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THE TUNE SPLITTING and the EMITTANCE CHANGE CAUSED by RANDOM TWISTS of QUADRUPOLES and RANDOM VERTICAL DISPLACEMENTS of SEXTUPOLES in the AGS Booster

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1. INTRODUCTION

Assume that a sextupole magnet is displaced vertically by Δy. This means that in the magnetic field expansions

$$B_x = B_o \left[a_o + a_1 x + b_1 y + b_2 2xy + \ldots\right] , \tag{1.1}$$

and

$$B_y = B_o \left[b_o + b_1 x - a_1 y + b_2 (x^2 - y^2) + \ldots\right] , \tag{1.2}$$

the y- coordinate is replaced by $-y + \Delta y$.

As the result the a_1 - term gets the contribution

$$a_1 x \rightarrow \left(a_1 + 2 b_2 \Delta y\right) x , \tag{1.3}$$

(from the sextupole term - $b_2 2xy$).

Similarly, assume that the ordinary quads are randomly twisted by a small angle $\Delta\theta$. This induces the following transformation of the coordinates, and the field components itself:

$$\begin{aligned} x &\rightarrow x + \Delta\theta \, y , \\ y &\rightarrow y - \Delta\theta \, x , \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} B_x &\rightarrow B_x + \Delta\theta \, B_y , \\ B_y &\rightarrow B_y - \Delta\theta \, B_x . \end{aligned} \tag{1.5}$$

We infer from this the following change in the a_1, coming from the quadrupole terms - b_1y, b_1x,

$$a_1 \rightarrow a_1 - 2b_1\Delta\theta \ . \tag{1.6}$$

If both factors, vertical displacements and twists, are present at the same time one gets for the a_1-term

$$a_1 \rightarrow a_1 - 2b_1\Delta\theta + 2b_2\Delta y \ . \tag{1.7}$$

The $\Delta\theta$-terms are located at the regular quads while the Δy-terms are located at the sextupoles. In case of the AGS Booster the distance between the both is .75 m, which is small in comparison to the characteristic length, the circumference, C = 201.78m[1], and will be ignored. Thus we assume, for simplicity, that the both errors have the same locations, at the main quads:

$$s_1, s_2, ..., s_N \ , \quad N = 48, \quad (\textit{see Ref.}1). \tag{1.8}$$

Furthermore, for purpose of the present note, we assume that the whole skew-quadrupole field is produced by the above two errors alone. This would mean that the original a_1-term is corrected to zero, so are the systematic errors in the vertical placement of the sextupoles and the rolls of the regular quadrupoles. Hence, our goal is to estimate the tune-splitting produced by the skew-quadrupole term of the form

$$a_1 = 2b_2\Delta y - 2b_1\Delta\theta \ , \tag{1.9}$$

where Δy and $\Delta\theta$ are mutually independent random variables distributed normally around the ring. Thus we assume that at each quad the following conditions are satisfied

$$\begin{gathered} < \Delta y > = < \Delta\theta > = 0 \ , \\ < \Delta y \ \Delta\theta > = 0 \ , \end{gathered} \tag{1.10}$$

while the dispersions

$$\langle (\Delta y)^2 \rangle , \quad \langle (\Delta\theta)^2 \rangle \ ,$$

are assumed to be independent of the quad's locations.

This problem was considered earlier by S. Peggs[4] and we will follow his method.

2. THE MODEL

We shall use the 4 x 4 matrix formalism [2,6] describing coasting beam in presence of the linear coupling. Passing to the circular representation (normalized coordinates) the single-turn transfer matrix can be presented as follows

$$\mathring{T} = \begin{bmatrix} \mathring{M} & \mathring{n} \\ \mathring{m} & \mathring{N} \end{bmatrix} = \begin{bmatrix} B_x & O \\ O & B_y \end{bmatrix} \begin{bmatrix} M & n \\ m & N \end{bmatrix} \begin{bmatrix} B_x^{-1} & O \\ O & B_y^{-1} \end{bmatrix}, \tag{2.1}$$

where the 2 $\times$ 2 submatrices $B_{x,y}$, $B_{x,y}^{-1}$ are

$$B_{x,y} = \begin{bmatrix} \beta^{-1/2} & O \\ \alpha\beta^{-1/2} & \beta^{1/2} \end{bmatrix}_{x,y}, \tag{2.2}$$

and

$$B_{x,y}^{-1} = \begin{bmatrix} \beta^{1/2} & O \\ -\alpha\beta^{-1/2} & \beta^{-1/2} \end{bmatrix}_{x,y}. \tag{2.3}$$

The submatrices $\mathring{n}$ *and* $\mathring{m}$, which describe the coupling, are, (cf. formulae (6-17) of (Ref. 6),

$$\mathring{n} = R(\mu_x)\, F, \qquad \mathring{m} = R(\mu_y)\, F\,, \tag{2.4}$$

where

$$F = \sum_{K=1}^{N} q_k\, R\left(-\mu_x^k\right) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} R\left(\mu_y^k\right)\,, \tag{2.5}$$

and

$$\mu_{x,y}^k = \int_o^{s_k} \frac{ds}{\beta_{x,y}} \text{ - are phase advances,} \tag{2.6}$$

and the strengths of the skew-quadrupole fields

$$q_k = \frac{l}{\rho} \sqrt{\beta_x \, \beta_y} \left(2b_2 \Delta y - 2b_1 \Delta \theta\right)\Big|_{s=sk} . \tag{2.7}$$

Here $(\alpha, \beta, \gamma, \mu)_{x,y}$ are Twiss parameters and the tunes of the perfect lattice, and $R(\phi)$ are rotations

$$R(\phi) = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} . \tag{2.8}$$

A crucial role is played by the matrix

$$\bar{\mathring{m}} + \mathring{n} = R(\mu_x)\, F - F\, R(-\mu_y) , \tag{2.9}$$

where $\bar{A}$ denotes the symplectic conjugate matrix to the matrix A, viz.

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ,$$

then

$$\bar{A} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .$$

It was shown by Edwards and Teng[3], and by Talman[5], that one may exactly decouple the motion by passing to, so called, normal modes

$$T = R \begin{bmatrix} A & O \\ O & B \end{bmatrix} R^{-1}, \tag{2.10}$$

where A, B symplectic submatrices are given in terms of the new Twiss parameters and new tunes $(\alpha, \beta, \gamma, \mu)_{A,B}$. In case of a single-turn transfer matrix T we have

$$A = M + (t + \delta)^{-1}\left(\bar{m} + n\right) m = \tag{2.11}$$

$$= \begin{bmatrix} \cos\mu_A + \alpha_A \sin\mu_A , & \beta_A \sin\mu_A \\ -\gamma_A \sin\mu_A , & \cos\mu_A - \alpha_A \sin\mu_A \end{bmatrix} ,$$

and

$$B = N - (t + \delta)^{-1}\left(\bar{m} + n\right) m = \tag{2.13}$$

$$= \begin{bmatrix} \cos \mu_B + \alpha_B \sin \mu_B , & \beta_B \sin \mu_B \\ -\gamma_B \sin \mu_B , & \cos \mu_B - \alpha_B \sin \mu_B \end{bmatrix} , \tag{2.14}$$

where

$$t = \frac{1}{2} tr(M - N) , \tag{2.15}$$

and

$$\delta = \frac{1}{2} tr(A - B) . \tag{2.16}$$

In particular, we have the relations

$$\frac{1}{2}\ tr(A) = \cos \mu_A = \cos (2\pi\nu_A) , \qquad (\nu_A \equiv Q_A) , \tag{2.17}$$

and similar for the B matrix, and

$$\begin{aligned} &\frac{1}{2} tr(M) = \cos \mu_x , \\ &\frac{1}{2} tr(N) = \cos \mu_y , \quad \mu_{x,y} = 2\pi\nu_{x,y} . \end{aligned} \tag{2.18}$$

In the Thin Lens Model the tunes coincide, to the first order, with those of perfect lattice [cf. (2.4)]. The above quantities are connected by the equality[3],

$$\left(\cos \mu_A - \cos \mu_B \right)^2 = \left(\cos \mu_x - \cos \mu_y \right)^2 + Det\left(\bar{m} + n \right) . \tag{2.19}$$

3. THE TUNE SPLITTING

Assuming that the coupling is small one may write

$$
\begin{aligned}
\nu_x &= \nu_o + \frac{1}{2}\Delta\nu \,, \\
\nu_y &= \nu_o - \frac{1}{2}\Delta\nu \,, \\
\nu_A &\equiv \nu_o + \frac{1}{2}\Delta Q \,, \qquad (\nu_o = Q_o) \\
\nu_B &= \nu_o - \frac{1}{2}\Delta Q \,,
\end{aligned}
\tag{3.1}
$$

where $\Delta\nu$ and ΔQ are small. It follows now from the previous equality that

$$\Delta Q \approx \left[(\Delta\nu)^2 + \frac{Det\left(\bar{m} + n \right)}{4\pi^2 \sin^2(2\pi Q_o)} \right]^{\frac{1}{2}} . \tag{3.2}$$

On the resonance: $\nu_x = \nu_y = \nu_o$, $\Delta\nu = 0$, the tune-splitting reaches its minimum value

$$\Delta Q_{\min} \approx (2\pi |\sin (2\pi Q_o) |)^{-1} \left[Det\left(\bar{m} + n \right) \right]^{\frac{1}{2}} \Big|_{\nu_x = \nu_y = Q_o} . \tag{3.3}$$

Putting $\nu_x = \nu_y = Q_o$ in the formula (2.9) one finds, using (2.5) and (2.8), the result

$$\bar{\mathring{m}} + \mathring{n} \Big|_{\nu_x = \nu_y = Q_o} \approx \sin (2\pi Q_o) \begin{bmatrix} p & r \\ -r & p \end{bmatrix} , \tag{3.4}$$

where the quantities -p and -r are given by the expressions

$$p = \sum_{k=1}^{N} q_k \cos\left(\mu_y^k - \mu_x^k\right) , \tag{3.5}$$

and

$$r = \sum_{k=1}^{N} q_k \sin\left(\mu_y^k - \mu_x^k\right) . \tag{3.6}$$

Hence, taking into account the equalities

$$Det\left(\bar{m} + n \right) = Det\left(\bar{\mathring{m}} + \mathring{n} \right) \approx \sin^2 (2\pi Q_o) \left(p^2 + r^2 \right) , \tag{3.7}$$

one gets the following formula for the minimal tune splitting produced by the skew-quadrupole errors

$$\Delta Q_{\min} \approx \frac{1}{2\pi} \left(p^2 + r^2 \right)^{\frac{1}{2}} . \tag{3.8}$$

In view of the random character of those errors it is useful to evaluate random mean square of this quantity. This can be done by taking into account the independence of Δy and $\Delta\theta$ random variables

$$\begin{aligned} &< q_k > = 0 , \\ &< q_k \, q_l > = \delta_{kl} < q_k^2 > , \\ &< \cos^2 \mu^k > = < \sin^2 \mu^k > = \frac{1}{2} , \quad k, l = 1, \ldots , N . \end{aligned} \tag{3.9}$$

Assuming further that all the quads are identical and are placed in a lattice composed of the regular FODO cells one finds, using (3.5) and (3.6)

$$< p^2 > = < r^2 > = \frac{N}{2} < q^2 > , \tag{3.10}$$

and, as the result

$$(\Delta Q_{\min})_{rms} = \frac{N^{\frac{1}{2}}}{2\pi} < q^2 >^{\frac{1}{2}} , \tag{3.11}$$

where, according to the formula (2.7), we have

$$< q^2 >^{\frac{1}{2}} = \frac{lb_1}{\rho} \left(\beta_x \beta_y\right)^{\frac{1}{2}} \left[\left(\frac{2b_2}{b_1}\right)^2 \langle (\Delta y)^2 \rangle + 4 \langle (\Delta\theta)^2 \rangle \right]^{\frac{1}{2}}. \tag{3.12}$$

Assuming that the sextupoles are correcting the natural chromaticity of the adjacent quadrupoles one gets the following relation between their strengths and the dispersion

$$2b_2 = \frac{b_1}{\eta} . \tag{3.13}$$

To estimate the order of magnitude of the tune-splitting we use some typical values for the dispersion and the β functions:

$$\begin{aligned} \beta_x \approx \beta_y \approx \beta_{typ} \approx \frac{R}{Q_o} , \\ \eta \approx < \eta_{typ} > \approx \frac{\beta_{typ}}{Q_o} \approx \frac{R}{Q_o^2} . \end{aligned} \tag{3.14}$$

Upon substitution of these relations into the formula (2.24) one gets finally

$$(\Delta Q_{\min})_{rms} =$$

$$\approx \frac{K}{2\pi} N^{\frac{1}{2}} \beta_{typ} \left[4 \langle (\Delta\theta)^2 \rangle + \frac{\langle (\Delta y)^2 \rangle}{\langle \eta_{typ} \rangle^2} \right]^{\frac{1}{2}}, \tag{3.15}$$

$$\approx \frac{K}{2\pi} N^{\frac{1}{2}} \left[4 \frac{R^2}{Q_o^2} \langle (\Delta\theta)^2 \rangle + Q_o^2 \langle (\Delta y)^2 \rangle \right]^{\frac{1}{2}}, \tag{3.16}$$

where we denoted

$$K = \frac{lb_1}{\rho}, \tag{3.17}$$

the strength of a regular quadrupole.

This formula should be compared with formula 41 of Ref. 4, which is similar but not identical.

4. EMITTANCE CHANGE DUE TO LINEAR COUPLING

This problem was considered earlier by K. Brown and R. Servranckx[7] for a transport line case and we follow their method supplemented by our treatment of Δ-terms[8], (see below). We also consider a ring case which requires an extension of their formalism. When the linear coupling is present one considers a single 4-dimensional ellipsoid instead of two separate invariant ellipses

$$\tilde{z}\sigma^{-1}z = 1, \tag{4.1}$$

where

$$z = \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}, \tag{4.2}$$

and

$$\sigma = \begin{bmatrix} \sigma_x & \chi \\ \tilde{\chi} & \sigma_y \end{bmatrix}, \tag{4.3}$$

is a symmetric and positive definite matrix while σ_x, σ_y are symmetric, positive-definite 2×2 submatrices describing projected emittance and χ represents the linear coupling. When passing from a point s_o to another s_1 in a ring the σ matrix transforms as follows

$$\sigma_1 = T\sigma_o \tilde{T}, \tag{4.4}$$

when

$$z_1 = T z_o .$$

Assuming that the initial beam is decoupled, ($\chi_o = 0$) one gets the relations

$$\sigma_{x1} = M\sigma_{xo}\tilde{M} + n\sigma_{yo}\tilde{n} , \tag{4.5}$$

and

$$\sigma_{y1} = N\sigma_{yo}\tilde{N} + m\sigma_{xo}\tilde{m} , \tag{4.6}$$

and

$$\chi_1 = M\sigma_{xo}\tilde{m} + n\sigma_{yo}\tilde{N} . \tag{4.7}$$

The initial condition $\chi_o = 0$ is pertinent to a transport line. For a ring rather periodic boundary condition is appropriate and this requires suitable extension of the formalism as it is given in the Section 4.B, and an Appendix. Denoting the initial projected emittance as ϵ_{xo}, ϵ_{yo} we have at the point s_o

$$\epsilon_{xo}^2 = Det(\sigma_{xo}) , \tag{4.8}$$

and

$$\epsilon_{yo}^2 = Det(\sigma_{yo}) , \tag{4.9}$$

and at the point s_1

$$\epsilon_{x1}^2 = Det(\sigma_{x1}) , \tag{4.10}$$

and

$$\epsilon_{y1}^2 = Det(\sigma_{y1}) , \tag{4.11}$$

where σ_{x1}, σ_{y1} are given by (4.5) and (4.6), respectively.

Let the initial beam ellipses be upright and match perfectly the machine ellipses

$$\sigma_{xo} = \begin{bmatrix} \sigma_{11} & O \\ O & \sigma_{22} \end{bmatrix} = \epsilon_{xo} \begin{bmatrix} \beta_{xo} & O \\ O & \beta_{xo}^{-1} \end{bmatrix} , \tag{4.12}$$

and

$$\sigma_{yo} = \begin{bmatrix} \sigma_{33} & O \\ O & \sigma_{44} \end{bmatrix} = \epsilon_{yo} \begin{bmatrix} \beta_{yo} & O \\ O & \beta_{yo}^{-1} \end{bmatrix} , \tag{4.13}$$

where we have assumed, for simplicity, that $\alpha_{xo} = \alpha_{yo} = 0$.

Using the identity,

$$Det(\ A\ +\ B\ ) = Det(\ A\ ) + Det(\ B\ ) + \begin{bmatrix} A_{11} & A_{12} \\ B_{21} & B_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ A_{21} & A_{22} \end{bmatrix} , \tag{4.14}$$

one finds for the projected emittance at some point s_1 downstream

$$\epsilon_{x1}^2 = \epsilon_{xo}^2\ Det^2(\ M\ ) + \epsilon_{yo}^2\ Det^2(\ n\ ) + \Delta_x\ , \tag{4.15}$$

and

$$\epsilon_{y1}^2 = \epsilon_{xo}^2\ Det^2(\ m\ ) + \epsilon_{yo}^2\ Det^2(\ N\ ) + \Delta_y\ , \tag{4.16}$$

where

$$\begin{aligned} \Delta_x = {} & \sigma_{11}\sigma_{33} \begin{bmatrix} M_{11} & n_{11} \\ M_{21} & n_{21} \end{bmatrix}^2 + \sigma_{11}\sigma_{44} \begin{bmatrix} M_{11} & n_{12} \\ M_{21} & n_{22} \end{bmatrix}^2 + \\ & + \sigma_{22}\sigma_{33} \begin{bmatrix} M_{12} & n_{11} \\ M_{22} & n_{21} \end{bmatrix}^2 + \sigma_{22}\sigma_{44} \begin{bmatrix} M_{12} & N_{12} \\ M_{22} & n_{22} \end{bmatrix}^2 , \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \Delta_y = {} & \sigma_{11}\sigma_{33} \begin{bmatrix} m_{11} & N_{11} \\ m_{21} & N_{21} \end{bmatrix}^2 + \sigma_{11}\sigma_{44} \begin{bmatrix} m_{11} & N_{12} \\ m_{21} & N_{22} \end{bmatrix}^2 + \\ & + \sigma_{22}\sigma_{33} \begin{bmatrix} m_{12} & N_{11} \\ m_{22} & N_{21} \end{bmatrix}^2 + \sigma_{22}\sigma_{44} \begin{bmatrix} m_{12} & N_{12} \\ m_{22} & N_{22} \end{bmatrix}^2 . \end{aligned} \tag{4.18}$$

It follows from the symplecticity of the matrix -T that

$$Det(\ m\ ) = Det(\ n\ ) \equiv \kappa\ , \tag{4.19}$$

and

$$Det(\, M\,) = Det(\, N\,) = 1 - \kappa \;, \tag{4.20}$$

and

$$\Delta_x = \Delta_y \equiv \Delta \;. \tag{4.21}$$

Thus the projected emittance, at the point s_1, is

$$\epsilon_{x1}^2 = (\, 1 - \kappa\,)^2\, \epsilon_{xo}^2 + \kappa^2\, \epsilon_{yo}^2 + \Delta \;, \tag{4.22}$$

and

$$\epsilon_{y1}^2 = \kappa^2\, \epsilon_{xo}^2 + (\, 1 - \kappa\,)^2\, \epsilon_{yo}^2 + \Delta \;, \tag{4.23}$$

where[8]

$$\begin{aligned} \Delta = \epsilon_{xo}\epsilon_{yo} \Big\{ \; & \beta_{xo}\beta_{yo} \begin{bmatrix} M_{11} & n_{11} \\ M_{21} & n_{21} \end{bmatrix}^2 + \beta_{xo}\beta_{yo}^{-1} \begin{bmatrix} M_{11} & n_{12} \\ M_{21} & n_{22} \end{bmatrix}^2 + \\ & + \beta_{xo}^{-1}\beta_{yo} \begin{bmatrix} M_{12} & n_{11} \\ M_{22} & n_{21} \end{bmatrix}^2 + \beta_{xo}^{-1}\beta_{yo}^{-1} \begin{bmatrix} M_{12} & n_{12} \\ M_{22} & n_{22} \end{bmatrix}^2 \Big\} \geq 0 \;. \end{aligned} \tag{4.24}$$

As a result we obtain the relation[7]

$$\epsilon_{x1}^2 - \epsilon_{y1}^2 = (\, 1 - 2\kappa\,) \left(\epsilon_{xo}^2 - \epsilon_{yo}^2 \right) . \tag{4.25}$$

Following Brown and Servranckx[7] we consider simple consequences of this relation for a transport line.

4.A. APPLICATION FOR A TRANSPORT LINE

Case 1°. It follows now that, if at the point s_o

$$\epsilon_{xo} = \epsilon_{yo} \;, \tag{4.26}$$

then

$$\epsilon_{x1} = \epsilon_{y1} \;, \tag{4.27}$$

at all points downstream.

Case 2°. If at any point downstream

$$\kappa = ½ , \tag{4.28}$$

then, at such the location

$$\epsilon_{x1} = \epsilon_{y1} = \left[\frac{1}{4} \left(\epsilon_{xo}^2 + \epsilon_{yo}^2 \right) + \Delta \right]^{\frac{1}{2}} . \tag{4.29}$$

Case 3°. If

$$\begin{gathered} \epsilon_{xo} \neq 0 \quad but \quad \epsilon_{yo} = 0 \quad , \\ (\sigma_{33} = \sigma_{44} = 0) \quad , \end{gathered} \tag{4.30}$$

then

$$\Delta = 0 ,$$

and

$$\begin{gathered} \epsilon_{x1} = | 1 - \kappa | \epsilon_{xo} , \\ and \\ \epsilon_{y1} = | \kappa | \epsilon_{xo} . \end{gathered} \tag{4.31}$$

If

$$0 \leq \kappa \leq 1 , \tag{4.32}$$

then

$$\epsilon_{x1} = (1 - \kappa) \epsilon_{xo} , \qquad \epsilon_{y1} = \kappa \epsilon_{xo} , \tag{4.33}$$

and, as the result the equality holds

$$\epsilon_{x1} + \epsilon_{y1} = \epsilon_{xo} . \tag{4.34}$$

Both projected emittances are bounded by the initial emittance ϵ_{xo}.

However, if at the point s_1

$$\kappa < 0 , \tag{4.35}$$

then

$$\epsilon_{x1} = (1 - \kappa)\,\epsilon_{xo}\,, \quad \epsilon_{y1} = -\kappa\,\epsilon_{xo}\,, \tag{4.36}$$

and the emittance can be large

$$\epsilon_{x1} - \epsilon_{y1} = \epsilon_{xo}\,. \tag{4.37}$$

Similarly, if at the point s_1

$$\kappa > 1\,, \tag{4.38}$$

then

$$\epsilon_{x1} = -(1 - \kappa)\,\epsilon_{xo}\,, \quad \epsilon_{y1} = \kappa\,\epsilon_{xo}\,, \tag{4.39}$$

and again the projected emittance can be large

$$\epsilon_{y1} - \epsilon_{x1} = \epsilon_{xo}\,. \tag{4.40}$$

Case 4°. If

$$\epsilon_{xo} \neq 0\,, \quad \mathit{and} \quad \epsilon_{yo} \neq 0\,,$$

one can show that quite generally that [8]

$$\epsilon_{x1} \geq |\,1 - \kappa\,|\,\epsilon_{xo} + |\,\kappa\,|\,\epsilon_{yo}\,, \tag{4.41}$$

and

$$\epsilon_{y1} \geq |\,\kappa\,|\,\epsilon_{xo} + |\,1 - \kappa\,|\,\epsilon_{yo}\,. \tag{4.42}$$

Hence, the sum of projected emittances can only grow, in general

$$\epsilon_{x1} + \epsilon_{y1} \geq \left(|1-\kappa| + |\kappa|\right)\left(\epsilon_{xo} + \epsilon_{yo}\right) \geq \epsilon_{xo} + \epsilon_{yo}\,, \tag{4.43}$$

depending on the magnitude of the $-\kappa$ at the point s_1 in a transfer line.

4.B. APPLICATION FOR A RING

Inside a circular accelerator, the proper condition in constraining the beam ellipsoid matrix is the periodic condition. In other words, formulae (4.4) now becomes

$$\sigma_1 = T \sigma \tilde{T} = \sigma = \begin{bmatrix} \sigma_x & \chi \\ \tilde{\chi} & \sigma_y \end{bmatrix}, \tag{4.44}$$

where now T stands for a single turn transfer matrix T(0), and the subscript "o" at σ was dropped since now $s_o = 0$ is any point in a ring, (which we also choose as a reference point in our calculations). As the result σ becomes determined, and thus it is not any more possible to assume that the submatrix χ vanishes, as it was done for the transfer line case.

Solving the above equation, (see Appendix for details), one finds in particular the expressions

$$\sigma_x = g^2(\rho_x + R_B \rho_y \tilde{R}_B) , \tag{4.45}$$

and

$$\sigma_y = g^2(\rho_y + R_A \rho_x \tilde{R}_A) , \tag{4.46}$$

and for the submatrix χ, which describes the coupling, one gets the result

$$\chi = g^2(\rho_x \tilde{R}_A + R_B \rho_y) . \tag{4.47}$$

Here we have used the notations:

$$\rho_x = \epsilon_A \begin{bmatrix} \beta_A & -\alpha_A \\ -\alpha_A & \gamma_A \end{bmatrix} , \tag{4.48}$$

is a symmetric and positive definite matrix, and similar for ρ_y, with A replaced by B, and

$$R_A = (t + \delta)^{-1} (m + \bar{n}) = -\tilde{R}_B , \tag{4.49}$$

and

$$g^2 = \frac{t + \delta}{2\delta} \quad , \tag{4.50}$$

with all the quantities being calculated at the reference point s = 0.

Notice that σ_x and σ_y agree with those given previously [cf. (4.12) and (4.14)] when the coupling is absent. It is so because $g^2 = 1$, and $R_{A,B} = 0$, and the new Twiss parameters coincide with those of the ideal lattice when the coupling vanishes. The same is true for the parameters $\epsilon_{A,B}$ which coincide with the projected emittance $\epsilon_{x,y}$. The χ submatrix vanishes, obviously, when the coupling is absent.

The formulae (4.45) and (4.46) enable us to calculate the projected emittance, at the point s = 0, using the formulae, (4.8) and (4.9) and the identity (4.14)

$$\epsilon_x^2 = g^4(\epsilon_A^2 + | R_B |^2 \epsilon_B^2 + \Delta'_x) \quad , \tag{4.51}$$

and

$$\epsilon_y^2 = g^4(\epsilon_B^2 + | R_A |^2 \epsilon_A^2 + \Delta'_y) \quad , \tag{4.52}$$

where

$$\Delta'_x = \Delta'_y = \Delta' \quad . \tag{4.53}$$

In the Thin Lens Model one finds

$$\Delta' \approx \frac{1}{4\pi^2(\Delta\nu)^2} \left[\sum_{k,l=1}^{N} q_k q_l \cos(\mu_x^k - \mu_x^l)\cos(\mu_y^k - \mu_y^l) + \frac{1}{2}\kappa \right] 2\epsilon_A\epsilon_B \tag{4.54}$$

where

$$\Delta\nu = \nu_x - \nu_y \quad . \tag{4.55}$$

Also, using the formulae (21) of Ref. 6, one finds the result

$$\kappa \approx \sum_{1 \le k < l \le N} q_k q_l \sin (\mu_x^k - \mu_x^l) \sin (\mu_y^k - \mu_y^l) \quad . \tag{4.56}$$

Denoting

$$\zeta = \frac{t - \delta}{2\delta} \quad , \tag{4.57}$$

one gets from (2.19), and (4.49) and (4.50), the results

$$g^4 \mid R_A \mid^2 = \zeta^2 \quad , \tag{4.58}$$

and

$$g^4 = (1 - \zeta)^2 \quad , \tag{4.59}$$

Hence, we obtain finally the projected emittance at some point s = 0 in a ring

$$\epsilon_x^2 = (1 - \zeta)^2 \epsilon_A^2 + \zeta^2 \epsilon_B^2 + \Delta' \quad , \tag{4.60}$$

and

$$\epsilon_y^2 = (1 - \zeta)^2 \epsilon_B^2 + \zeta^2 \epsilon_A^2 + \Delta' \quad . \tag{4.61}$$

Subtracting both equations, one gets the relation which is analogous to that for a transfer line [cf. (4.25)], only now ς is the relevant parameter instead of κ

$$\epsilon_x^2 - \epsilon_y^2 = (1 - 2\zeta) (\epsilon_A^2 - \epsilon_B^2) \quad . \tag{4.62}$$

Therefore, one may repeat the analysis and consider different cases of interest:

Case 1. If the new emittances are equal

$$\epsilon_A = \epsilon_B \quad , \tag{4.63}$$

then, at any point in the ring the project emittance coincide

$$\epsilon_x = \epsilon_y \quad . \tag{4.64}$$

Case 2. If at s = 0

$$\zeta = \frac{1}{2} \quad , \tag{4.65}$$

then, at this point

$$\epsilon_x = \epsilon_y = \left[\frac{1}{4} (\epsilon_A^2 + \epsilon_B^2) + \Delta' \right]^{\frac{1}{2}} . \tag{4.66}$$

Case 3. If

$$\epsilon_A \neq 0 \quad but \quad \epsilon_B = 0 \quad , \tag{4.67}$$

then

$$\Delta' = 0,$$

and

$$\epsilon_x = | 1 - \zeta | \epsilon_A$$

and (4.68)

$$\epsilon_y = | \zeta | \epsilon_A \quad .$$

If

$$0 \leq \zeta \leq 1 \quad , \tag{4.69}$$

then

$$\epsilon_x = (1 - \zeta) \epsilon_A \quad , \qquad \epsilon_y = \zeta \epsilon_A \quad ,$$

and, as the result the sum becomes constant

$$\epsilon_x + \epsilon_y = \epsilon_A \quad . \tag{4.70}$$

However, if at s = 0

$$\zeta < 0 \quad , \tag{4.71}$$

then

$$\epsilon_x = (1 - \zeta) \epsilon_A \quad , \qquad \epsilon_y = -\zeta \epsilon_A \quad ,$$

and now the difference becomes constant

$$\epsilon_x - \epsilon_y = \epsilon_A \quad . \tag{4.72}$$

Hence, the projected emittance can become large.

Similarly, if at the point s = 0

$$\zeta > 1 \quad , \tag{4.73}$$

then

$$\epsilon_x = -\,(\,1 - \zeta\,)\,\epsilon_A \,, \quad \epsilon_y = \zeta\,\epsilon_A \quad , \tag{4.74}$$

and again the projected emittance can be large

$$\epsilon_y - \epsilon_x = \epsilon_A \quad . \tag{4.75}$$

Similar situation arises when $\epsilon_A = 0$ but $\epsilon_B \neq 0$.

The sum of projected emittances can only exceed the sum of new emittances, in general, in accordance with inequalities analogous to (4.41), (4.42), and (4.43).

In the case when the coupling is produced by random errors one should consider rather mean values of these quantities. For this one finds first, using (3.11),

$$<\Delta'> \approx (\,\Delta\nu\,)^{-2}\left[(\,\Delta Q_{\min})_{rms}\,\right]^2 2\epsilon_A\epsilon_B \quad , \tag{4.76}$$

since

$$<\kappa> = 0 \quad . \tag{4.77}$$

Using the formulae (A.15) of Ref. 6 one finds the average

$$<Det(\,\bar{m} + n)> \approx 4\pi^2\sin^2(\,2\pi Q_o\,)\,[\,(\Delta Q_{\min})_{rms}\,]^2 \quad , \tag{4.78}$$

and as the result

$$<\zeta^2> \approx \frac{1}{2}[\,1 - sgn(t)\,]a \quad , \tag{4.79}$$

and

$$< (1 - \zeta)^2 > \approx \frac{1}{2}[1 + sgn(t)]a \quad , \tag{4.80}$$

where a is given by the expression

$$a = 1 - \frac{1}{2 (\Delta\nu)^2}[(\Delta Q_{\min})_{rms}]^2 \quad . \tag{4.81}$$

From this one finds for the average of the projected emittance

$$< \epsilon_{xo}^2 > \approx \epsilon_{A+}^2 + \epsilon_{B-}^2 + \tilde{\Delta} \quad , \tag{4.82}$$

where we have denoted

$$\tilde{\Delta} = \frac{1}{(\Delta\nu)^2} [(\Delta Q_{\min})_{rms}]^2 \left[2\epsilon_A\epsilon_B - \frac{1}{2} (\epsilon_{A+}^2 + \epsilon_{B-}^2) \right] \quad , \tag{4.83}$$

and

$$\epsilon_{A+} = \frac{1}{2}[1 + sgn(t)]\epsilon_A = \begin{array}{ll} \epsilon_A \, , & t > 0 \, , \\ 0 \, , & t < 0 \, , \end{array} \tag{4.84}$$

and

$$\epsilon_{B-} = \frac{1}{2}[1 - sgn(t)]\epsilon_B = \begin{array}{ll} 0 \, , & t > 0 \, , \\ \epsilon_B \, , & t < 0 \, . \end{array} \tag{4.85}$$

One gets similar results for the vertically projected emittance

$$< \epsilon_y^2 > = < \epsilon_x^2 >|\ A \leftrightarrow B \quad . \tag{4.86}$$

We can also perform relevant calculations of the projected emittance at some other point s_1 in a ring with the known emittance founded for position s_o. For that we have used the expressions which generalize formulae (4.5) and (4.6) to the case of a ring.

$$\sigma_{x1} = M\,\sigma_x\tilde{M} + n\,\sigma_y\,\tilde{n} + n\tilde{\chi}\,\tilde{M} + M\,\chi\,\tilde{n} \quad , \tag{4.87}$$

$$\sigma_{y1} = N\,\sigma_y\,\tilde{N} + m\,\sigma_x\,\tilde{m} + N\tilde{\chi}\tilde{m} + m\chi\,\tilde{N} \quad , \tag{4.88}$$

$$\chi_1 = M\,\chi\,\tilde{M} + n\tilde{\chi}\,\tilde{m} + M\sigma_x\,\tilde{m} + n\sigma_y\,\tilde{N} \quad , \tag{4.89}$$

where σ_x, σ_y and χ were given before [cf. (4.45), (4.46) and (4.47)] while the submatrices M, N, m, n belong now to the transfer matrix $T(s_1,0)$. After some calculations we get the expressions

$$\epsilon_{x1}^2 = (\,1 - \kappa\,)^2\,\epsilon_x^2 + \kappa^2\epsilon_y^2 + 2\kappa(\,1 - \kappa\,)Det(\,\chi\,) + \Delta''_x \quad , \tag{4.90}$$

and

$$\epsilon_{y1}^2 = (\,1 - \kappa\,)^2\,\epsilon_y^2 + \kappa^2\epsilon_x^2 + 2\kappa(\,1 - \kappa\,)Det(\,\chi\,) + \Delta''_y \quad , \tag{4.91}$$

where the two Δ'' - terms coincide only in part and are given by formulae which are, however, too lengthy to be presented here.

5. APPLICATION TO THE AGS BOOSTER

In the AGS Booster the sextuples are tightly connected to the adjacent quads so they share the vertical displacement errors and the twists, as well. The relevant parameters are:

$R = 32.1$ m	average radius
$Q_x = 4.63$	horizontal tune
$Q_y = 4.58$	vertical tune
$Q_o = 4.6$	average tune
$N = 48$	number of regular quads
$K = K_1 l = .55 \times .5 = .27$	quadrupole strength.

Assuming that the average errors are:

$$(\Delta y)_{rms} = 10^{-3}\ \text{m} \ , \tag{5.1}$$

and

$$(\Delta\theta)_{rms} = 10^{-3}\ \text{rad} \ , \tag{5.2}$$

we get from the formula (3.16) the result

$$(\Delta Q_{min})_{rms} \approx 4.4 \times 10^{-3} . \tag{5.3}$$

Using the formulae (4.68), (4.69), and (4.72), we get the results

$$(\epsilon_x)_{rms} \approx .99\ \epsilon_B + 7.7 \times 10^{-3} \epsilon_A \quad , \tag{5.4}$$

$$(\epsilon_y)_{rms} \approx .99\ \epsilon_A + 7.7 \times 10^{-3}\ \epsilon_B \quad . \tag{5.5}$$

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APPENDIX *Solution of the equation for σ*

Using the Edwards-Teng decomposition (2.10) one may transform the equation for σ

$$\sigma = T\ \sigma\ \tilde{T} \quad , \tag{A.1}$$

into the equation for $\rho = R^{-1}\ \sigma\ \tilde{R}^{-1}$ as follows

$$\rho = U\ \rho\ \tilde{U} = \begin{bmatrix} \rho_x & \eta \\ \tilde{\eta} & \rho_y \end{bmatrix} , \tag{A.2}$$

where

$$U = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} , \tag{A.3}$$

and

$$R = g \begin{bmatrix} 1 & R_B \\ R_A & 1 \end{bmatrix} , \tag{A.4}$$

with $R_{A,B}$ given by (4.49), and g by the formulae (4.50), while A and B are given by the formulae (2.11) and (2.14), respectively. The equation (A.2) is equivalent to the following set of equations for submatrices of ρ

$$\rho_x = A\ \rho_x\ \tilde{A} \quad , \tag{A.5}$$

and

$$\rho_y = B\ \rho_y\ \tilde{B} \quad , \tag{A.6}$$

and

$$\eta = A\ \eta\ \tilde{B} \quad . \tag{A.7}$$

Passing to the circular representation we get

$$A = B_A^{-1}\ R\ (\mu_A)\ B_A \quad , \tag{A.8}$$

and

$$B = B_B^{-1}\ R\ (\ \mu_B\ )\ B_B \quad , \tag{A.9}$$

where $B_{A,B}$ and $B_{A,B}^{-1}$ are given by the new Twiss parameters [cf. (2.2) and (2.3)], and R $(\mu_{A,B})$ are rotations.

The equations (A.5) and (A.6) yield for the matrices

$$\omega_x = B_A\ \rho_x\ B_A^{-1} \quad , \tag{A.10}$$

and

$$\omega_y = B_B\ \rho_y\ B_B^{-1} \quad , \tag{A.11}$$

the conditions

$$\omega_x = R\ (\mu_A)\ \omega_x\ \tilde{R}\ (\mu_A) \quad , \tag{A.12}$$

and

$$\omega_y = R\ (\mu_B)\ \omega_y\ \tilde{R}\ (\mu_B) \quad . \tag{A.13}$$

It follows now that both matrices ω_x, ω_y are proportional to the unit matrices since the angles $\mu_{A,B}$ are arbitrary. Hence, we get the results

$$\omega_x = \epsilon_A \cdot 1 \quad , \quad \omega_y = \epsilon_B \cdot 1 \quad , \tag{A.14}$$

where ϵ_A, ϵ_B are some non-negative numbers since ρ_x and ρ_y are positive-definite matrices. As the result we get from (A.10) and (A.11) the formula

$$\rho_x = \begin{bmatrix} \beta_A & -\alpha_A \\ -\alpha_A & \gamma_A \end{bmatrix} \epsilon_A \quad , \tag{A.15}$$

and similar for ρ_y, with A↔B.

The analysis of the equation (A.7) yields only trivial solution for η since the relevant determinant does not vanish, in general

$$\eta = 0 \quad . \tag{A.16}$$

Having ρ one finds σ and the formulae (4.45)-(4.47).

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