BNL-105148-2014-TECH

Booster Technical Note No. 104;BNL-105148-2014-IR

Review of space charge calculations

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January 1988

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U.S. Department of Energy

USDOE Office of Science (SC)

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REVIEW OF SPACE CHARGE CALCULATIONS

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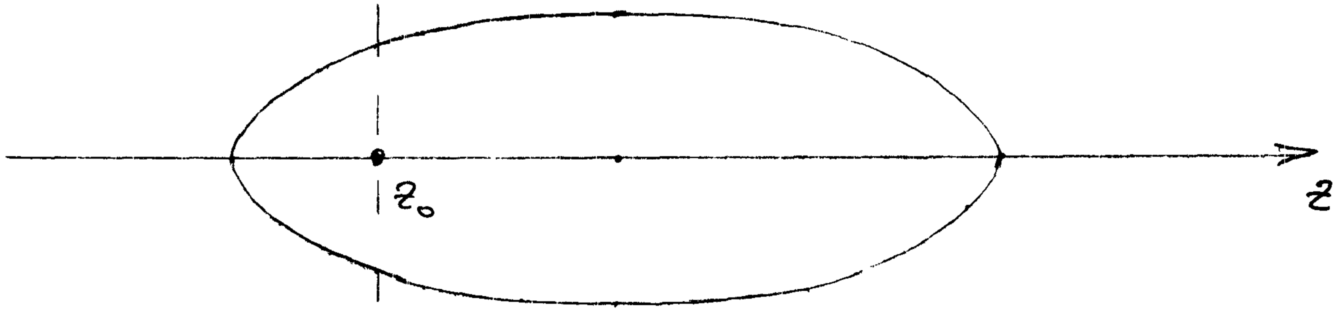
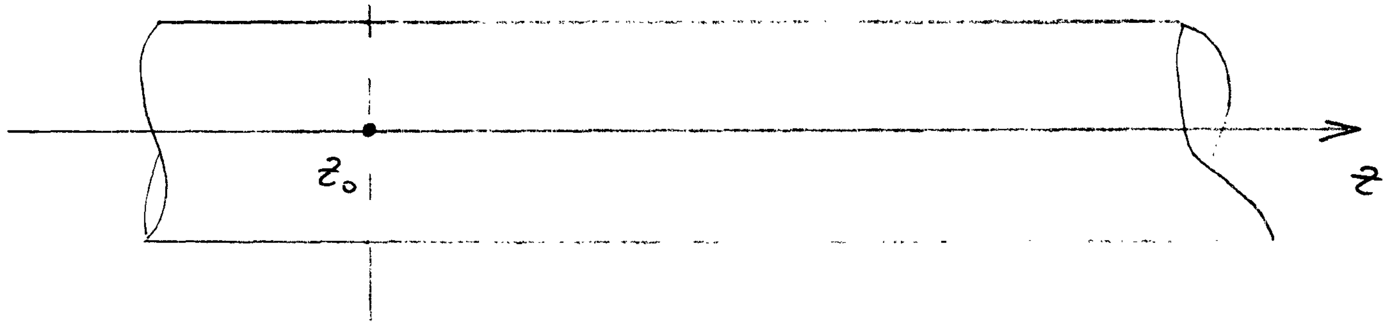
INTRODUCTION

In this note we review the calculation of the tune depression caused by the beam space charge. In the past calculations were done either for a uniform transverse distribution with elliptical cross-section or for a gaussian distribution with circular cross-section. The generalization to any charge distribution and geometry combinations was then applied "ad hoc".

In this paper we solve the case of a beam with elliptical cross-section and arbitrary transverse distribution. In particular we examine the case of uniform and gaussian distribution. The solution is found by employing, instead of the usual rectangular coordinates, a pair of elliptical coordinates for a constant aspect ratio.

The results are finally applied to the Booster and the AGS.





$$\begin{cases} \rho = Ne\, f(z-vt)\, g(x,y) \\ \vec{j} \equiv (0,0,j) \end{cases} \qquad j = \beta c \rho$$

$f(z-vt) \Rightarrow f(z_0) = f_0$, constant

$\mathrm{div}\,\vec{j} + \frac{1}{c}\frac{\partial \rho}{\partial t} = 0$ satisfied $\frac{\partial j}{\partial z} = \frac{\partial \rho}{\partial t} = \phi$

N, number of particles per bunch

V, scalar potential

$\vec{A} \equiv (0, 0, A)$, vector potenzial

"Free-Space" calculations - No boundary conditions -
Only longitudinal component of $\vec{A}$ different from ϕ -

$$\begin{cases} \nabla^2 V - \dfrac{1}{c^2} \dfrac{\partial^2 V}{\partial t^2} = -4\pi\rho \\ \nabla^2 A - \dfrac{1}{c^2} \dfrac{\partial^2 A}{\partial t^2} = -\dfrac{4\pi}{c} j = -4\pi\beta\rho \end{cases}$$

Thus $\qquad A = \beta V$

Also

$$\operatorname{div} \vec{A} + \frac{1}{c} \frac{\partial V}{\partial t} = 0$$

But $\qquad \dfrac{\partial A}{\partial z} = \dfrac{\partial V}{\partial t} = \phi$

and

$$\frac{\partial A}{\partial t} = \frac{\partial V}{\partial z} = \phi$$

$$\vec{F} = e\vec{E} + e\frac{\vec{v}}{c} \times \vec{H}$$

$$\vec{v} = (0, 0, v) \qquad \begin{cases} \vec{E} = -\operatorname{grad} V - \underbrace{\frac{1}{c}\frac{\partial \vec{A}}{\partial t}}_{=\phi} \\ \vec{H} = \operatorname{rot} \vec{A} \end{cases}$$

$$\begin{cases} E_x = -\dfrac{\partial V}{\partial x} \\ E_y = -\dfrac{\partial V}{\partial y} \end{cases}$$

$$\vec{H} \equiv \begin{Vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi & \phi & A \end{Vmatrix} \equiv \left\| \frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, \phi \right\|$$

$$\vec{v} \times \vec{H} \equiv \begin{Vmatrix} 0 & 0 & v \\ \frac{\partial A}{\partial y} & -\frac{\partial A}{\partial x} & 0 \end{Vmatrix} \equiv \left\| v\frac{\partial A}{\partial x}, v\frac{\partial A}{\partial y}, \phi \right\|$$

$$F_x = -e\frac{\partial V}{\partial x} + e\beta\frac{\partial A}{\partial x} = -e(1-\beta^2)\frac{\partial V}{\partial x}$$

$$F_y = -e\frac{\partial V}{\partial y} + e\beta\frac{\partial A}{\partial y} = -e(1-\beta^2)\frac{\partial V}{\partial y}$$

Equations of Motion :

$$\begin{cases} m_0 \gamma \dfrac{d^2 x}{dt^2} = F_x \\ m_0 \gamma \dfrac{d^2 y}{dt^2} = F_y \end{cases}$$

replace $ds = v\,dt$

$$\begin{cases} \dfrac{d^2 x}{ds^2} = \dfrac{F_x}{m_0 c^2 \beta^2 \gamma} = - \dfrac{e}{m_0 c^2 \beta^2 \gamma^3} \dfrac{\partial V}{\partial x} \\ \dfrac{d^2 y}{ds^2} = \dfrac{F_y}{m_0 c^2 \beta^2 \gamma} = - \dfrac{e}{m_0 c^2 \beta^2 \gamma^3} \dfrac{\partial V}{\partial y} \end{cases}$$

Tune Depression :

$$\begin{cases} \Delta \nu_x = \dfrac{e}{4\pi m_0 c^2 \beta^2 \gamma^3} \displaystyle\oint \dfrac{\beta_x}{x} \dfrac{\partial V}{\partial x}\, ds \\ \Delta \nu_y = \dfrac{e}{4\pi m_0 c^2 \beta^2 \gamma^3} \displaystyle\oint \dfrac{\beta_y}{y} \dfrac{\partial V}{\partial y}\, ds \end{cases}$$

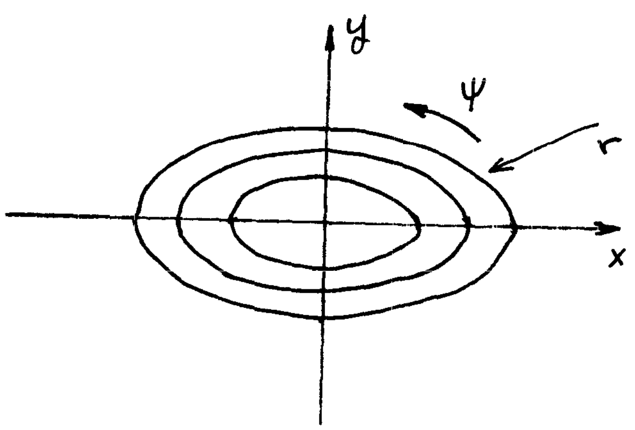
We need the calculation of $V = V(x, y)$

Equation for V

$$\nabla_\perp^2 V = -4\pi\rho$$

in rectangular coordinates : $\nabla_\perp^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

Let us use elliptical coordinates :



$$\begin{cases} x = r\cos\psi \\ y = \eta r \sin\psi \end{cases}$$

$$r^2 = x^2 + \frac{y^2}{\eta^2}$$

$$\eta \equiv \text{aspect ratio} = \frac{\text{vertical axis}}{\text{horizontal axis}}$$

$$d\tau \equiv \text{surface element} \equiv dx\,dy = g_r^{\frac{1}{2}}\, g_\psi^{\frac{1}{2}}\, dr\, d\psi$$

$$g_r = \cos^2\psi + \eta^2 \sin^2\psi$$

$$g_\psi = r^2\left(\sin^2\psi + \eta^2\cos^2\psi\right)$$

For $\eta = 1$ $\quad g_r = 1$ and $g_\psi = r^2$

The charge density can be written as

$$\rho = \frac{N e f_0 h(r)}{2\pi \alpha g_r}$$

It can be easily verified that

$$\int \rho \, d\tau = N e f_0$$

provided that

$$2\pi\alpha = \int_{-\pi}^{+\pi} \sqrt{\frac{\sin^2\varphi + \eta^2 \cos^2\varphi}{\cos^2\varphi + \eta^2 \sin^2\varphi}} \, d\varphi$$

and

$$\int_0^\infty r \, h(r) \, dr = 1$$

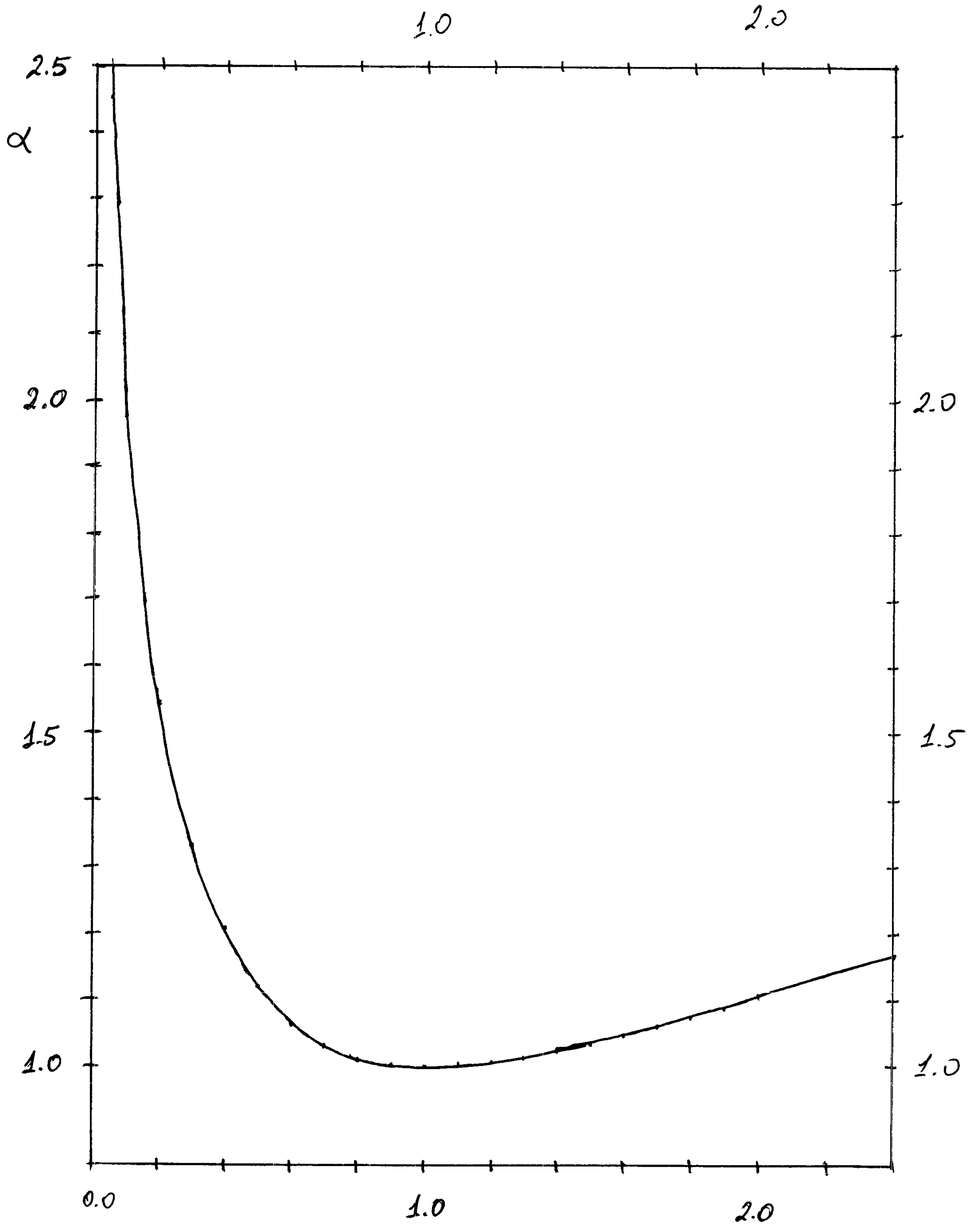


Figure 1

$$\nabla_\perp^2 V = -4\pi\rho = -\frac{2Ne f_o h(r)}{\alpha g_r}$$

If $\partial V/\partial\psi = \phi$ then

$$\nabla_\perp^2 V = \frac{1}{r g_r} \frac{d}{dr}\left(r \frac{dV}{dr}\right)$$

and with one integration

$$\frac{dV}{dr} = -\frac{2Ne f_o}{\alpha r} \int_0^r h(r') r' dr'$$

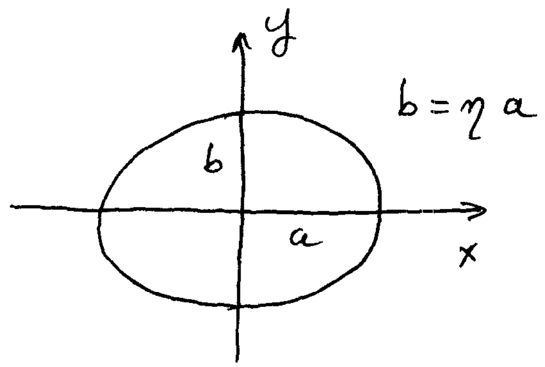
Also

$$\begin{cases} \dfrac{\partial V}{\partial x} = \dfrac{x}{r} \dfrac{dV}{dr} \\ \\ \dfrac{\partial V}{\partial y} = \dfrac{y}{\eta^2 r} \dfrac{dV}{dr} \end{cases}$$

$$\Delta \nu_x = - \frac{N f_0 r_0}{2\pi \beta^2 \gamma^3} \oint \frac{\beta_x \, ds}{\alpha r^2} \int_0^r h(r')\, r' dr'$$

$$\Delta \nu_y = - \frac{N f_0 r_0}{2\pi \beta^2 \gamma^3} \oint \frac{\beta_y \, ds}{\alpha \eta^2 r^2} \int_0^r h(r')\, r' dr'$$

$$r_0 = e^2 / m_0 c^2$$



<u>Uniform Distribution</u>

$$h(r) = 2/a^2 \quad \text{for} \quad r < a$$
$$= \phi \quad \text{for} \quad r > a$$

then

$$\frac{1}{r^2} \int_0^r h(r')\, r' dr' = \frac{1}{a^2} \quad , \quad \underline{\text{constant}}$$

$$\begin{cases} \Delta\nu_x = -\dfrac{N r_0 f_0}{2\pi\beta^2\gamma^3} \oint \dfrac{\beta_x}{\alpha\, a^2}\, ds & a^2 = 2\sigma_x^2 \\[2ex] \Delta\nu_y = -\dfrac{N r_0 f_0}{2\pi\beta^2\gamma^3} \oint \dfrac{\beta_y}{\alpha\, b^2}\, ds & b^2 = 2\sigma_y^2 \end{cases}$$

<u>Gaussian Distribution</u>

$$h(r) = \frac{\exp(-r^2/2\sigma_x^2)}{\sigma_x^2} \qquad \sigma_y = \eta\, \sigma_x$$

$$\frac{1}{r^2}\int_0^r h(r')r'dr' = \frac{1 - e^{-r^2/2\sigma_x^2}}{r^2}$$

$$\approx \frac{1}{2\sigma_x^2} \quad \text{for} \quad r^2 \ll 2\sigma_x^2$$

$$\approx \frac{1}{r^2} \quad \text{for} \quad r^2 \gg 2\sigma_x^2$$

Longitudinal Distribution

* Uniform, full bunch length L

$$f_0 = 1/L \qquad \text{independent of position } z_0$$

* Gaussian, rms bunch length σ

$$f_0 = \frac{e^{-z_0^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma} \qquad \text{which varies with location } z_0$$

M bunches equally spaced

Bunching Factor

$$B = \frac{ML}{2\pi R} \qquad \text{uniform distribution}$$

$$= \frac{\sqrt{2\pi}\,\sigma}{2\pi R} \qquad \text{gaussian distribution}$$

$2\pi R$, circumference

$N_{tot} = M N$

We have

$$\begin{cases} \Delta\nu_x = \dfrac{N_{tot}\, r_0}{2\beta^2\gamma^3 B\, \varepsilon_x} J_x & (1) \\[2em] \Delta\nu_y = \dfrac{N_{tot}\, r_0}{2\beta^2\gamma^3 B\, \varepsilon_y} J_y & (2) \end{cases}$$

where

$$J_x = \frac{1}{2\pi R} \oint \frac{\beta_x\, \varepsilon_x}{\alpha\, a^2 \pi}\, ds$$

$$J_y = \frac{1}{2\pi R} \oint \frac{\beta_y\, \varepsilon_y}{\alpha\, b^2 \pi}\, ds$$

We assume that there is no dispersion in the vertical plane, then

$$\varepsilon_y = \pi b^2 / \beta_y$$

and

$$J_y = \frac{1}{2\pi R} \oint \frac{ds}{\alpha} \qquad (3)$$

On the other hand, if X_p denotes the dispersion in the horizontal plane,

$$a^2 = \frac{\varepsilon_x \beta_x}{\pi} + \Delta^2 X_p^2$$

and

Δ = half of the relative momentum spread for a uniform distribution, otherwise

$\Delta^2 = 2\delta^2$ for a gaussian distribution with rms width δ.

$$J_x = \frac{1}{2\pi R} \oint \frac{ds}{\left(1 + \frac{\Delta^2 X_p^2}{\varepsilon_x \beta_x} \pi\right) \alpha} \qquad (4)$$

Discussion :

1. The tune-shift depends on the location of the particle within the bunch (z_0). For a uniform longitudinal distribution the dependence vanishes. For a gaussian longitudinal distribution there is a variation between the centre of the bunch and the location at the tail which equals the tune-shift value at the centre. Clearly synchrotron oscillations will cause a tune-modulation which depends on the amplitude of the oscillation. This should be studied with particle tracking.

To minimize the effect, one should aim for a uniform longitudinal distribution during rf capture ("painting").

2. If the transverse charge distribution is uniform the tune-shift is independent of the amplitudes of the betatron oscillations. The tune-shifts are given by (1) and (2) where ε_x and ε_y are the "total" beam emittances. This is the most favourable case and one should aim to a uniform transverse distribution during multi-turn injection ("painting").

If the transverse charge distribution is gaussian the tune-shifts are still given by (1) and (2) but now ε_x and ε_y are the r.m.s. values of emittance

$$\varepsilon_x = 2\pi \sigma_x^2/\beta_x \qquad \varepsilon_y = 2\pi \sigma_y^2/\beta_y$$

Moreover there is a variation of the tune-shift with the amplitude of the betatron oscillation. Eventually the tune-spread equals the tune-shift at the centre of the beam.

3. The tune-shifts given by (1) and (2) are a factor of two larger than those obtained with more conventional formulae (see for instance the RHIC CDR). The difference is caused by our assumption here that the distribution in x is closely correlated to the distribution in y. If the two distributions were completely decoupled, as it has been assumed in other places, the resulting tune-shifts would be smaller. Thus the degree of coupling could affect crucially the results.

4. The two form factors (3) and (4) show the dependence of the tune-shifts on the beam aspect ratio and on the relative contribution of the beam momentum spread to the beam size. Observe that the aspect ratio

$$\eta = \sqrt{\frac{\varepsilon_y \beta_y}{\varepsilon_x \beta_x \left(1 + \frac{\Delta^2 X_p^2}{\varepsilon_x \beta_x} \pi \right)}}$$

which is independent of the particular shape of the distribution (uniform, gaussian). During the rf-capture process and the multiturn injection this quantity can vary considerably from turn to turn and there could be a significant variation from one machine to another depending on their repetition rates.

To calculate J_x and J_y we need to estimate α as a function of η. This can be derived from the plot of Figure 1.

Application to the AGS - Booster

For the case of a proton beam, multi-turn injection is achieved by the method of charge exchange. Thus, turn after turn, the beam "roundness" and dimensions are preserved, that is $\varepsilon_x = \varepsilon_y$. We will estimate the tune-shifts at the end of the capture process and assume uniform distribution in all dimensions.

Beam parameters are

$$N_{tot} = 1.5 \times 10^{13}$$
$$\beta = 0.56616$$
$$\gamma = 1.21316$$

and the tune-depression is (for each plane)

$$\Delta\nu = \frac{6.4\,J}{B\,\varepsilon}$$

where ε is in π mm·mrad units.

The beam aspect ratio η varies between 0.5 and 2. Thus, from Figure 1 we see that α will vary only between 1 and 1.1. It derives that $J_y \sim 1$ and that J_x is even smaller. Thus the largest tune-shift is on the vertical plane.

The results are shown in Figure 2. They apply also to the case of the AGS with the present mode of injection at 200 MeV. Indeed also in the AGS $J_y \sim 1$.

Booster : $N = 5 \times 10^{12}$ in 3 bunches

AGS : $N = 1.25 \times 10^{12}$ in 12 bunches

($\varepsilon_N = 20\pi$ mm·mrad $\rightarrow$ $\varepsilon = 30\pi$ mm·mrad)

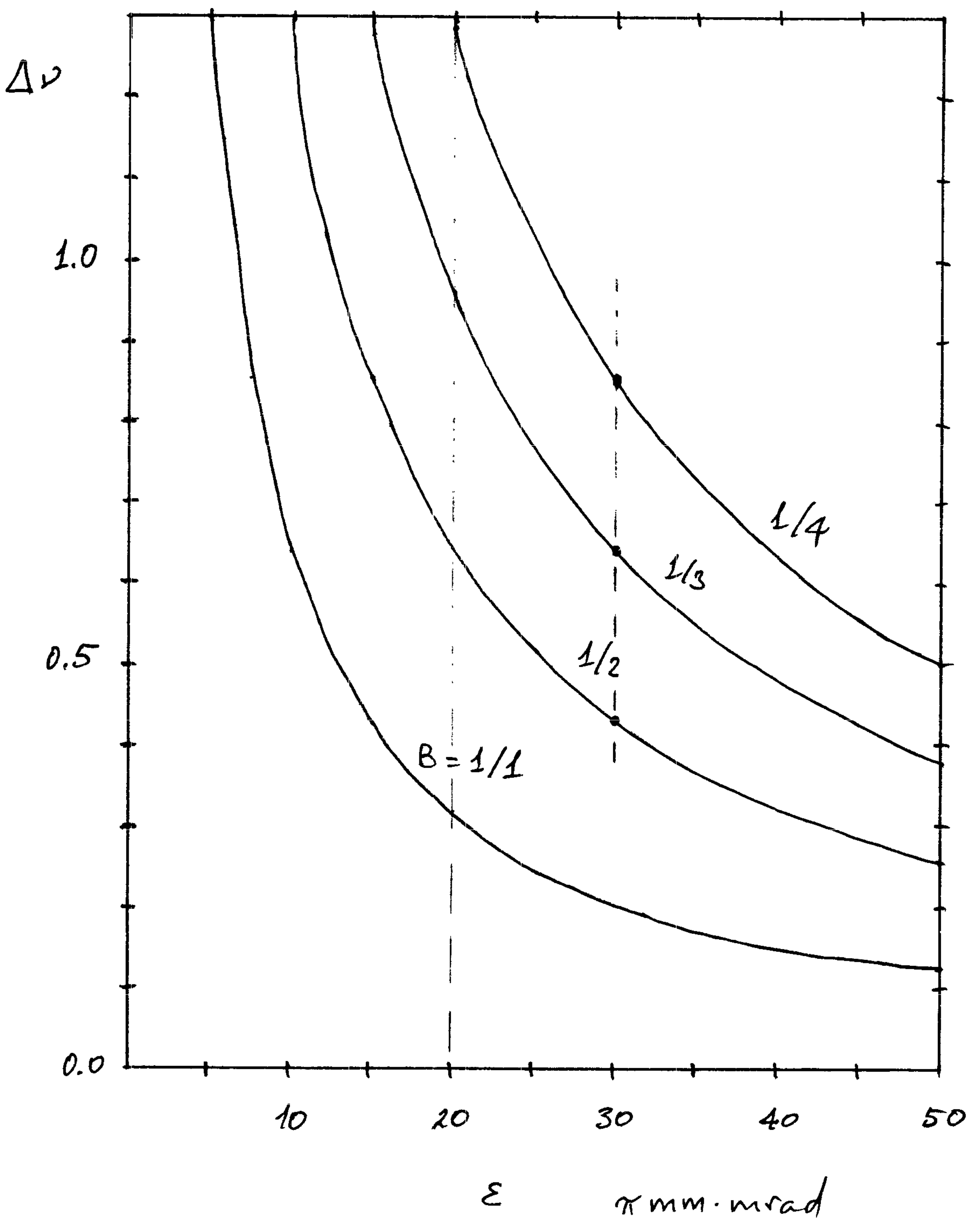


Figure 2