED FROM LSQFIT;
C                 IN THE ORDER OF THE COEFFICIENTS OF THE EQUATION.
C
C                 SIGUV2(1,1)     SIGUV2(1,2)     SIGUV2(1,3)
C    SIGUV2 =     SIGUV2(2,1)     SIGUV2(2,2)     SIGUV2(2,3)
C                 SIGUV2(3,1)     SIGUV2(3,2)     SIGUV2(3,3)
C
C   WHERE:
C        SIGUV2(1,1) = VARIANCE OF A1 OR (STD. DEVIATION OF A1)**2
C        SIGUV2(2,2) =  "    "      A2  "    "      "    "   A2    "
C        SIGUV2(3,3) = "      "     A3  "     "     "    "   A3    "
C        SIGUV2(1,2) = COVARIANCE BETWEEN A1 AND A2
C        SIGUV2(1,3) = COVARIANCE BETWEEN A1 AND A3... ETC.
C   SIGUV2 IS A SYMMETRICAL ARRAY -- SIGUV2(1,2) = SIGUV2(2,1)
C
C
C        DERIV -- VECTOR OF DIMENSION NTERMS OF THE PARTIAL DERIVATIVES--
C
C                 IF  X = F(A1,A2,A3) THEN:
C
C         DERIV(1) = (PARTIAL DERIVATIVE OF X)/(PARTIAL DERIVATIVE OF A1)
C         DERIV(2) = "      "     "     "   "    "     "    "    "  A2
C         DERIV(3) = "        "          "          "         "      " A3
C
C
C        OUTPUT PARAMETER
C        STDEV -- STANDARD DEVIATION OR SQRT(VARIANCE) OF X FOR
C                X = F(A1,A2,A3...)
C
C   PROCEDURE:
C   (STD. DEV)**2 = VARIANCE = ((DERIV))T * ((SIGUV2)) * ((DERIV))
C   WHERE ((DERIV))T IS THE TRANSPOSE OF THE DERIV VECTOR.
C
        FUNCTION STDEV(DERIV)
C
        COMMON/LSQ/ NTERMS,SIGUV2(10,10)
        DIMENSION DERIV(10),WORK(10)
C
        STDEV = 0.
        VARIAN = 0.
        DO 10 J=1,NTERMS
10      WORK(J) = 0.
C
        DO 20 J= 1,NTERMS
        DO 20 I = 1,NTERMS
20      WORK(J) = WORK(J) + SIGUV2(J,I) * DERIV(I)
C
        DO 40 J = 1,NTERMS
40      VARIAN = VARIAN + DERIV(J) * WORK(J)
```

Figure B5. The Fortran Source Routines

```
STDEV = SQRT(VARIAN)
RETURN
END
```

Figure B6. The Fortran Source Routines

```
C  SUBROUTINE MATINV
C
C  PURPOSE -- INVERT A SYMMETRIC MATRIX & CALCULATE ITS DETERMINANT.
C
C  USAGE:
C   CALL MATINV ( ARRAY,NORDER, DET)
C
C  DESCRIPTION OF PARAMETERS
C  ARRAY -- INPUT MATRIX WHICH IS REPLACED BY ITS INVERSE
C  NORDER -- DEGREE OF MATRIX (ORDER OF DETERMINANT)
C  DET -- DETERMINANT OF INPUT MATRIX
C
C  SUBROUTINES REQUIRED -- NONE
C
C  COMMENTS: -- DIMENSION STATEMENT VALID FOR NORDER UP TO 10
C
C
        SUBROUTINE MATINV (ARRAY, NORDER, DET)
        DOUBLE PRECISION ARRAY,AMAX,SAVE
        DIMENSION ARRAY(10,10), IK(10),JK(10)
C
10      DET = 1.
C
C ------------------
11      DO 100 K= 1,NORDER
C
C   FIND LARGEST ELEMENT ARRAY(I,J) IN REST OF MATRIX
C
        AMAX = 0.
C ------
21      DO 30 I=K, NORDER
        DO 30 J=K, NORDER
23      IF( DABS(AMAX) - DABS( ARRAY(I,J) ) )24,24,30
24      AMAX = ARRAY(I,J)
        IK(K) = I
        JK(K) = J
30      CONTINUE
C ------
C
C  INTERCHANGE ROWS & COLUMNS TO PUT AMAX IN ARRAY(K,K)
C
31      IF(AMAX)41, 32,41
32      DET = 0.
        GO TO 140
41      I = IK(K)
        IF(I-K) 21,51,43
C ---
43      DO 50 J= 1, NORDER
        SAVE = ARRAY(K,J)
        ARRAY(K,J) = ARRAY(I,J)
50      ARRAY(I,J) = -SAVE
C ---
51      J = JK(K)
        IF(J-K) 21,61,53
C ---
53      DO 60 I =1,NORDER
        SAVE = ARRAY(I,K)
        ARRAY(I,K) = ARRAY(I,J)
60      ARRAY(I,J) = -SAVE
```

Figure B7. The Fortran Source Routines

```
C ---
C
C  ACCUMULATE ELEMENTS OF INVERSE MATRIX
C
C ---
61       DO 70 I=1, NORDER
         IF(I-K) 63,70,63
63       ARRAY(I,K) = -ARRAY(I,K) / AMAX
70       CONTINUE
C ---
C ------
71       DO 80 I=1,NORDER
         DO 80 J=1,NORDER
         IF(I-K) 74,80,74
74       IF(J-K) 75,80,75
75       ARRAY(I,J) = ARRAY(I,J) + ARRAY(I,K) * ARRAY(K,J)
80       CONTINUE
C ------
C ---
81       DO 90 J=1, NORDER
         IF(J-K) 83,90,83
83       ARRAY(K,J) = ARRAY(K,J) / AMAX
90       CONTINUE
C ---
         ARRAY(K,K) = 1.0 / AMAX
100      DET = DET * AMAX
C ------------------
C
C   RESTORE ORDERING OF MATRIX
C
C ------------
101      DO 130 L=1,NORDER
         K= NORDER -L + 1
         J = IK(K)
         IF(J-K) 111,111,105
105      DO 110 I =1,NORDER
         SAVE = ARRAY(I,K)
         ARRAY(I,K) = -ARRAY(I,J)
110      ARRAY(I,J) = SAVE
C
111      I = JK(K)
         IF(I-K) 130,130,113
113      DO 120 J=1, NORDER
         SAVE = ARRAY(K,J)
         ARRAY(K,J) = -ARRAY(I,J)
120      ARRAY(I,J) = SAVE
130      CONTINUE
C ------------
140      RETURN
         END
```

Figure B8. The Fortran Source Routines